

Inversion Formula For Generalized Three-Dimensional Fractional Cosine Transform

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Abstract :

The fractional Fourier transform (FRFT) is an extension of the conventional Fourier transform. The fractional cosine transform is strongly associated with the fractional Fourier transform and is now used in optical and signal processing applications. Most often used in the field of image processing, particularly when dealing with compression. This paper describes an extension of the fractional cosine transform in three dimensions. This paper defines the test function space and generalises the 3D fractional cosine transform. Concurrently, the inverse formula and uniqueness theorem for the 3D fractional cosine transform are established.

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1. Introduction:

V. Namias invented fractional operator of Fourier transform (FT) in 1980 [1]. He originally defined and mathematically framed Fractional Fourier Transform (FRFT). The phase plane angle α determines the Fractional Fourier transform (FRFT). This generalises signal processing notions like space (or time) and frequency domain. In signal and image processing, the Cosine and Sine transformations and their discrete equivalents are important for signal coding [2], watermarking [3], and de-focused picture restoration [4]. In 1996, Lohmann [6] first defined FrFST and FrFCT with signal processing applications. Pei and Yeh [7] expanded the Cosine transform to the discrete fractional cosine transform (DFrCT) and the discrete fractional sine transform (DFrST). Both of them exhibit the DFrFT's angle additivity feature nicely. Furthermore, the DFrCT and DFrST are employed in the digital computation of FrFT to reduce the DFrFT's computing burden. Sharma and Khapre [8]

established an inversion formula for generalized two-dimensional fraction cosine and sine transform.

Definition 1.1 Two-dimensional fractional Cosine transform with parameter α of $f(t_1, t_2)$ denoted by $F_C^\alpha(t_1, t_2)$ perform a linear operation given by the integral transform.

$$F_C^\alpha\{f(t_1, t_2)\}(l_1, l_2) = \int_0^\infty \int_0^\infty f(x, y)K_\alpha(t_1, t_2, l_1, l_2)dt_1 dt_2 \quad (1.1)$$

Where the Kernel,

$$K_C^\alpha(t_1, t_2, l_1, l_2) = \sqrt{\frac{1-icota}{2\pi}} e^{\frac{i(t_1^2+t_2^2+l_1^2+l_2^2)cota}{2}} \cos(\text{coseca}.l_1t_1). \cos(\text{coseca}.l_2t_2)$$

Definition 1.2 (Three-Dimensional Generalized Fractional Cosine Transform)

Three-dimensional fractional Cosine transform with parameter α of $f(t_1, t_2, t_3)$ denoted by $F_C^\alpha(t_1, t_2, t_3)$ perform a linear operation given by the integral transform.

$$F_C^\alpha\{f(t_1, t_2, t_3)\}(l_1, l_2, l_3) = \int_0^\infty \int_0^\infty \int_0^\infty f(t_1, t_2, t_3)K_\alpha(t_1, t_2, t_3, l_1, l_2, l_3)dt_1 dt_2 dt_3 \quad (1.2)$$

where the Kernel,

$$K_C^\alpha(t_1, t_2, t_3, l_1, l_2, l_3) = \sqrt{\frac{1-icota}{2\pi}} e^{\frac{i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cota}{2}} \cos(\text{coseca}.l_1t_1). \cos(\text{coseca}.l_2t_2)\cos(\text{coseca}.l_3t_3) \quad (1.3)$$

2. The Test Function Space E

An infinitely differentiable complex-valued function ψ on R^n belong to $E(R^n)$ if for each compact set $I \subset S_{p,q,r}$,

$$\text{where, } S_{p,q,r} = \{t_1, t_2, t_3: t_1, t_2, t_3 \in R^n, |t_1| \leq p, |t_2| \leq q, |t_3| \leq r, q, p, r > 0\}, I \in R^n \quad (2.1)$$

$$Y_{E,a,b,c}(\psi) = \text{Sup}_{t_1,t_2,t_3} |D_{t_1,t_2,t_3}^{a,b,c} \psi(t_1, t_2, t_3)| < \infty \text{ and, } a, b, c = 1,2,3,4,5, \dots$$

Thus $E(R^n)$ will denote the space of all $\psi \in E(R^n)$ with support contained in $S_{p,q,r}$.

Note that the space E is complete and, therefore, a Frechet space. Moreover, we say that f is a fractional Cosine transformable if it is a member of E^* , the dual space of E .

2.1 Distributional Three-Dimensional Fractional Cosine Transform

The three-dimensional distributional fractional cosine transform of $f(t_1, t_2, t_3) \in E^*(R^n)$ defined by $F_C^\alpha\{f(t_1, t_2, t_3)\} = F^\alpha(l_1, l_2, l_3) = \langle f(t_1, t_2, t_3), K_\alpha(t_1, t_2, t_3, l_1, l_2, l_3) \rangle$

$$K_C^\alpha(t_1, t_2, t_3, l_1, l_2, l_3) =$$

$$\sqrt{\frac{1-icot\alpha}{2\pi}} e^{\frac{i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \cos(\text{cosec}\alpha \cdot l_1 t_1) \cos(\text{cosec}\alpha \cdot l_2 t_2) \cos(\text{cosec}\alpha \cdot l_3 t_3) \quad (2.2)$$

Where, RHS of the above equation has a meaning as the application of $f \in E^*$ to $K_\alpha(t_1, t_2, t_3, l_1, l_2, l_3) \in E^*$

3. Inversion Formula for Generalized 3Dimensional Fractional Cosine Transform (3DFrCT):

Theorem 3.1: If the three-dimensional fractional cosine transform is given by

$$F_C^\alpha\{f(t_1, t_2, t_3)\}(l_1, l_2, l_3) = \int_0^\infty \int_0^\infty \int_0^\infty f(t_1, t_2, t_3) K_\alpha(t_1, t_2, t_3, l_1, l_2, l_3) dt_1 dt_2 dt_3 \quad (3.1)$$

Then its inverse $f(t_1, t_2, t_3)$ is given by

$$f(t_1, t_2, t_3) = \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty F_C^\alpha(l_1, l_2, l_3) \overline{K}_\alpha(t_1, t_2, t_3, l_1, l_2, l_3) dl_1 dl_2 dl_3 \quad (3.2)$$

Where,

$$\overline{K}_\alpha = e^{\frac{-i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \left(\sqrt{\frac{1-icot\alpha}{2\pi}} \right)^{-1} \text{cosec}^3 \alpha \cos(\text{cosec}\alpha \cdot l_1 t_1) \cos(\text{cosec}\alpha \cdot l_2 t_2) \cos(\text{cosec}\alpha \cdot l_3 t_3)$$

Proof:

$$\begin{aligned} F_C^\alpha(l_1, l_2, l_3) e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \\ = C_k \int_0^\infty \int_0^\infty \int_0^\infty f(t_1, t_2, t_3) e^{\frac{i(t_1^2+t_2^2+t_3^2)cot\alpha}{2}} \cos(\text{cosec}\alpha \cdot l_1 t_1) \cos(\text{cosec}\alpha \cdot l_2 t_2) \\ \cos(\text{cosec}\alpha \cdot l_3 t_3) dt_1 dt_2 dt_3, \end{aligned} \quad (3.3)$$

where $C_k = \sqrt{\frac{1-icot\alpha}{2\pi}}$.

$$\begin{aligned} F_C^\alpha(l_1, l_2, l_3) e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} = \\ C_k \int_0^\infty \int_0^\infty \int_0^\infty g(t_1, t_2, t_3) \cos(\text{cosec}\alpha \cdot l_1 t_1) \cos(\text{cosec}\alpha \cdot l_2 t_2) \cos(\text{cosec}\alpha \cdot l_3 t_3) dt_1 dt_2 dt_3 \end{aligned} \quad (3.4)$$

where, $g(t_1, t_2, t_3) = C_k e^{\frac{i(t_1^2+t_2^2+t_3^2)cot\alpha}{2}} f(t_1, t_2, t_3)$.

$$F_C^\alpha(l_1, l_2, l_3) e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} = [Cg(t_1, t_2, t_3)] \cos(\text{cosec}\alpha \cdot l_1) \cos(\text{cosec}\alpha \cdot l_2) \cos(\text{cosec}\alpha \cdot l_3) \quad (3.5)$$

Let $\text{cosec}\alpha \cdot l_1 = \eta, \quad \text{cosec}\alpha \cdot l_2 = \zeta, \quad \text{cosec}\alpha \cdot l_3 = \mu$

$\therefore d\eta = \text{cosec}\alpha \cdot dl_1, \quad d\zeta = \text{cosec}\alpha \cdot dl_2, \quad d\mu = \text{cosec}\alpha \cdot dl_3$

$$F_C^\alpha(l_1, l_2, l_3) e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} = [Cg(t_1, t_2, t_3)](\eta, \zeta, \mu) \quad (3.6)$$

$$F_C^\alpha(l_1, l_2, l_3)e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} = G(\eta, \zeta, \mu) \quad (3.7)$$

The R.H.S is the cosine transform of $g(t_1, t_2, t_3)$ with argument η, ζ, μ .

Applying the inverse cosine transform we can write ,

$$g(t_1, t_2, t_3) = \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty G(\eta, \zeta, \mu) \cos(\eta t_1) \cos(\zeta t_2) \cos(\mu t_3) d\eta d\zeta d\mu \quad (3.8)$$

$$\begin{aligned} C_k e^{\frac{i(t_1^2+t_2^2+t_3^2)cot\alpha}{2}} f(t_1, t_2, t_3) &= \\ \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty F_C^\alpha(l_1, l_2, l_3) e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \cos(\eta x) \cos(\zeta y) \cos(\mu z) d\eta d\zeta d\mu &= \\ C_k e^{\frac{i(t_1^2+t_2^2+t_3^2)cot\alpha}{2}} f(t_1, t_2, t_3) &= \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty F_C^\alpha(u, v, w) e^{\frac{-i(l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \cos(\text{cosec}\alpha.l_1 t_1) \\ &\cdot \cos(\text{cosec}\alpha.l_2 t_2) \cos(\text{cosec}\alpha.l_3 t_3) \text{cosec}^3\alpha dl_1 dl_2 dl_3. \end{aligned} \quad (3.9)$$

$$\begin{aligned} f(t_1, t_2, t_3) &= \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty F_C^\alpha(u, v, w) C_k^{-1} e^{\frac{-i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \cos(\text{cosec}\alpha.l_1 t_1) \\ f(t_1, t_2, t_3) &= \frac{8}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty F_C^\alpha(l_1, l_2, l_3) \overline{K}_\alpha(t_1, t_2, t_3, l_1, l_2, l_3) dl_1 dl_2 dl_3, \end{aligned} \quad (3.10)$$

where,

$$\begin{aligned} \overline{K}_\alpha(t_1, t_2, t_3, l_1, l_2, l_3) &= \\ \sqrt{\frac{1-icot\alpha}{2\pi}} e^{\frac{-i(t_1^2+t_2^2+t_3^2+l_1^2+l_2^2+l_3^2)cot\alpha}{2}} \cos(\text{cosec}\alpha.l_1 t_1) \cos(\text{cosec}\alpha.l_2 t_2) \cos(\text{cosec}\alpha.l_3 t_3) \text{cosec}^3\alpha \end{aligned}$$

Theorem 3.2: Uniqueness Theorem for Generalized Three-Dimensional Fractional Cosine Transform :

$$\text{If } [F_C^\alpha f(t_1, t_2, t_3)](l_1, l_2, l_3) = F(l_1, l_2, l_3) [F_C^\alpha g(t_1, t_2, t_3)](l_1, l_2, l_3) = G(l_1, l_2, l_3) \quad (3.11)$$

for $0 < \alpha \leq \frac{\pi}{2}$ and $Supf \subset S_{p,q,r}, Supg \subset S_{p,q,r}$, where $S_{p,q,r} = \{t_1, t_2, t_3: t_1, t_2, t_3 \in R, |t_1| \leq p, |t_2| \leq q, |t_3| \leq r, p, q, r > 0\}$. If $F_\alpha(l_1, l_2, l_3) = G_\alpha$ then $f = g$ in the sense of equality in $D^*(I)$.

Proof. By inversion theorem

$$f - g = \frac{8}{\pi^3} \lim_{x \rightarrow \infty} \int_{-x}^x \int_{-x}^x \int_{-x}^x \overline{K}_\alpha(t_1, t_2, t_3, l_1, l_2, l_3) dl_1 dl_2 dl_3 = 0. \quad (3.12)$$

Thus $f = g$ in $D^*(I)$.

4. Conclusion

This research focuses on developing the space of testing functions and the distributional generalized three-dimensional fractional cosine transform. The study establishes the foundational framework for these transformations and proves the inversion and uniqueness theorems associated with the generalized three-dimensional fractional cosine transform. These results demonstrate the robustness of the transform's mathematical properties and ensure its applicability to broader analytical contexts. The work lays a solid theoretical groundwork for future research and practical implementations in signal processing and related fields by providing a rigorous approach to inversion and uniqueness. This advancement opens new avenues for exploration in mathematical analysis.

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