Exact Solutions for Stochastic Fractional Zhiber-Shabat Equations

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Abstract

This paper is devoted to give exact solutions of the variable coefficient fractional Zhiber -Shabat equation with space-time-fractional derivatives. Moreover, by using the Hermite transform and the homogeneous balance principle, the white noise functional solutions for the Wick-type stochastic fractional Zhiber-Shabat equation are explicitly shown. Detailed computations and implemented examples are explicitly provided.

Keywords: Fractional Zhiber-Shabat equations; White noise; Stochastic; Hermite transform.

MSC: 60H30; 60H15; 35R60

1 Introduction

The main task of this paper is to explore exact solutions for the following fractional Zhiber-Shabat equation with variable coefficients:

$$\partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}u + p(t)e^{u} + q(t)e^{-u} + r(t)e^{-2u} = 0 \tag{1.1}$$

where $\partial_{x^{\alpha_1}}, \partial_{t^{\alpha_2}} (0 < \alpha_1, \alpha_2 < 0)$ are the modified Riemann-Liouville fractional derivatives defined by Jumarie [6] and $q(t), p(t)$ and $r(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Random waves is an important subject of random fractional partial differential equations. Recently, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear integer-order differential equation. In the past decades, an important progress has been made in the research of the exact solutions of nonlinear partial differential equations (PDEs). To seek various exact solutions of multifarious physical models described by nonlinear

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PDEs, various methods have been proposed. There are many authers studied this subject. Wadati first introduced and studied the stochastic KdV equation and gave the diffusion of soliton of the KdV equation under Gaussian noise in ([10]-[12]). Xie firstly researched Wick-type stochastic KdV equation on white noise space and showed the auto-Bachlund transformation and the exact white noise functional solutions in [14], furthermore, Chen and Xie ([1]-[3]) and Xie ([15]-[17]) researched some Wick-type stochastic wave equations using white noise analysis method. Recently, Uğurlu and Kaya[9] gave the tanh function method, Wazzan [13] showed the modified tanh-coth method, these methods have been applied to derive nonlinear transformations and exact solutions of nonlinear PDEs in mathematical physics. If Eqn.(1.1) is considered in random environment, we can get random fractional Zhiber-Shabat equation with space-fractional derivatives. In order to give the exact solutions of random fractional Zhiber-Shabat equation with space-fractional derivatives, we only consider this problem in white noise environment. Wick-type stochastic generalized fractional Zhiber-Shabat equations with space-fractional derivatives is the perturbation of Eqn.(1.1) by random force $W(t) \diamond R^{\diamond}(U, U_{xt})$, which represented by:

$$\partial_{x^{\alpha_1}} \partial_{t^{\alpha_2}} U + P(t) \diamond e^{\diamond U} + Q(t) \diamond e^{\diamond(-U)} + R(t) e^{\diamond(-2U)} = W(t) \diamond R^{\diamond}(U, U_{x^{\alpha_1} t^{\alpha_2}}) \tag{1.2}$$

where $W(t)$ is Gaussian white noise, i.e., $W(t) = B^{\cdot}(t)$ and B(t) is a Brownian motion, $R(u, u_{x^{\alpha_1} t^{\alpha_2}})$ $= -\beta_1 \partial_{x^{\alpha_1}} \partial_{t^{\alpha_2}} u - \beta_2 e^u - \beta_3 e^{-u} - \beta_4 e^{-2u}$ is a functional of $u, \partial_{x^{\alpha_1}} \partial_{t^{\alpha_2}} u := \frac{\partial^{\alpha_1+\alpha_2} u}{\partial x^{\alpha_1} \partial x^{\alpha_2}} = u_{x^{\alpha_1} x^{\alpha_2}}$ for some constants $\beta_1, ..., \beta_4$ and $R^{\diamond}$ is the Wick version of the functional R. " $\diamond$ " is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$ and $P(t), Q(t)$ and $R(t)$ are white noise functionals. Eqn.(1.2) can be seen as the perturbation of the coefficients $p(t), q(t)$ and $r(t)$ of Eqn.(1.1) by white noise functionals.

This paper is devoted to give white noise functional solution for Wick-type stochastic generalized fractional Zhiber-Shabat equations with space-fractional derivatives. Moreover, the Hermite transform and the homogenous balance principle are employed to find the exact solution for stochastic fractional Zhiber-Shabat equation with variable coefficient. Finally, implemented examples are explicitly shown.

2 Preliminaries

There are different definitions for fractional derivatives, for more details (see [5, 6]). In our paper we use the modified Riemann-Liouville derivative defined by Jumarie [6]:

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha-1} [f(y) - f(0)] dy, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \int_0^x (x-y)^{-\alpha} [f(y) - f(0)] dy, & 0 < \alpha < 1, \\ \left[f^{(\alpha-n)}(x) \right]^{(n)}, & n \leq \alpha < n+1, \quad n \in \mathbb{N} \end{cases} \tag{2.1}$$

which has merits over the original one, for example, the α-order derivative of a constant is zero. Some properties of the modified Riemann-Liouville derivative were summarized in [5] , three useful

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formulas of them are

$$\begin{cases} D_x^\alpha x^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta > 0, \\ D_x^\alpha (u(x)v(x)) = u(x) D_x^\alpha v(x) + v(x) D_x^\alpha u(x), \\ D_x^\alpha [f(u(x)] = \dfrac{df}{du} D_x^\alpha u(x) = \left(\dfrac{du}{dx}\right)^\alpha D_u^\alpha f(u). \end{cases} \tag{2.2}$$

Now, we outline the main idea of the modified fractional sub-equation method. Many authors considered nonlinear FPDE, say, in two variables

$$F(u, u_x, u_t, D_x^\alpha u, D_t^\alpha u, ...) = 0, \qquad 0 < \alpha \leq 1 \tag{2.3}$$

where F is a nonlinear function with respect to the indicated variables. To determine the solution $u = u(x,t)$ explicitly, we first introduce the following transformation

$$u = u(\xi), \qquad \xi = \xi(x,t) \tag{2.4}$$

which converts Eq.(2.3) into a fractional ordinary differential equation

$$G(u, u', u'', D_\xi^\alpha u, D_\xi^{2\alpha} u, ...) = 0. \tag{2.5}$$

Next we introduce a new variable $Y = Y(\xi)$ which is a solution of the fractional Riccati equation

$$D_\xi^\alpha Y = h_0 + h_1 Y + h_2 Y^2, \quad 0 < \alpha \leq 1, \tag{2.6}$$

where h_0, h_1 and h_2 are arbitrary constants. Eq.(2.6) is the fractional Riccati differential equation, where α is a parameter describing the order of the fractional derivative. In the case of $\alpha = 1$ Eq.(2.6) is reduced to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems. The feed back gain of the linear quadratic optimal control depends on a solution of a Riccati differential equation which has to be found for the whole time horizon of the control process [18, 19]. Then we propose the following series expansion as a solution of Eq.(2.3)

$$u(x,t) = u(\xi) = \sum_{k=0}^{n} a_k(x,t) Y^k(\xi) + \sum_{k=1}^{n} b_k(x,t) Y^{-k}(\xi), \tag{2.7}$$

where $a_k (k = 0, 1, ..., n), b_k (k = 1, ..., n)$ are functions to be determined later and n is a positive integer which can be determined via the balancing of the highest derivative term with the nonlinear term in equation Eq.(2.5). Inserting Eq.(2.7) into Eq.(2.5) and using Eq.(2.6) will give an algebraic equation in powers of Y . Since all coefficients of Y^k must vanish, this will give a system of algebraic equations with respect to a_k and b_k . With the aid of Mathematica, we can determine a_k and b_k . According to the recent paper by Zhang et al. [19], we can deduce the following set of solutions of Eq.(2.6).

$$\begin{cases} Y_1(\xi) = E_\alpha(\xi) - 1, & h_0 = h_1 = 1,\ h_2 = 0, \\ Y_2(\xi) = \coth_\alpha(\xi) \pm \mathrm{csch}_\alpha(\xi),\ Y_3(\xi) = \tanh_\alpha(\xi) \pm i\ \mathrm{sech}_\alpha(\xi), & h_0 = -h_2 = \frac{1}{2},\ h_1 = 0, \\ Y_4(\xi) = \frac{1}{2}\tan_\alpha(2\xi),\ Y_5(\xi) = \frac{1}{2}\cot_\alpha(2\xi), & h_0 = \frac{1}{4}h_2 = 1,\ h_1 = 0, \end{cases} \tag{2.8}$$

with the generalized hyperbolic and trigonometric functions

$$\tanh_\alpha(x) = \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)},\ \coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)},\ \operatorname{csch}_\alpha(x) = \frac{1}{\sinh_\alpha(x)},\ \operatorname{sech}_\alpha(x) = \frac{1}{\cosh_\alpha(x)},$$

$$\sinh_\alpha(x) = \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2},\ \cosh_\alpha(x) = \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2},\ \tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)},$$

$$\cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)},\ \sin_\alpha(x) = \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2i},\ \cos_\alpha(x) = \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2},$$

defined by the Mittag-Leffler function $E_\alpha(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(1+j\alpha)}$. For more details about the generalized exponential, hyperbolic and trigonometric functions see [8].

3 Exact Solutions of Eqn. (1.2).

Many authors considered nonlinear equations of the form

$$P(u, u_t, u_x, u_{xt}, u_{xx}, u_{xxx}, ...) = 0 \tag{3.1}$$

where P is a nonlinear function with respect to the indicated variables. Introducing the one wave variable $\zeta = x - ct$ carry out the two independent partial differential equation (3.1) into an ODE

$$N(u, u', u'', u''', ...) = 0 \tag{3.2}$$

Equation (3.2) is then integrated as long as all terms contain derivatives. The tanh technique is based on the priori assumption that the travelling wave solutions can be expressed in terms of the tanh function [7]. We therefor introduce a new independent variable

$$Y = tanh(\mu\zeta)$$

that leads to the change of derivatives:

$$\frac{d}{d\zeta} = \mu(1 - Y^2)\frac{d}{dY},$$

$$\frac{d^2}{d\zeta^2} = \mu^2(1 - Y^2)(-2Y\frac{d}{dY} + (1 - Y^2\frac{d^2}{dY^2}))$$

The solution can be proposed by the tanh method as a finite power series in Y in the form:

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{3.3}$$

limiting them to solitary and shock wave profiles. However, the extended tanh method admits the use of the finite expansion

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} a_k Y^{-k}, \tag{3.4}$$

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where M is a positive integer, in most cases, that will be determined. Expansion (3.4) reduces to the standard tanh method [7] for $a_k = 0, 1 \leqslant k \leqslant M$. Substituting (3.3) or (3.4) into the ODE (3.2) results in an algebraic equation in powers of Y. In this section, we will give exact solutions of Eqn(3.2). Taking the Hermite transform of Eqn.(3.2), we get

$$\partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}\widetilde{U}(x,t,z)+\lambda_2(t,z)e^{\widetilde{U}(x,t,z)}+\lambda_2(t,z)e^{-\widetilde{U}(x,t,z)}+\lambda_3(t,z)e^{-2\widetilde{U}(x,t,z)}=0 \tag{3.5}$$

where $z=(z_1,z_2,...)\in C^{\mathbb{N}}$ is a parameter. Using the transformation

$$\zeta=\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)},$$

that will carry out Eqn.(3.5) into

$$\lambda_1\widetilde{U}_{\zeta\zeta}+\lambda_2(t,z)e^{\widetilde{U}(\zeta,z)}+\lambda_3(t,z)e^{-\widetilde{U}(\zeta,z)}+\lambda_4(t,z)e^{-2\widetilde{U}(\zeta,z)}=0. \tag{3.6}$$

where, $\lambda_1=\mu\nu$, $\lambda_2=:\lambda_2(t,z)=\frac{1}{1+\beta_1}\{\widetilde{P}(t,z)+\beta_2\}$, $\lambda_3=:\lambda_3(t,z)=\frac{1}{1+\beta_1}\{\widetilde{Q}(t,z)+\beta_3\}$ and $\lambda_4=:\lambda_4(t,z)=\frac{1}{1+\beta_1}\{\widetilde{R}(t,z)+\beta_4\}$. Denote $u(\zeta,z)=\widetilde{U}(\zeta,z)$ and assume that the solutions of (3.6) is the form

$$u(\zeta,z)=\frac{\partial^2F(\phi(\zeta,z))}{\partial\zeta^2}+V(\zeta,z)$$

Let $v(\zeta,z)=e^{u(\zeta,z)}$, then Eqn.(3.6) becomes

$$\lambda_1\{vv''-v'^2\}+\lambda_2v^3+\lambda_3v+\lambda_4=0; \tag{3.7}$$

Considering the homogeneous balance between vv'' and v^3 in (3.7), gives M=2, hence we set the tanh-coth assumption by

$$v(x,t,z)=S(Y)=a_0(t,z)+a_1(t,z)Y(\zeta)+a_2(t,z)Y^2(\zeta)+b_1(t,z)Y^{-1}(\zeta)+b_2(t,z)Y^{-2}(\zeta) \tag{3.8}$$

where $Y(\zeta)$ satisfies the Riccati equation

$$Y'=c_1+c_2Y+c_3Y^2, \tag{3.9}$$

and c_1,c_2,c_3 are constant to be prescribed later. By virtue of (3.8) and (3.9) with observation of the linear independence of $Y^n(n=-6,-5,...,6)$, Eqn.(3.7) implies the following system of linear equations

$$\begin{cases}
\lambda_4+\lambda_3 a_0+\lambda_2[a_0(a_0^2+2a_1b_1+2a_2b_2)+a_1(2a_0b_1+2a_1b_2)+a_2(b_1^2+2a_0b_2)+\\
b_1(2a_0a_1+2a_2b_1)+b_2(a_1^2+2a_0a_2)]+\lambda_1[D_0a_0+D_8a_1+D_7a_2+D_1b_1+D_2b_2-(a_1c_1-b_1c_3)^2+\\
(a_1c_2+2a_2c_1)(b_1c_2+2b_2c_3)+(a_1c_3+2a_2c_2)(b_1c_1+2b_2c_2)+4a_2c_3b_2c_1]=0;\\
\lambda_3a_1+\lambda_2[a_0(2a_0a_1+2a_2b_1)+a_1(a_0^2+2a_1b_1+2a_2b_2)+a_2(2a_0b_1+2a_1b_2)+\\
b_1(a_1^2+2a_0a_2)+2a_1a_2b_2]+\lambda_1[D_0a_1+D_1a_0+D_8a_2+D_2b_1+D_3b_2\\
-(a_1c_1-b_1c_3)(a_1c_2+2a_2c_1)+(a_1c_3+2a_2c_2)(b_1c_2+2b_2c_3)]=0;\\
\lambda_3a_2+\lambda_2[a_0(a_1^2+2a_0a_2)+a_1(2a_0a_1+2a_2b_1)+a_2(a_0^2+2a_1b_1+2a_2b_2)+2a_1a_2b_1+a_2^2b_2]+\\
\lambda_1[D_2a_0++D_1a_1+D_3b_1+D_4b_2-(a_1c_1-b_1c_3)(a_1c_3+2a_2c_2)\\
-(a_1c_2+2a_2c_1)^2+2a_2c_3(b_1c_2+2b_2c_3)]=0;\\
\lambda_2[2a_0a_1a_2+a_1(a_2^2+2a_0a_2)+a_2(2a_0a_1+2a_2b_1)+b_1a_2^2]+\\
\lambda_1[D_3a_0+D_2a_1+D_1a_2+D_4b_1-2a_2c_3(a_1c_1-b_1c_3)-(a_1c_2+2a_2c_1)(a_1c_3+2a_2c_2)]=0;\\
\lambda_2[a_0a_2^2+2a_1^2a_2+a_2(a_1^2+2a_0a_2)]+\lambda_1[D_4a_0+D_3a_1+D_2a_2-2a_2c_3(a_1c_2+2a_2c_1)\\
-(a_1c_3+2a_2c_2)^2]=0;\\
\lambda_2[a_1a_2^2+2a_1a_2^2]+\lambda_1[D_4a_1+D_3a_2-2a_2c_3(a_1c_3+2a_2c_2)]=0;\\
\lambda_2[a_2^3]+\lambda_1[D_4a_2-4a_2^2c_3^2]=0;\\
\lambda_3b_2+\lambda_2[a_0(b_1^2+2a_0b_2)+b_1(2a_0b_1+2a_1b_2)+b_2(a_0^2+2a_1b_1+2a_2b_2)+\\
2a_1b_1b_2+a_2b_2^2]+\lambda_1[D_0b_2+D_7a_0+D_6a_1+D_5a_2+D_8b_1-(b_1c_2+2b_2c_3)^2+\\
2b_2c_1(a_1c_2+2a_2c_1)+(a_1c_1-b_1c_3)(b_1c_1+2b_2c_2)]=0;\\
\lambda_3b_1+\lambda_2[a_0(2a_0b_1+2a_1b_2)+b_1(a_0^2+2a_1b_1+2a_2b_2)+b_2(2a_0a_1+2a_2b_1)+\\
a_1(b_1^2+2a_0b_2)+2a_2b_2b_1]+\lambda_1[D_0b_1+D_8a_0+D_7a_1+D_6a_2+\\
D_1b_2+(a_1c_1-b_1c_3)(b_1c_2+2b_2c_3)+(a_1c_2+2a_2c_1)(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[2a_0b_1b_2+b_1(b_1^2+2a_0b_2)+b_2(2a_0b_1+2a_1b_2)+a_1b_2^2]+\lambda_1[D_6a_0+D_5a_1+D_7b_1+D_8b_2\\
+2b_2c_1(a_1c_1-b_1c_3)-(b_1c_2+2b_2c_3)(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[a_0b_2^2+2a_0b_1^2b_2+b_2(b_1^2+2a_0b_2)]+\lambda_1[D_5a_0+D_6b_1+D_7b_2-2b_2c_1(b_1c_2+2b_2c_3)\\
-(b_1c_1+2b_2c_2)^2]=0;\\
\lambda_2[b_1b_2^2+2a_0b_1b_2^2]+\lambda_1[D_5b_1+D_6b_2-2b_2c_1(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[b_2^3]+\lambda_1[D_5b_2-4b_2^2c_1^2]=0.
\end{cases}$$

where, $D_0=c_1(a_1c_2+2a_2c_1)+c_3(b_1c_2+2b_2c_3)$, $D_1=c_2(a_1c_2+2a_2c_1)+2c_1(a_1c_3+2a_2c_2)$, $D_2=c_3(a_1c_2+2a_2c_1)+2c_2(a_1c_3+2a_2c_2)+6a_2c_3c_1$, $D_3=2c_3(a_1c_3+2a_2c_2)+6a_2c_3c_2$, $D_4=6a_2c_3^2$, $D_5=6a_2b_2c_1^2$, $D_6=2c_1(b_1c_1+2b_2c_2)+6b_2c_1c_2$, $D_7=c_1(b_1c_2+2b_2c_3)+2c_2(b_1c_1+2b_2c_2)+6b_2c_3c_1$, and $D_8=c_2(b_1c_2+2b_2c_3)+2c_3(b_1c_1+2b_2c_2)$. In the remaining part of this section we will discuss and solve our problem for some special cases for the Riccati equation as follows:

A. $c_1=c_2=1, c_3=0$.

This choice for the constants implies that

$$Y_1(\zeta)=exp(\zeta)-1 \tag{3.10}$$

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By the aid of Maple 12, the above system of equations can be solve for the following cases:

Case 1: $\lambda_4 = a_1 = a_2 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_0 = \pm i\sqrt{\frac{\lambda_3}{\lambda_2}}$; $b_1 = \frac{3}{\lambda_2 \pm i\sqrt{\lambda_2\lambda_3}}$; $b_2 = -\frac{2\lambda_1}{\lambda_2}$. By virtue of Eqn.(3.8), then Eqn.(3.5) have the solution

$$u_1 = ln\{\pm i\sqrt{\frac{\lambda_3}{\lambda_2}} + \frac{3}{\lambda_2 \pm i\sqrt{\lambda_2\lambda_3}} \times \frac{1}{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1} - \frac{2\lambda_1}{\lambda_2(exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1)^2}\} \tag{3.11}$$

Case 2: For $\lambda_4 = a_1 = a_2 = b_1 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_0 = \pm i\sqrt{\frac{\lambda_3}{\lambda_2}}$; $b_2 = -\frac{2\lambda_1}{\lambda_2}$. Eqn.(3.5) have the solution

$$u_2 = ln\{\pm i\sqrt{\frac{\lambda_3}{\lambda_2}} - \frac{2\lambda_1}{\lambda_2(exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1)^2}\} \tag{3.12}$$

B. $c_1 = -c_3 = 0.5, c_2 = 0$.

This choice for the constants implies that

$$Y_2(\zeta) = coth(\zeta) \pm csch(\zeta) \tag{3.13}$$

or

$$Y_3(\zeta) = tanh(\zeta) \pm isech(\zeta) \tag{3.14}$$

By the aid of Maple 12, the above system of equations can be solve for the following cases:

Case 3: $\lambda_4 = a_0 = a_1 = a_2 = b_1 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $b_2 = -\frac{\lambda_1}{2\lambda_2}$. By virtue of Eqn.(3.8), then Eqn.(3.5) have the solution

$$u_3 = ln\{-\frac{\lambda_1}{2\lambda_2(coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2}\} \tag{3.15}$$

or

$$u_4 = ln\{-\frac{\lambda_1}{2\lambda_2(tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2}\} \tag{3.16}$$

Case 4: For $\lambda_4 = a_0 = a_1 = b_1 = b_2 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_2 = -\frac{\lambda_1}{2\lambda_2}$. Eqn.(3.5) have the solution

$$u_5 = ln\{-\frac{\lambda_1}{2\lambda_2}(coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2\} \tag{3.17}$$

or

$$u_6 = ln\{-\frac{\lambda_1}{2\lambda_2}(tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2\} \tag{3.18}$$

4 White noise functional solutions of (1.2)

In this section, we will use Theorem 2.1 of Xie [17] for $d = 1$ to obtain white noise functional solutions of Eqs.(1.2). The properties of hyperbolic functions yield that there exists a bounded open set $\mathbf{S} \subset \mathbb{R}_+ \times \mathbb{R}, m > 0$ and $n > 0$ such that $u(x,t,z), u_{xt}(x,t,z)$ are uniformally bounded for all $(t,x,z) \in \mathbf{S} \times \mathbb{K}_m(n)$, continuous with respect to $(t,x) \in \mathbf{S}$ for all $z \in \mathbb{K}_m(n)$ and analytic with respect to $z \in \mathbb{K}_m(n)$ for all $(t,x) \in \mathbf{S}$. Using Theorem 2.1 of Xie [17], there exists a stochastic process $U(t,x)$ such that the Hermite transformation of $U(t,x)$ is $u(t,x,z)$ for all $\mathbf{S} \times \mathbb{K}_m(n)$, and $U(t,x)$ is the solution of (1.2). This implies that $U(t,x)$ is the inverse Hermite transformation of $u(t,x,z)$. Hence, for $\Lambda_1\Lambda_2\Lambda_3 \neq 0$ the white noise functional solutions of Eqn.(1.2) as follows:

$$\begin{aligned} U_1(x,t) = ln^{\diamond}\{\pm i\sqrt{\frac{\Lambda_3(t)}{\Lambda_2(t)}} + \frac{3\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-1}}{\Lambda_2(t) \pm i\sqrt{\Lambda_2(t)\Lambda_3(t)}} - \\ \frac{2\mu\nu}{\Lambda_2(t)}\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \end{aligned} \tag{4.1}$$

$$U_2(x,t) = ln^{\diamond}\{\pm i\sqrt{\frac{\Lambda_3(t)}{\Lambda_2(t)}} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \tag{4.2}$$

$$U_3(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \tag{4.3}$$

$$U_4(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \tag{4.4}$$

$$U_5(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \tag{4.5}$$

$$U_6(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \tag{4.6}$$

We observe that for different form of $\Lambda_2(t)$ and $\Lambda_3(t)$, we can get different solutions of (1.2) from (3.1)-(3.6).

5 Example and Concluding Remarks

Let $\dot{B_t}$ be the Gaussian white noise, where B_t is Brown motion. We have the Hermite transform $\tilde{\dot{B}}(t,z) = \sum_{k=1}^{\infty} z_k \int_0^t \eta_k(s)ds$. Science $\exp^{\diamond}(B_t) = \exp(B_t - t^2/2)$, we have $tanh^{\diamond}(B_t) = tanh(B_t - t^2/2)$, $coth^{\diamond}(B_t) = cot(B_t - t^2/2)$, $sech^{\diamond}(B_t) = sech(B_t - t^2/2)$ and $csch^{\diamond}(B_t) = csch(B_t - t^2/2)$. Suppose $\Lambda_3(t) = \alpha\Lambda_2(t)$ and $\Lambda_2(t) = \lambda_2(t) + \beta\dot{B_t}$, where α, β are arbitrary

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constants and $\lambda_2(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$. The white noise functional solutions of (1.2) are as follows: If $\Lambda_1(t)\Lambda_2(t)\Lambda_3(t) \neq 0$

$$U_7(x,t) = ln\{\pm i\sqrt{\alpha} + \frac{3\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-1}}{\Lambda_2(t)(1+\pm i\sqrt{\alpha})} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \tag{5.1}$$

$$U_8(x,t) = ln\{\pm i\sqrt{\alpha} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \tag{5.2}$$

$$U_9(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \tag{5.3}$$

$$U_{10}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm i\, sech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \tag{5.4}$$

$$U_{11}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \tag{5.5}$$

$$U_{12}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm i\, sech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \tag{5.6}$$

Finally, we remark that for $\alpha_1 = \alpha_2 = 0$, $p(t) = 1$ and $q(t) = r(t) = 0$, Eqn.(1.1) reduces to the Liouville equation. For $\alpha_1 = \alpha_2 = 0$, $r(t) = 0$ and $q(t) = p(t) = 1$, Eqn.(1.1) reduces to the Sinh-Gordon equation. For $\alpha_1 = \alpha_2 = 0$, $p(t) = r(t) = 1$ and $q(t) = 0$, Eqn.(1.1) reduces to the the well known Dodd-Bullough-Mikhailov equation. Moreover, for $\alpha_1 = \alpha_2 = 0$, $p(t) = 0$, $q(t) = -1$ and $r(t) = 1$, gives Tzitzeica-Dodd-Bullough equation. Hence, our results in this work can be considered as a continuation of our results in our previous papers [4,5], this work gives directly exact solutions for wick-type stochastic form to each one of the above equations. Also, we remark that, since the Riccati equation has other solution if select other values of c_1, c_2 and c_3, there are many other exact solutions of variable coefficient and wick-type stochastic Zhiber-Shabat equations

References

[1] Chen. B, Xie. Y.C., 2005, Exact solutions for wick-type stochastic coupled Kadomtsev-Petviashili equations, J. Phys. A, 38, 815-22.

[2] Chen. B, Xie. Y.C., 2005, Exact solutions for generalized stochastic Wick-type KdV-mKdV equations, Chaos Solitons Fractals, 23,281-7.

[3] Chen. B, Xie. Y.C., 2007, Periodic-like solutions of variable coefficient and Wick-type stochastic NLS equations, J. Comput. Appl. Math., 203,249-63.

[4] Ghany H. A., 2011, Exact Solutions for Stochastic Generalized Hirota-Satsuma Coupled KdV Equations, Chin. J. Phys., 49,926-940.

[5] Ghany H. A., Mohammed S. A., 2012, White Noise Functional Solutions for the Wick-type Stochastic Fractional KdV-Burgers-Kuramoto Equations, Chin. J. Phys., 50,619-627.

[6] Jumarie G.,2006, Modifed Riemann-Liouville derivative and fractional Taylor series of non differentiable functions further results, Comp. Math. Appl., 51, 1367-1376.

[7] Malfliet W., 1992, Solitary wave solutions of nonlinear wave equations, Am. J. Phys, 60(7),6504.

[8] Podlubny. I., 1999, Fractional Differential Equations, Academic Press, San Diego.

[9] Uğurlu. Y., Kaya. D., 2007, Analytic method for solitary solutions of some partial differential equations, Phys. Lett. A, 370,251-9.

[10] Wadati. M., 1983, Stochastic Korteweg de Vries equation, J. Phys. Soc. Jpn, 52,2642-8.

[11] Wadati. M., 1990, Deformation of solitons in random media, J. Phys. Soc. Jpn, 59,4201-3.

[12] Wadati. M., Akutsu. Y., 1984, Stochastic Korteweg de Vries equation, J. Phys. Soc. Jpn, 53,3342-50.

[13] Wazzan. L., A modified tanh-coth method for solving the KdV and the KdV-Burgers equations, to appear on Commu. Nonlinear Sci. Numer. Simul.

[14] Xie. Y. C., 2003, Exact solutions for stochastic KdV equations, Phys. Lett. A, 310,161-7.

[15] Xie. Y. C., 2004, Positonic solutions for Wick-type stochastic KdV equations, Chaos Solitons Fractals, 20,337-42.

[16] Xie. Y. C., 2004, An auto-Backlund transformation and exact solutions for wick type stochastic generalized KdV equations, J. Phys. A: Math. Gen., 37,5229-36.

[17] Xie. Y. C., 2004, Exact solutions of the Wick-type stochastic Kadomtsev-Peviashvili equations, Chaos Solitons Fractals, 21,473-80.

[18] Zhang. J.L., Ren. D.F. and Wang. M.L., 2003, The periodic wave solutions for the generalized Nizhnik-Novikov-Veselov equation, Chin. Phys, 12(8), 825-830.

[19] Zhang. J.L, Zong. Q.A., Liu. D. and Gao. Q., 2010, A generalized exp-function method for fractional Riccati differential equations, Communication Fractional Calculus, 1, 48-56.