CONTINUOUS ON A GENERALIZED TOPO-SPACE

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Abstract: The content of this chapter are $\widehat{D_\alpha}$-continuous maps, their relations with variousgeneralized continuous maps, $\widehat{D_\alpha}$-irresolute maps, strongly $\widehat{D_\alpha}$-continuous maps, perfectly$\widehat{D_\alpha}$-continuous maps, totally $\widehat{D_\alpha}$-continuous maps, contra $\widehat{D_\alpha}$-continuous maps and several comparisons are made inorder to justify my topic.

1 Introduction

Many authors working in the field of general topology have shown more interest in studying the concepts of generalizations of continuous maps. A weak from of continuous maps called g-continuous maps was introduced by Balachandran, Sundaram and Maki [1]. Veerakumar [12] introduced and studied another form of generalized continuous maps called pre-semi continuous maps.Irresolute maps are introduced and studied by Crossley and Hildebrand [3]. R. A. Mohmoud and Abd EL-Monsef [7] have investigated _-irresolute maps. In this section the concepts of $\widehat{D_\alpha}$-irresolute maps, strongly $\widehat{D_\alpha}$-continuous maps and perfectly $\widehat{D_\alpha}$-continuous maps in topological spaces and their properties are studied.

The notion of contra _-continuity has been introduced by Caldas and Jafari [2]. T. M. Nour [9] introduced the concept of totally semi continuous maps in topological spaces. This section contains the concept of totally $\widehat{D_\alpha}$-continuous map which is stronger than $\widehat{D_\alpha}$-continuous and weaker than perfectly $\widehat{D_\alpha}$-continuous and the concepts of contra $\widehat{D_\alpha}$- continuous maps. Further the characterizations of these maps are obtained.

The content of this chapter are $\widehat{D_\alpha}$-continuous maps, their relations with variousgeneralized continuous maps, $\widehat{D_\alpha}$-irresolute maps, strongly $\widehat{D_\alpha}$-continuous maps, perfectly$\widehat{D_\alpha}$-continuous maps, totally $\widehat{D_\alpha}$-continuous maps, contra $\widehat{D_\alpha}$-continuous maps.

2. Preliminaries:

Definition 2.1:$\widehat{D_\alpha}$-closed set [11] if $\alpha cl(A) \subset U$ whenever $A \subset U$ and U is $\widehat{D}$-open in (X, τ).

Proposition:2.2

1) Every $\widehat{D_\alpha}$-closed set is gspr- closed (resp. gpr-open, gsp-open).
2) Every closed set is $\widehat{D_\alpha}$- closed
3) Every ω- closed set is $\widehat{D_\alpha}$- closed

Definition 2.3. A map f : (X, τ) → (Y, σ) is called

1. an α -continuous [8] if $f^{-1}(F)$ is a α - closed subset of (X, τ) for each closed subset F of (Y, σ).
2. a g-continuous [6] if f^{-1} (F) is a g - closed subset of (X, τ) for each closed subset F of (Y, σ).
3. a $gspr$-continuous [6] if f^{-1} (F) is a $gspr$ - closed subset of (X, τ) for each closed subset of (Y, σ).
4. a rg - continuous [10] if f^{-1} (F) is a rg - closed subset of (X, τ) for every closed set F of (Y, σ).
5. a gpr - continuous [5] if f^{-1} (F) is a gpr -closed subset of (X, τ) for each closed subset of (Y, σ).
6. a gsp - continuous [4] if f^{-1} (F) is a gsp -closed subset of (X, τ) for each closed subset of (Y, σ).
7. a $*gs$ - continuous [13] if f^{-1} (F) is a $*gs$ - closed subset of (X, τ) for each closed subset (Y, σ).
8. a D - continuous [14] if f^{-1} (F) is a D - closed subset of (X, τ) for every closed set F of (Y, σ).

3. $\widehat{D_\alpha}$-continuous maps

In this section, the concept of $\widehat{D_\alpha}$-continuous maps in topological spaces has been introduced and the composition of two $\widehat{D_\alpha}$-continuous maps need not be $\widehat{D_\alpha}$-continuous is also proved. Further some characterizations of $\widehat{D_\alpha}$-continuous maps under certain conditions have been studied.

Definition 3.1. A function f : (X, τ) → (Y, σ) is said to be $\widehat{D_\alpha}$-continuous if $f^{-1}(H)$ is $\widehat{D_\alpha}$-closed in (X, τ) for every closed set H in Y .

Example 3.2. Let X = Y = {p, q, r, s}, τ = {φ, {r}, {s}, {p, r}, {r, s}, {p, r, s},X} and τ = {φ, {p}, {p, q, r}, Y }. Define the map f : (X, τ) → (Y, σ) by f(p) = q, f(q) = r, f(r) = p and f(s) = s is $\widehat{D_\alpha}$-continuous.

Theorem 3.3. Every continuous is $\widehat{D_\alpha}$-continuous.

Proof. Let f :X → Y be a continuous function. Let M be closed in (Y, σ). Since f is continuous, $f^{-1}(M)$ is closed in (X, τ). By **Proposition 2.2**, $f^{-1}(M)$ is $\widehat{D_\alpha}$-closed in (X, τ). Hence f is $\widehat{D_\alpha}$-continuous.

Remark 3.4. The converse of the above theorem need not be true as seen from the following example.

Example 3.5. Let X = {p, q, r, s, t} and Y = {p, q, r, s}, τ = {φ, {p, q}, {p, q, s}, {p, q, r, s}, {p, q, s, t},X} and σ= {φ, {p}, {p, q, r}, Y }. Define the map f : (X, τ) → (Y, σ) by f(p) = p, f(q) = f(r) = q, f(s) = r and f(t) = s. Then f is $\widehat{D_\alpha}$-continuous but not continuous, since for every closed set N = {q, r, s}, f^{-1}(N) = {q, r, s, t} is $\widehat{D_\alpha}$-closed but not closed.

Proposition 3.6. Every α-continuous (ω-continuous) is $\widehat{D_\alpha}$-continuous.

Proof. By **Proposition 2.2**, every α-closed (ω-closed) is $\widehat{D_\alpha}$-closed. Hence the proof follows.

Remark 3.7. The converse of the above theorem need not be true as seen from the following example.

Example 3.8. Let X = {p, q, r, s, t} and Y = {p, q, r, s}, τ = {φ, {p}, {p, q}, {r, s}, {p, r, s}, {p, q, r, s},X} and σ = {φ, {r}, {s}, {p, r}, {r, s}, {p, r, s}, Y }. Define the map f : (X, τ) → (Y, σ) by f(p) = p, f(q) = q, f(r) = r, and f(s) = f(t) = s.
Then f is $\widehat{D_\alpha}$-continuous but not ω-continuous, since for every closed set M = {p, q} in (Y, σ), f^{-1}(M) = {p, q} is $\widehat{D_\alpha}$-closed but not α-closed (not ω-closed) in (X, τ).

Proposition 3.9. Every $\widehat{D_\alpha}$-continuous is gspr-continuous

Proof. By **Proposition 2.2**, every $\widehat{D_\alpha}$-closed set is gspr-closed. Hence the proof follows.

Remark 3.10. The converse of the above theorem need not be true as seen from the following example.

Example 3.11. Let X = {p, q, r, s, t} and Y = {p, q, r, s, t}, τ = {φ, {p, q}, {p, q, s}, {p, q, r, s}, {p, q, s, t},X} and σ = {φ, {p}, {p, q}, {r, s}, {p, r, s}, {p, q, r, s}, Y }. Then the identity function f : (X, τ) → (Y, σ) is gspr-continuous but not $\widehat{D_\alpha}$-continuous, since for every closed set M = {p, q, t} in (Y, σ), f^{-1}(M) = {p, q, t} is gspr-closed but not $\widehat{D_\alpha}$-closed in (X, τ).

Proposition 3.12. Every $\widehat{D_\alpha}$-continuous is gsp-continuous.

Proof. By **Proposition 2.2**, every $\widehat{D_\alpha}$-closed set is gsp-closed. Hence the proof follows.

Remark 3.13. The converse of the above theorem need not be true as seen from the following example.

Example 3.14. Let X = {p, q, r, s} and Y = {p, q, r}, τ = {φ, {p}, {p, q, r},X} and σ = {φ, {p}, {q}, {p, q}, Y }. Define f : (X, τ) → (Y, σ) by f(p) = f(s) = p, f(q) = q and f(r) = r. Then f is gsp-continuous but not $\widehat{D_\alpha}$-continuous. Since for every closed set M = {p, r} in (Y, σ), f^{-1}(M) = {p, r, s} is gsp-closed but not $\widehat{D_\alpha}$-closed in (X, τ).

Remark 3.15. D_α-continuous is independent of rg-continuous.It is shown by the following example.

Example 3.16. Let X = Y = {u, v,w}, τ = {φ, {u}, {v}, {u, v},X} and σ = {φ, {v,w}, Y}. Then the identity function f : (X, τ) → (Y, σ) is $\widehat{D_\alpha}$-continuous but not rg-continuous. Since for every closed set M = {u} in (Y, σ), f^{-1}(M) = {u} is $\widehat{D_\alpha}$-closed in (X, τ) but not rg-closed.

Example 3.17. Let X = Y = {u, v,w}, τ = {φ, {u}, {v}, {u, v},X} and σ = {φ, {w}, Y }. Then the identity function f : (X, τ) → (Y, σ) is rg-continuous but not $\widehat{D_\alpha}$-continuous. Since for every closed set M = {u, v} in (Y, σ), f−1(M) = {u, v} is rg-closed but not $\widehat{D_\alpha}$-closed.

Remark 3.18. $\widehat{D_\alpha}$-continuous is independent of g-continuous and D-continuous. It is shown by the following example.

Example 3.19. Let X = Y = {p, q, r, s}, τ = {φ, {r}, {s}, {p, r}, {r, s}, {p, r, s},X} and σ = {φ, {p, q, r}, Y }. Then the identity function f : (X, τ) → (Y, σ) is $\widehat{D_\alpha}$-continuous but neither g-continuous nor D-continuous. Since for every closed set M = {s} in (Y, σ), f^{-1}(M) = {s} is $\widehat{D_\alpha}$-closed in (X, τ) but neither g-closed nor D-closed.

Example 3.20. Let X = Y = {p, q, r, s}, τ = {φ, {r}, {s}, {p, r}, {r, s}, {p, r, s},X} and σ = {φ, {p}, Y }. Then the identity function f : (X, τ) → (Y, σ) is g-continuous and D-continuous but not $\widehat{D_\alpha}$-continuous. Since for every closed set M = {q, r, s} in (Y, σ), f^{-1}(M) = {q, r, s} is g-closed and D-closed but not $\widehat{D_\alpha}$-closed.

Remark 3.21. $\widehat{D_\alpha}$-continuous is independent of *g-continuous, g*-continuous, g*p-continuous, α*g-continuous, ρ-continuous and gp-continuous. It is shown by the follow ing example.

Example 3.22. Let X = Y = {p, q, r, s}, τ = {φ, {r}, {s}, {p, r}, {r, s}, {p, r, s},X} and σ = {φ, {p, q, r}, Y }. Then the identity function f : (X, τ) → (Y, σ) is $\widehat{D_\alpha}$-continuous but not *g-continuous (resp.not g*-continuous, not g*p-continuous, not α*g-continuous, not ρ-continuous , not gp-continuous). Since for every closed set M = {s} in (Y, σ), f^{-1}(M) = {s} is $\widehat{D_\alpha}$-closed in (X, τ) but not *g-closed (resp.notg*-closed, not g*p-closed, not α*g-closed, not ρ-closed, not gp-closed).

Example 3.23. Let X = Y = {p, q, r, s}, τ = {φ, {r}, {s}, {p, r}, {r, s}, {p, r, s},X} and σ = {φ, {p}, Y }. Then the identity function f : (X, τ) → (Y, σ) is *g-continuous (resp. g*-continuous, g*p-continuous, α*g-continuous, ρ-continuous, gp-continuous). but not $\widehat{D_\alpha}$-continuous. Since for every closed set M = {q, r, s} in (Y, σ), f−1(M) = {q, r, s} is *g-closed (resp. g*-closed, g*p-closed, α*g-closed, ρ-closed, gp-closed) but not $\widehat{D_\alpha}$-closed.

Remark 3.24. $\widehat{D_\alpha}$-continuous is independent of *gs-continuous .It is shown by the following example.

Example 3.25. Let X = Y = {p, q, r, s}, τ = {φ, {p}, {p, q, r},X} and σ = {φ, {q, r}, Y }. Then the identity function f : (X, τ) → (Y, σ) is *gs-continuous but not $\widehat{D_\alpha}$-continuous. Since for every closed set M = {s} in (Y, σ), f^{-1}(M) = {s} is *gs-closed in (X, τ) but not $\widehat{D_\alpha}$-closed.

Example 3.26. Let X = Y = {p, q, r}, τ = {φ, {p}, {q}, {p, q},X} and σ = {φ, {q, r}, Y }. Then the identity function f : (X, τ) → (Y, σ) is $\widehat{D_\alpha}$-continuous but not *gs-continuous. Since for every closed set M = {p} in (Y, σ), f^{-1}(M) = {p} is $\widehat{D_\alpha}$-closed but not *gs-closed.

Remark 3.27. we have the following relationship between $\widehat{D_\alpha}$-continuous and other related generalized continuous. A → B (A = B) represents A implies B but not conversely (A and B are independent of each other).

Theorem 3.28. A function f : (X, τ) → (Y, σ) is $\widehat{D_\alpha}$-continuous if an only if f^{-1}(U) is $\widehat{D_\alpha}$-open in (X, τ) for every open set U in (Y, σ).

Proof. Let f : (X, τ) → (Y, σ) be $\widehat{D_\alpha}$-continuous andU be an open set in (Y, σ). Then $f^{-1}(U^c)$ is $\widehat{D_\alpha}$-closed in (X, τ). But $f^{-1}(U^c) = (f^{-1}(U))^c$ and so f^{-1}(U) is $\widehat{D_\alpha}$-open in (X, τ).

Conversely, suppose f^{-1}(U) is $\widehat{D_\alpha}$-open in (X, τ) for every open set U in (Y, σ). Let U be an open set in (Y, σ). Then U^c is closed in (Y, σ). Since f^{-1}(U) is $\widehat{D_\alpha}$-open in (X, τ) and $(f^{-1}(U))^c = f^{-1}(U^c)$, $f^{-1}(U^c)$ is $\widehat{D_\alpha}$-closed in (X, τ).

Remark 3.29. The composition of two $\widehat{D_\alpha}$-continuous functions need not be $\widehat{D_\alpha}$-continuous. It is shown by the following example.

Example 3.30. Let X = Y = Z = {p, q, r, s}, τ = {φ, {p}, {r}, {p, q}, {p, r}, {p, q, r}, {p, q, s},X}, σ = {φ, {r}, {s}, {p, r}, {r, s}, {p, r, s}, Y } and η= {φ, {q, s},Z}. Define f : (X, τ) → (Y, σ) by f(p) = r, f(q) = q, f(r) = s and f(s) = p and define an identity function g : (Y, σ) → (Z, η). Then both f and g are $\widehat{D_\alpha}$-continuous. Let F = {p, r} be a closed set in (Z, η). Then $(g \circ f)^{-1}(F) = f^{-1}(g-1(F))$ = {p, s} which is not $\widehat{D_\alpha}$-closed in (X, τ). Therefore g ◦ f is not $\widehat{D_\alpha}$-continuous.

Proposition 3.31. For a subset A of a topological space X, the following conditions are equivalent:

(i) $\widehat{D_\alpha}o$ (τ) is closed under any union,

(ii) A is $\widehat{D_\alpha}$-closed if and only if $\widehat{D_\alpha}cl(A) = A$,

(iii) A is $\widehat{D_\alpha}$-open if and only if $\widehat{D_\alpha}int(A) = A$.

Proof. (i) =⇒**(ii) :**Let A be a $\widehat{D_\alpha}$-closed set. Then by the definition of $\widehat{D_\alpha}$-closure, we get $\widehat{D_\alpha}cl(A) = A$. Conversely, assume that $\widehat{D_\alpha}cl(A) = A$. For each $x \in A^c$, $x \notin \widehat{D_\alpha}cl(A)$, there exists a $\widehat{D_\alpha}$-open set G_x such that $G_x \cap A = \varphi$ and hence $x \in G_x \subset A^c$. Therefore we obtain $A^c = \cup_{x\in A^c} G_x$. By (i) A^c is $\widehat{D_\alpha}$-open and hence A is $\widehat{D_\alpha}$-closed.

(ii) =⇒**(iii) :**obviously true

(iii) =⇒**(i) :**Let $\{U_\alpha / \alpha \in \Lambda\}$ be a family of $\widehat{D_\alpha}$open sets of X. Put $U = \cup_\alpha U_\alpha$. For each $x \in U$, there exist $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)} \subset U$. Since $U_{\alpha(x)}$ is $\widehat{D_\alpha}$-open, $x \in \widehat{D_\alpha}int(U)$ and so $U = \widehat{D_\alpha}int(U)$. By (iii), U is $\widehat{D_\alpha}$-open. Thus $\widehat{D_\alpha}o(\tau)$ is closed under any union.

Proposition 3.32. In a topological space X, assume that $\widehat{D_\alpha}o(\tau)$ is closed under any union. Then $\widehat{D_\alpha}cl(A)$ is a $\widehat{D_\alpha}$-closed set for every subset A of X.

Proof. Since $\widehat{D_\alpha}cl(A) = \widehat{D_\alpha}cl(\widehat{D_\alpha}cl(A))$ and by proposition 3.31, we get $\widehat{D_\alpha}cl(A)$ is a $\widehat{D_\alpha}$-closed set.

Theorem 3.33. Let f :X → Y be a map. Assume that $\widehat{D_\alpha}o(\tau)$ is closed under any union. Then the following are equivalent:

(i) The map f is $\widehat{D_\alpha}$-continuous;

(ii) The inverse of each open set is $\widehat{D_\alpha}$-open;

(iii) For each point x in X and each open set V in Y with $f(x) \in V$, there is an $\widehat{D_\alpha}$-open set U in X such that $x \in U$, $f(U) \subset V$;

(iv) For each subset A of X, $f(\widehat{D_\alpha}cl(A)) \subset cl(f(A))$;

(v) For each subset B of Y , $\widehat{D_\alpha}cl(f^{-1}(B)) \subset f^{-1}(cl(B))$;

(vi) For each subset B of Y ,$f^{-1}(int(B)) \subset \widehat{D_\alpha} int(f^{-1}(B))$.

Proof. (i) ⇐⇒(ii) : By Theorem 3.28.

(i) ⇐⇒(iii) : Suppose (iii) holds and let V be an open set in Y and $x \in f^{-1}(V)$.

Then $f(x) \in V$ and thus there exist an $\widehat{D_\alpha}$-open set U_x such that $x \in U_x$ and $f(U_x) \subset V$.

Now $x \in U_x \subset f^{-1}(V)$ and $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$. By assumption $f^{-1}(V)$ is $\widehat{D_\alpha}$-open in X and therefore f is $\widehat{D_\alpha}$-continuous.

Conversely, suppose that (i) holds. Let V be an open set in Y with $f(x) \in V$. Then $x \in f^{-1}(V) \in \widehat{D_\alpha}o(\tau)$, since f is $\widehat{D_\alpha}$-continuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subset V$.

(iv) ⇐⇒(i) : Suppose (i) holds and A be a subset of X. Since $A \subset f^{-1}(f(A))$, we have $A \subset f^{-1}(cl(f(A))$. Since $cl(f(A))$ is a closed set in Y , by assumption $f^{-1}(cl(f(A)))$ is a $\widehat{D_\alpha}$-closed set containing A. Consequently, $\widehat{D_\alpha}cl(A) \subset f^{-1}(cl(f(A)))$. Thus $f(\widehat{D_\alpha}cl(A)) \subset f(f^{-1}(cl(f(A)))) \subset cl(f(A))$.

Conversely, suppose that (iv) holds for any subset A of X. Let F be a closed subset of Y . Then by assumption, $f(\widehat{D_\alpha}cl(f^{-1}(F))) \subset cl(f(f^{-1}(F))) \subset cl(F) = F$. Thus $\widehat{D_\alpha}cl(f^{-1}(F)) \subset f^{-1}(F)$. But $\widehat{D_\alpha}cl(f^{-1}(F)) \supset f^{-1}(F)$. Therefore, $f^{-1}(F)$ is $\widehat{D_\alpha}$-closed.

(iv) ⇐⇒(v) : Suppose (iv) holds and B be any subset of Y then replacing A by f−1(B) in (iv) we get $f(\widehat{D_\alpha}cl(f^{-1}(B))) \subset cl(f(f^{-1}(B))) \subset cl(B)$. Thus $\widehat{D_\alpha}cl(f^{-1}(B)) \subset f^{-1}(cl(B))$.

Conversely, suppose that (v) holds. Let B = f(A) where A is a subset of X. Then we have $\widehat{D_\alpha}cl(A) \subset \widehat{D_\alpha}cl(f^{-1}(B)) \subset f^{-1}(cl(f(A)))$ and so $f(\widehat{D_\alpha}cl(A)) \subset cl(f(A))$.

(v) ⇐⇒(vi) : Let B be any subset of Y . Then by (v) we have $\widehat{D_\alpha}cl(f^{-1}(B^c)) \subset (f^{-1}cl(B^c))$ and hence $(\widehat{D_\alpha}intf^{-1}(B))^c \subset (f^{-1}int(B))^c$. Therefore, we obtain $f^{-1}(int(B)) \subset \widehat{D_\alpha}int(f^{-1}(B))$.

(vi) ⇐⇒(i) : Suppose (vi) holds. Let F be any closed subset of Y . We have $f^{-1}(F^c) = f^{-1}(int(F^c)) \subset \widehat{D_\alpha}int(f^{-1}(F^c)) = (\widehat{D_\alpha}cl(f^{-1}(F)))^c$ and hence $\widehat{D_\alpha}cl(f^{-1}(F)) \subset f^{-1}(F)$. By proposition 3.2.34 $f^{-1}(F)$ is $\widehat{D_\alpha}$-closed. Hence, f is $\widehat{D_\alpha}$-continuous.

4. $\widehat{D_\alpha}$-irresolute maps, strongly $\widehat{D_\alpha}$-continuous maps and perfectly $\widehat{D_\alpha}$-continuous maps

Definition 4.1. A map $f : X \rightarrow Y$ is called $\widehat{D_\alpha}$-irresolute if $f^{-1}(F)$ is $\widehat{D_\alpha}$-closed in X for every $\widehat{D_\alpha}$closed set F of Y .

Example 4.2. Let X = {a, b, c} = Y , τ = {φ, {a, b},X} and σ= {φ, {a}, Y }.

Here $\widehat{D_\alpha}c(\tau)$ = {φ, {a}, {b}, {c}, {b, c}, {a, c},X} and $\widehat{D_\alpha}c(\sigma)$ = {φ, {b}, {c}, {b, c},X}.

Define a map f :X → Y by f(a) = b, f(b) = a, f(c) = c. Clearly f is $\widehat{D_\alpha}$-irresolute, since every $\widehat{D_\alpha}$-closed subset of Y is $\widehat{D_\alpha}$-closed in X.

Proposition 4.3. If f :X → Y is $\widehat{D_\alpha}$-irresolute, then f is $\widehat{D_\alpha}$-continuous but not conversely.

Proof. Since every closed set is $\widehat{D_\alpha}$-closed. Hence f is $\widehat{D_\alpha}$-continuous.

Example 4.4. Let X = Y = {a, b, c}, τ = {φ, {a},X} and σ = {φ, {a, b}, Y }.
Here $\widehat{D_\alpha}$c(τ) = {φ, {b}, {c}, {b, c},X} and $\widehat{D_\alpha}$c(σ) = P(X) − {a, b}. Define a map f :X → Y by f(a) = b, f(b) = c and f(c) = a. Clearly f is $\widehat{D_\alpha}$-continuous but not $\widehat{D_\alpha}$-irresolute, since {b} is $\widehat{D_\alpha}$-closed in Y but f^{-1}({b}) = {a} is not $\widehat{D_\alpha}$-closed in X.

Proposition 4.5. Let f :X → Y and g : Y → Z be any two maps. Then

(a) g ◦ f is $\widehat{D_\alpha}$-irresolute if both f and g are $\widehat{D_\alpha}$-irresolute.

(b) g ◦ f is $\widehat{D_\alpha}$-continuous if g is $\widehat{D_\alpha}$-continuous and f is $\widehat{D_\alpha}$-irresolute.

Proof. (a) : Let f : X → Y and g : Y → Z be any two maps. Let F be a $\widehat{D_\alpha}$closed set in Z. Since g is $\widehat{D_\alpha}$-irresolute, g^{-1}(F) is $\widehat{D_\alpha}$-closed in Y . Since f is $\widehat{D_\alpha}$-irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\widehat{D_\alpha}$-closed in X. Thus g ◦ f is $\widehat{D_\alpha}$-irresolute.

(b) : Let F be a closed set in Z. Since g is $\widehat{D_\alpha}$-continuous, g^{-1}(F) is $\widehat{D_\alpha}$-closed in Y . Since f is $\widehat{D_\alpha}$-irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\widehat{D_\alpha}$-closed in X. Thus g ◦ f is $\widehat{D_\alpha}$-continuous.

Proposition 4.6. Let X be a topological space,Y be a $T_{\widehat{D_\alpha}}$-space and f :X → Y be a map. Then the following are equivalent:

(i) f is $\widehat{D_\alpha}$-irresolute,

(ii) f is $\widehat{D_\alpha}$-continuous.

Proof. (i) =⇒(ii) : Since every closed set is $\widehat{D_\alpha}$-closed. Hence f is $\widehat{D_\alpha}$-continuous.

(ii) =⇒(i) : Let F be a $\widehat{D_\alpha}$-closed set in Y . Since Y is a $T_{\widehat{D_\alpha}}$
-space, F is a closed
set in Y and by hypothesis, f^{-1}(F) is $\widehat{D_\alpha}$-closed in X. Thereforef is $\widehat{D_\alpha}$-irresolute.

Definition 4.7. A map f :X → Y is said to be strongly $\widehat{D_\alpha}$-continuous if the inverse image of every $\widehat{D_\alpha}$-open set of Y is open in X.

Proposition 4.8. If a map f :X → Y is strongly $\widehat{D_\alpha}$-continuous, then f is continuous but not conversely.

Proof. Since every open set is $\widehat{D_\alpha}$-open. Then f is continuous.

Example 4.9. Let X = Y = {a, b, c}, τ = {φ, {a}, {a, b}, {a, c},X} and σ =

{φ, {a}, Y }. Define a map f :X → Y by f(a) = a, f(b) = c, f(c) = b. Clearly f is continuous but not strongly $\widehat{D_\alpha}$-continuous. Since {b} is $\widehat{D_\alpha}$-open in Y but f^{-1}({b}) = {c} is not open in X.

Proposition4.10. Let X be a topological space,Y be a $T_{\widehat{D_\alpha}}$ -space and f :X → Y be a map. Then the following are equivalent:

(i) f is strongly $\widehat{D_\alpha}$-continuous,

(ii) f is continuous.

Proof. (i) =⇒(ii) : Since every open set is $\widehat{D_\alpha}$-open. Then f is continuous.

(ii) =⇒(i) : Let V be any $\widehat{D_\alpha}$-open set in Y . Since Y is a $T_{\widehat{D_\alpha}}$-space, V is open in Y . By (ii) f^{-1}(V) is open in X. Therefore, f is strongly $\widehat{D_\alpha}$-continuous.

Proposition 4.11. Every strongly $\widehat{D_\alpha}$-continuous map is SMPC, but not conversely.

Proof. Since every pre open set is $\widehat{D_\alpha}$-open. Hence every strongly $\widehat{D_\alpha}$-continuous map is SMPC.

Example 4.12. Let X = Y = {a, b, c}, τ = {φ, {a}, {c}, {a, c}, {b, c},X} and σ = {φ, {a}, {b}, {a, b},X}. Let f :X → Y be defined by f(a) = a, f(b) = c and f(c) = b. Clearly f is SMPC but not strongly $\widehat{D_\alpha}$-continuous, since {b} is$\widehat{D_\alpha}$-closed in (Y, σ) and f^{-1}({b}) = {c} is not closed in (X, τ).

Proposition 4.13. If a map f :X → Y is strongly continuous, then f is strongly $\widehat{D_\alpha}$-continuous but not conversely.

Proof. Every $\widehat{D_\alpha}$-open set is a subset itself. Hence f is strongly $\widehat{D_\alpha}$-continuous.

Example 4.14. Let X = Y = {a, b, c}, τ = {φ, {a}, {a, b}, {a, c},X} and σ = {φ, {b}, {a, b}, {b, c}, Y }. Let f :X → Y be a map defined by f(a) = b, f(b) = a and f(c) = c. Clearly f is strongly $\widehat{D_\alpha}$-continuous but not strongly continuous. Since f^{-1}({a, c}) = {b, c} is closed but not open in X.

Definition 4.15. A map f :X → Y is called perfectly $\widehat{D_\alpha}$-continuous if the inverse image of every $\widehat{D_\alpha}$-open set in Y is both open and closed in X.

Proposition 4.16. If a map f :X → Y is perfectly $\widehat{D_\alpha}$-continuous then f is perfectly continuous (resp. Continuous) but not conversely.

Proof. Let V be an open set in Y . Then V is $\widehat{D_\alpha}$-open in Y . Since f is perfectly $\widehat{D_\alpha}$-continuous, f^{-1}(V) is both open and closed in X. Thus f is perfectly continuous and also continuous.

Example 4.17. Let X = Y = {a, b, c}, τ = {φ, {a}, {b, c},X} and σ = {φ, {a}, Y }.

Clearly the identity map f :X → Y is perfectly continuous and continuous but not perfectly $\widehat{D_\alpha}$-continuous, since the set {c} is $\widehat{D_\alpha}$-open in Y but $f^{-1}(\{c\}) = \{c\}$ is neither closed nor open in X.

Proposition 4.18. If f :X → Y is perfectly $\widehat{D_\alpha}$-continuous then it is strongly $\widehat{D_\alpha}$-continuous but not conversely.

Proof. Let V be $\widehat{D_\alpha}$-open in Y . Since f is perfectly $\widehat{D_\alpha}$-continuous, $f^{-1}(V)$ is both open and closed in X. Thus f is strongly $\widehat{D_\alpha}$-continuous.

Example 4.19. Let (X, τ) and (Y, σ) be defined as in Example 3.14 then f is strongly $\widehat{D_\alpha}$-continuous but not perfectly $\widehat{D_\alpha}$-continuous, since the set {a, b} is $\widehat{D_\alpha}$-open in Y and $f^{-1}(\{a, b\}) = \{a, b\}$ is open but not closed in X.

Proposition 4.20. Every strongly $\widehat{D_\alpha}$-continuous map is $\widehat{D_\alpha}$-continuous but not conversely.

Proof. Let V be an open set in Y . Since f is strongly $\widehat{D_\alpha}$-continuous and every open set is $\widehat{D_\alpha}$-open, $f^{-1}(V)$ is an open set in X. Therefore $f^{-1}(V)$ is $\widehat{D_\alpha}$-open in X and so f is $\widehat{D_\alpha}$-continuous.

Example 4.21. Let (X, τ) and (Y, σ) be defined as in Example 3.2. Define a map f :X → Y by f(a) = b, f(b) = c and f(c) = a. Clearly f is $\widehat{D_\alpha}$-continuous but not strongly $\widehat{D_\alpha}$-continuous, since {a, b} is $\widehat{D_\alpha}$-open in Y but $f^{-1}(\{a, b\}) = \{a, c\}$ is not open in X.

From the above discussion we have the following implications:

Proposition 4.22. Let X be a discrete topological space,Y be any topological space and f :X → Y be a map. Then the following are equivalence:

1. f is perfectly $\widehat{D_\alpha}$-continuous,
2. f is strongly $\widehat{D_\alpha}$-continuous.

Proof. (i) =⇒**(ii) :**Let V be $\widehat{D_\alpha}$-open in Y . Since f is perfectly $\widehat{D_\alpha}$-continuous, $f^{-1}(V)$ is both open and closed in X. Thus f is strongly $\widehat{D_\alpha}$-continuous.

(ii) =⇒**(i) :**Let U1 be any $\widehat{D_\alpha}$-open set in Y . By hypothesis $f^{-1}(U)$ is open in X. Since X is a discrete space, $f^{-1}(U)$ is also closed in X and so f is perfectly $\widehat{D_\alpha}$-continuous.

Proposition 4.23. If f :X → Y and g : Y → Z are perfectly $\widehat{D_\alpha}$-continuous, then their composition g ◦ f :X → Z is also perfectly $\widehat{D_\alpha}$-continuous.

Proof. Let f :X → Y and g : Y → Z be two maps. Let V be a $\widehat{D_\alpha}$-open set in Z. Since g is perfectly $\widehat{D_\alpha}$-continuous. $g^{-1}(V)$ is both open and closed in Y . As f is

perfectly $\widehat{D_\alpha}$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is both open and closed in X.
Thus g ∘ f is perfectly $\widehat{D_\alpha}$-continuous.

Proposition 4.24. If f :X → Y and g : Y → Z be any two maps. Then their composition g ∘ f :X → Z is

i) $\widehat{D_\alpha}$-irresolute if g is perfectly $\widehat{D_\alpha}$-continuous and f is $\widehat{D_\alpha}$-continuous.

ii) strongly $\widehat{D_\alpha}$-continuous if g is perfectly $\widehat{D_\alpha}$-continuous and f is continuous.

iii) perfectly $\widehat{D_\alpha}$-continuous if g is strongly continuous and f is perfectly $\widehat{D_\alpha}$-continuous.

Proof. i) Let f :X → Y and g : Y → Z be two maps. Let V be a $\widehat{D_\alpha}$-open set in Z. Since g is perfectly $\widehat{D_\alpha}$-continuous. $g^{-1}(V)$ is both open and closed in Y . As f is $\widehat{D_\alpha}$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\widehat{D_\alpha}$-open in X. Thus g ∘ f is $\widehat{D_\alpha}$-irresolute.

ii) Let f :X → Y and g : Y → Z be two maps. Let V be a $\widehat{D_\alpha}$-open set in Z. Since g is perfectly $\widehat{D_\alpha}$-continuous. $g^{-1}(V)$ is both open and closed in Y . As f is continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is open in X. Thus g ∘ f is strongly $\widehat{D_\alpha}$-continuous.

iii) Let f :X → Y and g : Y → Z be two maps. Let V be a $\widehat{D_\alpha}$-open set in Z. Since g is strongly continuous. $g^{-1}(V)$ is both open and closed in Y . As f is perfectly $\widehat{D_\alpha}$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is both open and closed in X. Thus g ∘ f is perfectly $\widehat{D_\alpha}$-continuous.

5.Totally $\widehat{D_\alpha}$-continuous and Contra $\widehat{D_\alpha}$-continuous maps

Definition 5.1 A map $f:X \to Y$ is said to be totally $\widehat{D_\alpha}$-continuous, if the inverse image of every open subset of Y is a $\widehat{D_\alpha}$-clopen subset of X.

Example 5.2. Let $X = \{a, b, c\}$, $Y = \{p, q\}$, $\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\varphi, \{p\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ such that $f(a) = p$, $f(b) = f(c) = q$.
Clearly f is totally $\widehat{D_\alpha}$-continuous.

Proposition 5.3. Every perfectly $\widehat{D_\alpha}$-continuous map is totally $\widehat{D_\alpha}$-continuous, but not conversely.

Proof. Let $f:X \to Y$ be a perfectly $\widehat{D_\alpha}$-continuous map. Let U be an open set in Y . Then U is $\widehat{D_\alpha}$-open in Y . Since f is perfectly $\widehat{D_\alpha}$-continuous, $f^{-1}(U)$ is clopen in X implies $f^{-1}(U)$ is $\widehat{D_\alpha}$-clopen in X. Hence every perfectly $\widehat{D_\alpha}$-continuous map is totally $\widehat{D_\alpha}$-continuous.

Example 5.4. Let $X = Y = \{a, b, c\}$, $\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\varphi, \{a\}, \{b, c\}, Y\}$. Clearly the identity map $f : (X, \tau) \to (Y, \sigma)$ is totally $\widehat{D_\alpha}$-continuous but not perfectly $\widehat{D_\alpha}$-continuous, since the set $\{a\}$ is $\widehat{D_\alpha}$-open in Y but $f^{-1}(\{a\}) = \{a\}$

is open but not closed in (X, τ).

Remark 5.5. The following two examples show that totally $\widehat{D_\alpha}$-continuous and strongly $\widehat{D_\alpha}$-continuous are independent.

Example 5.6. Let (X, τ) and (Y, σ) be defined as in Example 5.4. Clearly the identity map $f : (X, \tau) \rightarrow (Y, \sigma)$ is totally $\widehat{D_\alpha}$-continuous but not strongly $\widehat{D_\alpha}$-continuous, since the set $\{b, c\}$ is $\widehat{D_\alpha}$-open in (Y, σ) but $f^{-1}(\{b, c\}) = \{b, c\}$ is not open in (X, τ).

Example 5.7. Let $X = Y = \{a, b, c\}$, $\tau = \{\varphi, \{a\}, \{a, b\}\{a, c\}, X\}$ and $\sigma = \{\varphi, \{a\}, \{a, b\}, Y\}$ Define a map $f : X \rightarrow Y$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$.

Clearly f is strongly $\widehat{D_\alpha}$-continuous but not totally $\widehat{D_\alpha}$-continuous, since $\{a, b\}$ is open in (Y, σ) and $f^{-1}(\{a, b\}) = \{a, c\}$ is $\widehat{D_\alpha}$-open but not $\widehat{D_\alpha}$-closed in (X, τ).

Proposition 5.8. Every totally $\widehat{D_\alpha}$-continuous map is $\widehat{D_\alpha}$-continuous.

Proof. Clearly follows from Definition 5.1 totally $\widehat{D_\alpha}$-continuous map is $\widehat{D_\alpha}$-continuous.

Remark 5.9. The converse of Proposition 5.8 is not true by the Example 5.7. Here the map f is $\widehat{D_\alpha}$-continuous but not totally $\widehat{D_\alpha}$-continuous.

Remark 5.10. It is clear that the totally $\widehat{D_\alpha}$-continuous map is stronger than $\widehat{D_\alpha}$-continuous map and weaker than perfectly $\widehat{D_\alpha}$- continuous map.

Theorem 5.11. If $f : X \rightarrow Y$ is a totally $\widehat{D_\alpha}$- continuous map from a $\widehat{D_\alpha}$- connected space X onto any space Y, then Y is an indiscrete space.

Proof. Suppose that Y is not indiscrete. Let A be a proper nonempty open subset of Y. Then $f^{-1}(A)$ is a nonempty proper $\widehat{D_\alpha}$-clopen subset of X, which is a contradiction to the fact that X is $\widehat{D_\alpha}$-connected.

Definition 5.12. A topological space X is called $\widehat{D_\alpha}-T_2$ if for each pair of distinct points x and y in X, there exist disjoint $\widehat{D_\alpha}$-open sets U and V in X containing x and y respectively.

Theorem 5.13. Let $f : X \rightarrow Y$ be a totally $\widehat{D_\alpha}$-continuous injection. If Y is T_0 then X is $\widehat{D_\alpha}- T_2$.

Proof. Let x and y be any pair of distinct points of X. Then $f(x) \neq f(y)$. Since Y is T_0, there exists an open set U containing say $f(x)$ but not $f(y)$. Then $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$. Since f is totally $\widehat{D_\alpha}$-continuous, $f^{-1}(U)$ is a $\widehat{D_\alpha}$-clopen subset of X. Also, $x \in f^{-1}(U)$ and $y \in (f^{-1}(U))^c$. Hence, X is $\widehat{D_\alpha}- T_2$.

Theorem 5.14. A topological space X is $\widehat{D_\alpha}$-connected if and only if every totally $\widehat{D_\alpha}$-continuous map from a space X into any T_0-space Y is constant.

Proof. We prove the only "if" part. Suppose that X is not $\widehat{D_\alpha}$-connected. By hypothesis, every $\widehat{D_\alpha}$-continuous map from X into Y is constant. Since X is not $\widehat{D_\alpha}$-connected, there exists a proper nonempty $\widehat{D_\alpha}$-clopen subset A of X. Let $Y = \{x, y\}$ and $\sigma = \{Y, \varphi, \{x\}, \{y\}\}$ be a topology for Y . Let $f : X \to Y$ be a map such that $f(A) = \{x\}$ and $f(Ac) = \{y\}$. Then f is non constant and totally $\widehat{D_\alpha}$-continuous such that Y is T_0, which is a contradiction. Hence X must be $\widehat{D_\alpha}$-connected.

Theorem 5.15. Let $f : X \to Y$ be a totally $\widehat{D_\alpha}$-continuous map and Y is a T_1-space. If A is a $\widehat{D_\alpha}$-connected subset of X, then $f(A)$ is a single point.

Proof. Obvious.

Definition 5.16. Let X be a topological space. Then the set of all points y in X such that x and y cannot be separated by a $\widehat{D_\alpha}$-separation of X is said to be the quasi-$\widehat{D_\alpha}$-component of X.

Theorem 5.17. Let $f : X \to Y$ be a totally $\widehat{D_\alpha}$-continuous map from a topological space X into T_1-space Y . Then f is constant on each quasi-$\widehat{D_\alpha}$-component of X.

Proof. Let x and y be two points of X that lie in the same quasi-$\widehat{D_\alpha}$-component of X. Assume that $f(x) = \alpha \neq \beta = f(y)$. Since Y is T_1, $\{\alpha\}$ is closed in Y and so $\{\alpha\}^c$ is an open set. Since f is totally $\widehat{D_\alpha}$-continuous, $f^{-1}(\{\alpha\})$ and $f{-}1(\{\alpha\}^c)$ are disjoint $\widehat{D_\alpha}$-clopen subsets of X. Further $x \in f{-}1(\{\alpha\})$ and $y \in f{-}1(\{\alpha\}^c)$, which is a contradiction in view of the fact that y belongs to the quasi-$\widehat{D_\alpha}$-component of x and hence y must belongs to every quasi-$\widehat{D_\alpha}$-clopen set containing X. Hence the result.

Now we introduction the concept of contra $\widehat{D_\alpha}$-continuous map.

Definition 5.18. A map $f : X \to Y$ is called contra $\widehat{D_\alpha}$-continuous if $f^{-1}(V)$ is $\widehat{D_\alpha}$-closed in X for each open set V of Y .

Theorem 5.19. Assume that $\widehat{D_\alpha}o(\tau)$ is closed under arbitrary union. Then for a map $f : X \to Y$, the following are equivalent:

i) f is contra $\widehat{D_\alpha}$-continuous;

ii) for every closed subset F of Y ,$f^{-1}(F) \in \widehat{D_\alpha}o(\tau)$;

iii) for each $x \in X$ and each closed set F of Y containing $f(x)$, there exists $U \in \widehat{D_\alpha}o(\tau)$ and $x \in U$ such that $f(U) \subset F$;

iv) $f(\widehat{D_\alpha}cl(A)) \subset kerf(A)$, for every subset A of X;

v) $\widehat{D_\alpha}cl(f^{-1}(B)) \subset f^{-1}(ker(B))$, for every subset B of Y .

Proof. (i) =⇒(ii) : Obvious.

(ii) =⇒(iii) : Let x be any point of X and F any closed set of Y containing $f(x)$.

By (ii), $f^{-1}(F) \in \widehat{D_\alpha}o(\tau)$ and $x \in f^{-1}(F)$. Put $U = f^{-1}(F)$. Then $U \in \widehat{D_\alpha}o(\tau)$ and $f(U) \subset F$.

(iii) =⇒(ii) : Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \widehat{D_\alpha}o(\tau)$ and $x \in U_x$ such that $f(U_x) \subset F$. Therefore, by hypothesis $f^{-1}(F) = \cup\{U_x / x \in f^{-1}(F)\} \in \widehat{D_\alpha}o(\tau)$.

(ii) =⇒(iv) : Let A be any subset of X. Suppose that $y \notin ker(f(A))$. Then there exists a closed set F of Y containing y such that $f(A) \cap F = \varphi$.

Thus we have $A \cap f^{-1}(F) = \varphi$ and $\widehat{D_\alpha}cl(A) \cap f^{-1}(F) = \varphi$. Therefore, we obtain $f(\widehat{D_\alpha}cl(A)) \cap F = \varphi$ and $y \notin f(\widehat{D_\alpha}cl(A))$. This implies that $f(\widehat{D_\alpha}cl(A)) \subset ker\, f(A)$.

(iv) =⇒(v) : Let B be any subset of Y . By (iv) and Lemma 1.3.20, we have $f(\widehat{D_\alpha}cl\, f^{-1}(B)) \subset ker f(f^{-1}(B)) \subset ker(B)$ and $\widehat{D_\alpha}cl(f^{-1}(B)) \subset f^{-1}(ker(B))$.

(v) =⇒(i) : Let V be any open set of Y . Then we have $\widehat{D_\alpha}cl(f^{-1}(V)) \subset f^{-1}(ker(V)) = f^{-1}(V)$. Thus $\widehat{D_\alpha}cl(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is $\widehat{D_\alpha}$-closed in X .

Theorem 5.20. If a map $f : X \to Y$ is contra $\widehat{D_\alpha}$-continuous and Y is regular then f is $\widehat{D_\alpha}$-continuous.

Proof. Let x be any arbitrary point of X and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $cl(W) \subset V$. Since f is contra $\widehat{D_\alpha}$-continuous, by Theorem 5.19 there exists a $\widehat{D_\alpha}$-open set U containing x such that $f(U) \subset cl(W) \subset V$. Hence f is $\widehat{D_\alpha}$-continuous.

Theorem 5.21. If a map $f : X \to Y$ is contra continuous then it is contra $\widehat{D_\alpha}$-continuous.

Proof. The proof follows from the fact that every closed set is $\widehat{D_\alpha}$-closed.

Remark 5.22. The converse of the above theorem need not be true as seen from the following example.

Example 5.23. Let (X, τ) and (Y, σ) be defined as in Example 5.4. Then the identity map $f : (X, \tau) \to (Y, \sigma)$ is contra $\widehat{D_\alpha}$-continuous but not contra continuous, since the set $\{a\}$ is open in Y but $f^{-1}(\{a\}) = \{a\}$ is not closed in (X, τ).

Proposition 5.24. Let X be a $T_{\widehat{D_\alpha}}$-space. If a map $f : X \to Y$ is contra $\widehat{D_\alpha}$-continuous, then f is contra continuous.

Proof. The proof follows

Remark 5.25. The following two examples show that contra $\widehat{D_\alpha}$-continuous and continuous are independent.

Example 5.26. Let (X, τ) and (Y, σ) be defined as in Example 5.4. Define a map $f : X \to Y$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Clearly f is contra $\widehat{D_\alpha}$-continuous but not continuous, since the set $\{b, c\}$ is open in Y but $f^{-1}(\{b, c\}) = \{b, c\}$ is not open in (X, τ).

Example 5.27. Let $X = Y = \{a, b, c\}$, $\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\varphi, \{a, b\}, Y\}$. Define a map $f : X \to Y$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Clearly f is continuous but not contra $\widehat{D_\alpha}$-continuous, since the set $\{a, b\}$ is open in Y but $f^{-1}(\{a, b\}) = \{a, b\}$ is not $\widehat{D_\alpha}$-closed in (X, τ).

Theorem 5.28. If $f : X \to Y$ is a contra $\widehat{D_\alpha}$-continuous map and $g : Y \to Z$ is a continuous map, then $g \circ f : X \to Z$ is contra $\widehat{D_\alpha}$-continuous.

Proof. Clearly follows from Definitions.

Remark 5.29. The product of two contra $\widehat{D_\alpha}$-continuous maps is not a contra $\widehat{D_\alpha}$-continuous map, it is shown by the following example.

Example 5.30. Let $X = Y = Z = \{p, q, r\}$, $\tau = \{\varphi, \{p\}, \{q\}, \{p, q\}, X\}$, $\sigma = \{\varphi, \{p\}, \{q, r\}, Y\}$ and $\eta = \{\varphi, \{p, q\}, Z\}$. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two identity maps. Then f and g are contra $\widehat{D_\alpha}$-continuous. Let $A = \{p, q\}$ be an open in Z. Then $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) = \{p, q\}$ which is not $\widehat{D_\alpha}$-closed in (X, τ). Therefore $g \circ f : (X, \tau) \to (Z, \eta)$ is not a contra $\widehat{D_\alpha}$-continuous.

Theorem 5.31. Assume that $\widehat{D_\alpha}o(\tau)$ is closed under any union. If X is a topological space and for each pair of distinct points $x1$ and $x2$ in X there exists a map f that maps X into a uryshon space Y such that $f(x1) \neq f(x2)$ and f is contra $\widehat{D_\alpha}$-continuous at x_1 and x_2. Then X is $\widehat{D_\alpha} - T2$.

Proof. Let x_1 and x_2 be any two distinct points in X. Then by hypothesis, there is a uryshon space Y and a map $f : X \to Y$ which satisfies the conditions of the theorem. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y1 \neq y2$. Since Y is uryshon, there exists open neighbourhoods U_{y_1} and U_{y_2} of y_1 and y_2 respectively in Y such that $cl(U_{y_1}) \cap cl(U_{y_2}) = \varphi$. Since f is contra $\widehat{D_\alpha}$-continuous at xi, there exist $\widehat{D_\alpha}$-open neighbourhoods W_{x_i} of xi in X such that $f(W_{x_i}) \subset cl(U_{y_i})$ for $i = 1, 2$. Hence we get that $W_{x_1} \cap W_2 = \varphi$, because $cl(U_{y_1}) \cap cl(U_{y_2}) = \varphi$. Thus X is $\widehat{D_\alpha} - T_2$.

Corollary 5.32. Assume that $\widehat{D_\alpha}o(\tau)$ is closed under any union. If f is a contra $\widehat{D_\alpha}$-continuous injection of a topological space X into a uryshon space Y. Then X is $\widehat{D_\alpha} - T_2$.

Proof. For each pair of distinct points x_1 and x_2 in X, f is a contra $\widehat{D_\alpha}$-continuous of X into a uryshon space Y such that $f(x_1) \neq f(x_2)$ because f is injective. Hence by the above Theorem, X is $\widehat{D_\alpha} - T_2$.

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