

On symmetries and solutions of certain sixth order difference equations

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Abstract

We use the Lie group analysis method to investigate the invariance properties and the solutions of

$$x_{n+1} = \frac{x_{n-5}x_{n-3}}{x_{n-1}(a_n + b_nx_{n-5}x_{n-3})}.$$

We show that this equation has a two-dimensional Lie algebra and that its solutions can be presented in a unified manner. Besides presenting solutions of the recursive sequence above where a_n and b_n are sequences of real numbers, some specific cases are emphasized.

Key words: Difference equation; symmetry; reduction; group invariant solutions, periodicity

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1 Introduction

Difference equations are important in mathematical modelling, especially where discrete time evolving variables are concerned. They also occur when studying discretization methods for differential equations. Countless results in the subject of difference equations have been recorded. For rational difference equations of order greater than 1, the study can be quite challenging at the same time rewarding. Rewarding in the sense that such a study lays ground for the theory of global properties of difference equations (not necessarily rational) of higher order.

In [4], the author developed an effective symmetry based algorithm to deal with the obtention of solutions of difference equations of any order. However, the calculation one deals with in this application to difference equations of order greater than one can become cumbersome but with great recompense often times. The method consists of finding a group of transformations that maps solutions onto themselves. Symmetry method is a valuable tool and it has been used to solve several difference equations [1–3, 7, 8].

In this paper, our objective is to obtain the symmetry operators of

$$x_{n+1} = \frac{x_{n-5}x_{n-3}}{x_{n-1}(a_n + b_nx_{n-5}x_{n-3})} \tag{1}$$

where a_n and b_n are real sequences and to find its solutions by way of symmetries. Without loss of generality, we equivalently study the forward difference equation

$$u_{n+6} = \frac{u_n u_{n+2}}{u_{n+4}(A_n + B_n u_n u_{n+2})}. \tag{2}$$

We refer the interested reader to [4, 9] for a deeper knowledge of Lie analysis.

2 Definitions and Notation

In this section, we briefly present some definitions and notation (largely from Hydon in [4]) indispensable for the understanding of Lie symmetry analysis of difference equations.

Definition 2.1 *Let G be a local group of transformations acting on a manifold M . A subset $\mathcal{S} \subset M$ is called G -invariant, and G is called symmetry group of \mathcal{S} , if whenever $x \in \mathcal{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$.*

Definition 2.2 *Let G be a connected group of transformations acting on a manifold M . A smooth real-valued function $\mathcal{V} : M \rightarrow \mathbb{R}$ is an invariant function for G if and only if*

$$X(\mathcal{V}) = 0 \quad \text{for all} \quad x \in M,$$

and every infinitesimal generator X of G .

Definition 2.3 *A parameterized set of point transformations,*

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x; \varepsilon), \tag{3}$$

where $x = x_i, i = 1, \dots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$
2. $\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0
3. Each \hat{x}_i can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore

$$\hat{x}_i(x; \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i = 1, \dots, p. \tag{4}$$

Assuming that the sixth-order difference equation has the form

$$u_{n+6} = \Psi(n, u_n, \dots, u_{n+5}), \quad n \in D \tag{5}$$

for some smooth function Ω and a regular domain $D \subset \mathbb{Z}$. To deduce the symmetry group of (5), we search for a one parameter Lie group of point transformations

$$\Gamma_\varepsilon : (n, u_n) \mapsto (n, u_n + \varepsilon Q(n, u_n)), \tag{6}$$

in which ε is the parameter and Q a continuous function, referred to as characteristic. Let

$$X = Q(n, u_n) \frac{\partial}{\partial u_n} + Q(n+1, u_{n+1}) \frac{\partial}{\partial u_{n+1}} + \dots + Q(n+5, u_{n+5}) \frac{\partial}{\partial u_{n+5}} \quad (7)$$

be the corresponding ‘prolonged’ infinitesimal generator and $S : n \mapsto n+1$ the shift operator. The linearized symmetry condition is given by

$$S^6 Q - X \Psi = 0. \quad (8a)$$

Upon knowledge of the characteristic Q , it is important to introduce the canonical coordinate

$$S_n = \int \frac{du_n}{Q(n, u_n)}, \quad (9)$$

a useful tool which allows one to obtain the invariant \mathcal{V} .

3 Main results

As earlier emphasized, our equation under study is

$$u_{n+6} = \Psi = \frac{u_n u_{n+2}}{u_{n+4} (A_n + B_n u_n u_{n+2})}. \quad (10)$$

Applying the criterion of invariance (8) to (10), we get

$$\begin{aligned} & Q(n+6, u_{n+6}) + \frac{u_n u_{n+2}}{u_{n+4}^2 (A_n + B_n u_n u_{n+2})} Q(n+4, u_{n+4}) \\ & - \frac{A_n u_n}{u_{n+4} ((A_n + B_n u_n u_{n+2})^2)} Q(n+2, u_{n+2}) \\ & - \frac{A_n u_{n+2}}{u_{n+4} (A_n + B_n u_n u_{n+2})^2} Q(n, u_n) = 0. \end{aligned} \quad (11)$$

In order to eliminate u_{n+3} , we invoke implicit differentiation with respect to u_n (regarding u_{n+4} as a function of u_n, u_{n+2} and u_{n+3}) via the operator

$$L = \frac{\partial}{\partial u_n} - \frac{\Psi_{u_n}}{\Psi_{u_{n+4}}} \frac{\partial}{\partial u_{n+4}}.$$

With some simplification, one gets

$$\begin{aligned} & (A_n + B_n u_n u_{n+2}) Q'(n+4, u_{n+4}) - \frac{(A_n + B_n u_n u_{n+2})}{u_{n+4}} Q(n+4, u_{n+4}) \\ & + B_n u_n Q(n+2, u_{n+2}) - (A_n + B_n u_n u_{n+2}) Q'(n, u_n) \\ & + \left(2B_n u_{n+2} + \frac{A_n}{u_n} \right) Q(n, u_n) = 0. \end{aligned} \quad (12)$$

Note that the symbol ' stands for the derivative with respect to the continuous variable. After twice differentiating (12) with respect to u_n , keeping u_{n+2} and u_{n+4} fixed, we are led to the equation

$$\begin{aligned}
 & -B_n u_n u_{n+2} Q'''(n, u_n) - A_n Q'''(n, u_n) + \frac{A_n}{u_n} Q''(n, u_n) - \frac{2A_n}{u_n^2} Q'(n, u_n) \\
 & + \frac{2A_n}{u_n^3} Q(n, u_n) = 0
 \end{aligned} \tag{13}$$

Note that the characteristic in (13) is not a function of u_{n+2} and so we split (13) up with respect to u_{n+2} to get the system

$$1 : Q'''(n, u_n) - \frac{1}{u_n} Q''(n, u_n) + \frac{2}{u_n^2} Q'(n, u_n) - \frac{2}{u_n^3} Q(n, u_n) = 0 \tag{14a}$$

$$u_{n+2} : Q'''(n, u_n) = 0. \tag{14b}$$

We find that the solution to (14) is

$$Q(n, u_n) = \alpha_n u_n^2 + \beta_n u_n \tag{15}$$

for some arbitrary functions α_n and β_n that depend on n . Substituting (16) and its first, second and third shifts in (11), and then replacing the expression of u_{n+3} given in (10) in the resulting equation yields

$$\begin{aligned}
 & B_n u_n^2 u_{n+2}^2 u_{n+4}^2 \alpha_{n+4} + B_n u_n^2 u_{n+2}^2 u_{n+4} (\beta_{n+4} + \beta_{n+3}) \\
 & - A_n u_n^2 u_{n+2} u_{n+4} \alpha_n - A_n u_n u_{n+2}^2 u_{n+4} \alpha_{n+2} + A_n u_n u_{n+2} u_{n+4}^2 \alpha_{n+4} \\
 & + u_n^2 u_{n+2}^2 \alpha_{n+2} - A_n (\beta_n + \beta_{n+2} - \beta_{n+4} - \beta_{n+3}) = 0.
 \end{aligned} \tag{16}$$

Equating all coefficients of all powers of shifts of u_n to zero and simplifying the resulting system, we get its reduced form

$$\alpha_n = 0, \tag{17}$$

$$\beta_n + \beta_{n+2} = 0. \tag{18}$$

The two independent solutions of the linear second-order difference equation above are given by

$$\beta_n = \beta^n \text{ and } \beta_n = \bar{\beta}^n, \tag{19}$$

where $\beta = \exp\{i\pi/2\}$ and $\bar{\beta} = -\exp\{i\pi/2\}$ is its complex conjugate. The characteristic functions are given by

$$Q_1(n, u_n) = \beta^n u_n \quad \text{and} \quad Q_2(n, u_n) = \bar{\beta}^n u_n, \tag{20}$$

and so the Lie algebra of (10) is generated by

$$X_1 = \beta^n u_n \frac{\partial}{\partial u_n} + \beta^{n+2} u_{n+2} \frac{\partial}{\partial u_{n+2}} + \beta^{n+4} u_{n+4} \frac{\partial}{\partial u_{n+4}} \tag{21}$$

$$X_2 = \bar{\beta}^n u_n \frac{\partial}{\partial u_n} + \bar{\beta}^{n+2} u_{n+2} \frac{\partial}{\partial u_{n+2}} + \bar{\beta}^{n+4} u_{n+4} \frac{\partial}{\partial u_{n+4}}. \tag{22}$$

Using the canonical coordinate

$$S_n = \int \frac{du_n}{Q_1(n, u_n)} = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n| \tag{23}$$

and (17), we derive the invariant function $\tilde{\mathcal{V}}_n$ as follows:

$$\tilde{\mathcal{V}}_n = S_n \beta^n + S_{n+2} \beta^{n+2}. \tag{24}$$

Actually,

$$X_1(\tilde{\mathcal{V}}_n) = \beta^n + \beta^{n+2} = 0 \tag{25}$$

and

$$X_2(\tilde{\mathcal{V}}_n) = \bar{\beta}^n + \bar{\beta}^{n+2} = 0. \tag{26}$$

For the sake of convenience, we use

$$|\mathcal{V}_n| = \exp\{-\tilde{\mathcal{V}}_n\} \tag{27}$$

instead/ In other words, $\mathcal{V}_n = \pm 1/(u_n u_{n+2})$. Using (10) and (27), one can prove that

$$\mathcal{V}_{n+4} = A_n \mathcal{V}_n \pm B_n. \tag{28}$$

Utilizing the plus sign, the solution of (28) can be written as

$$\mathcal{V}_{4n+j} = \mathcal{V}_j \left(\prod_{k_1=0}^{n-1} A_{4k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{4l+j} \prod_{k_2=l+1}^{n-1} A_{4k_2+j} \right), \tag{29}$$

where $j = 0, 1, 2, 3$. From here, obtaining the solution of (10) is straightforward. We first employ (23) to get

$$|u_n| = \exp(\beta_n S_n). \tag{30}$$

Secondly, we employ (24) to obtain

$$|u_n| = \exp \left(\beta^n c_1 + \bar{\beta}^n c_2 - \frac{1}{2} \sum_{k_1=0}^{n-1} \beta^n \bar{\beta}^{k_1} \tilde{\mathcal{V}}_{k_1} - \frac{1}{2} \sum_{k_2=0}^{n-1} \bar{\beta}^n \beta^{k_2} \tilde{\mathcal{V}}_{k_2} \right). \tag{31}$$

Lastly, we use (27) to get

$$|u_n| = \exp \left(\beta^n c_1 + \bar{\beta}^n c_2 + \frac{1}{2} \sum_{k_1=0}^{n-1} \beta^n \bar{\beta}^{k_1} \ln |\mathcal{V}_{k_1}| + \frac{1}{2} \sum_{k_2=0}^{n-1} \bar{\beta}^n \beta^{k_2} \ln |\mathcal{V}_{k_2}| \right), \tag{32a}$$

in which V_k is given in (28) with $\gamma(n, k) = \beta^n \bar{\beta}^k$. Note that the constants c_1 and c_2 satisfy

$$c_1 + c_2 = \ln |u_0| \quad \text{and} \quad \beta(c_1 - c_2) = \ln |u_1|. \tag{32b}$$

Note. Equations in (32) give the solutions of (2) in a unified manner. On a further note, $\gamma(n, k) = \beta^n \bar{\beta}^k$ satisfies

$$\begin{aligned} \gamma(0, 1) &= \bar{\beta}, \gamma(0, 2) = -1, \gamma(1, 0) = \beta, \gamma(1, 2) = -\beta, \gamma(1, 3) = -1, \\ \gamma(n, n) &= 1, \gamma(n + 2, k) = -\gamma(n, k), \\ \gamma(n, k + 2) &= -\gamma(n, k), \gamma(4n, k) = \gamma(0, k), \gamma(n, 4k) = \gamma(n, 0). \end{aligned} \tag{33}$$

From u_n given in (32a) and equation (33), observe that

$$|u_{4n+j}| = \exp\left(H_j + \sum_{k_1=0}^{4n+j-1} \operatorname{Re}(\gamma(0, k_1)) \ln |\mathcal{V}_{k_1}|\right) \tag{34}$$

in which

$$H_j = \beta^j c_1 + \bar{\beta}^j c_2.$$

For $j = 0$, we have,

$$\begin{aligned} |u_{4n}| &= \exp(H_0 + \ln |\mathcal{V}_0| - \ln |\mathcal{V}_2| + \dots + \ln |\mathcal{V}_{4n-4}| - \ln |\mathcal{V}_{4n-2}|) \\ &= \exp(H_0) \prod_{s=0}^{n-1} \left| \frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}} \right|. \end{aligned} \tag{35}$$

It can be shown that there is no need for the absolute values via the utilization of the fact that

$$\mathcal{V}_i = \frac{1}{u_i u_{i+2}}. \tag{36}$$

In order to deduce $\exp(H_0)$, we set $n = 0$ in (35) and note that $|u_0| = \exp(H_0)$. Thus

$$u_{4n} = u_0 \prod_{s=0}^{n-1} \frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}}.$$

Similarly, replacing n with $4n + j$ for $j = 0, 1, 2, 3$, we obtain

$$U_{4n+j} = u_j \prod_{s=0}^{n-1} \frac{\mathcal{V}_{4s+j}}{\mathcal{V}_{4s+j+2}}. \tag{37}$$

Nevertheless, from (28), using the plus sign we are led to

$$\mathcal{V}_{4n+j} = \mathcal{V}_j \left(\prod_{k_1=0}^{n-1} A_{4k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{4l+j} \prod_{k_2=l+1}^{n-1} A_{4k_2+j} \right), \tag{38}$$

for $j = 0, 1, 2, 3$, where $\mathcal{V}_0 = \frac{1}{u_0 u_2}$. Thus, using (37) with $j = 0$, we get

$$\begin{aligned}
 U_{4n} &= u_0 \prod_{s=0}^{n-1} \frac{V_{4s}}{V_{4s+2}} \\
 &= u_0 \prod_{s=0}^{n-1} \frac{V_0 \left(\prod_{k_1=0}^{s-1} A_{4k_1} \right) + \sum_{l=0}^{s-1} \left(B_{4l} \prod_{k_2=l+1}^{s-1} A_{4k_2} \right)}{V_2 \left(\prod_{k_1=0}^{s-1} A_{4k_1+2} \right) + \sum_{l=0}^{s-1} \left(B_{4l+2} \prod_{k_2=l+1}^{s-1} A_{4k_2+2} \right)} \\
 &= u_0 \prod_{s=0}^{n-1} \frac{u_4 \left(\prod_{k_1=0}^{s-1} A_{4k_1} \right) + u_0 u_2 \sum_{l=0}^{s-1} \left(B_{4l} \prod_{k_2=l+1}^{s-1} A_{4k_2} \right)}{u_0 \left(\prod_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum_{l=0}^{s-1} \left(B_{4l+2} \prod_{k_2=l+1}^{s-1} A_{4k_2+2} \right)} \\
 &= u_0^{1-n} u_4^n \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1} \right) + u_0 u_2 \sum_{l=0}^{s-1} \left(B_{4l} \prod_{k_2=l+1}^{s-1} A_{4k_2} \right)}{\left(\prod_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum_{l=0}^{s-1} \left(B_{4l+2} \prod_{k_2=l+1}^{s-1} A_{4k_2+2} \right)}
 \end{aligned}$$

For $j = 1$, we have

$$\begin{aligned}
 U_{4n+1} &= u_1 \prod_{s=0}^{n-1} \frac{V_{4s+1}}{V_{4s+3}} \\
 &= u_1^{1-n} u_5^n \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1+1} \right) + u_1 u_3 \sum_{l=0}^{s-1} \left(B_{4l+1} \prod_{k_2=l+1}^{s-1} A_{4k_2+1} \right)}{\left(\prod_{k_1=0}^{s-1} A_{4k_1+3} \right) + u_3 u_5 \sum_{l=0}^{s-1} \left(B_{4l+3} \prod_{k_2=l+1}^{s-1} A_{4k_2+3} \right)}
 \end{aligned}$$

For $j = 2$, we have

$$\begin{aligned}
 U_{4n+2} &= u_2 \prod_{s=0}^{n-1} \frac{V_{4s+2}}{V_{4s+4}} \\
 &= u_0^n u_4^{-n} u_2 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum_{l=0}^{s-1} \left(B_{4l+2} \prod_{k_2=l+1}^{s-1} A_{4k_2+2} \right)}{\left(\prod_{k_1=0}^s A_{4k_1} \right) + u_0 u_2 \sum_{l=0}^s \left(B_{4l} \prod_{k_2=l+1}^s A_{4k_2} \right)}
 \end{aligned}$$

For $j = 3$, we get

$$\begin{aligned}
 U_{4n+3} &= u_3 \prod_{s=0}^{n-1} \frac{V_{4s+3}}{V_{4s+5}} \\
 &= u_1^n u_5^{-n} u_3 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1+3} \right) + u_3 u_5 \sum_{l=0}^{s-1} \left(B_{4l+3} \prod_{k_2=l+1}^{s-1} A_{4k_2+3} \right)}{\left(\prod_{k_1=0}^s A_{4k_1+1} \right) + u_1 u_3 \sum_{l=0}^s \left(B_{4l+1} \prod_{k_2=l+1}^s A_{4k_2+1} \right)}
 \end{aligned}$$

Hence, our solution in terms of x_n ($n > 0$) is given by

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^n \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} a_{4k_1} \right) + x_{-5} x_{-3} \sum_{l=0}^{s-1} \left(b_{4l} \prod_{k_2=l+1}^{s-1} a_{4k_2} \right)}{\left(\prod_{k_1=0}^{s-1} a_{4k_1+2} \right) + x_{-3} x_{-1} \sum_{l=0}^{s-1} \left(b_{4l+2} \prod_{k_2=l+1}^{s-1} a_{4k_2+2} \right)}, \tag{39}$$

$$x_{4n-4} = x_{-4}^{1-n} x_0^n \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} a_{4k_1+1} \right) + x_{-4} x_{-2} \sum_{l=0}^{s-1} \left(b_{4l+1} \prod_{k_2=l+1}^{s-1} a_{4k_2+1} \right)}{\left(\prod_{k_1=0}^{s-1} a_{4k_1+3} \right) + x_{-2} x_0 \sum_{l=0}^{s-1} \left(b_{4l+3} \prod_{k_2=l+1}^{s-1} a_{4k_2+3} \right)}, \tag{40}$$

$$x_{4n-3} = x_{-5}^n x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} a_{4k_1+2} \right) + x_{-3} x_{-1} \sum_{l=0}^{s-1} \left(b_{4l+2} \prod_{k_2=l+1}^{s-1} a_{4k_2+2} \right)}{\left(\prod_{k_1=0}^s a_{4k_1} \right) + x_{-5} x_{-3} \sum_{l=0}^s \left(b_{4l} \prod_{k_2=l+1}^s a_{4k_2} \right)} \tag{41}$$

and

$$x_{4n-2} = x_{-4}^n x_0^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} a_{4k_1+3} \right) + x_{-2} x_0 \sum_{l=0}^{s-1} \left(b_{4l+3} \prod_{k_2=l+1}^{s-1} a_{4k_2+3} \right)}{\left(\prod_{k_1=0}^s a_{4k_1+1} \right) + x_{-4} x_{-2} \sum_{l=0}^s \left(b_{4l+1} \prod_{k_2=l+1}^s a_{4k_2+1} \right)} \tag{42}$$

In the following sections, we specifically look at some special cases.

4 The case a_n , b_n are 1-periodic

Let $a_n = a$ and $b_n = b$, where a and b are non-zero constants.

4.1 Case: $a \neq 1$

We have

$$\begin{aligned}
 x_{4n-5} &= x_{-5}^{1-n} x_{-1}^n \prod_{s=0}^{n-1} \frac{a^s + bx_{-5}x_{-3} \frac{1-a^s}{1-a}}{a^s + bx_{-3}x_{-1} \frac{1-a^s}{1-a}}, \\
 x_{4n-4} &= x_{-4}^{1-n} x_0^n \prod_{s=0}^{n-1} \frac{a^s + bx_{-4}x_{-2} \frac{1-a^s}{1-a}}{a^s + bx_{-2}x_0 \frac{1-a^s}{1-a}}, \\
 x_{4n-3} &= x_{-5}^n x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{a^s + bx_{-3}x_{-1} \frac{1-a^s}{1-a}}{a^{s+1} + bx_{-5}x_{-3} \frac{1-a^{s+1}}{1-a}}, \\
 x_{4n-2} &= x_{-4}^n x_0^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{a^s + bx_{-2}x_0 \frac{1-a^s}{1-a}}{a^{s+1} + bx_{-4}x_{-2} \frac{1-a^{s+1}}{1-a}}
 \end{aligned}$$

as long as any of the denominators does not vanish.

Case: $a = -1$

We have

$$\begin{aligned}
 x_{4n-5} &= \begin{cases} x_{-5}^{1-n} x_{-1}^n \left(\frac{-1+bx_{-5}x_{-3}}{-1+bx_{-3}x_{-1}} \right)^{\lfloor \frac{n-1}{2} \rfloor}, & \text{if } n \text{ is odd;} \\ x_{-5}^{1-n} x_{-1}^n \left(\frac{-1+bx_{-5}x_{-3}}{-1+bx_{-3}x_{-1}} \right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases} \\
 x_{4n-4} &= \begin{cases} x_{-4}^{1-n} x_0^n \left(\frac{-1+bx_{-4}x_{-2}}{-1+bx_{-2}x_0} \right)^{\lfloor \frac{n-1}{2} \rfloor}, & \text{if } n \text{ is even;} \\ x_{-4}^{1-n} x_0^n \left(\frac{-1+bx_{-4}x_{-2}}{-1+bx_{-2}x_0} \right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \end{cases} \\
 x_{4n-3} &= \begin{cases} \frac{x_{-5}^n x_{-1}^{-n} x_{-3}}{-1+bx_{-5}x_{-3}} \left(\frac{-1+bx_{-3}x_{-1}}{-1+bx_{-5}x_{-3}} \right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \\ x_{-5}^n x_{-1}^{-n} x_{-3} \left(\frac{-1+bx_{-3}x_{-1}}{-1+bx_{-5}x_{-3}} \right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 x_{4n-2} &= \begin{cases} \frac{x_{-4}^n x_0^{-n} x_{-2}}{-1+bx_{-4}x_{-2}} \left(\frac{-1+bx_{-2}x_0}{-1+bx_{-4}x_{-2}} \right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \\ x_{-4}^n x_0^{-n} x_{-2} \left(\frac{-1+bx_{-2}x_0}{-1+bx_{-4}x_{-2}} \right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases}
 \end{aligned}$$

where $bx_{-i}x_{2-i} \neq 1$ for $i = 2, 3, 4, 5$.

4.2 Case: $a = 1$

The solution is given by

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^n \prod_{s=0}^{n-1} \frac{1 + bx_{-5}x_{-3}s}{1 + bx_{-3}x_{-1}s}, x_{4n-4} = x_{-4}^{1-n} x_0^n \prod_{s=0}^{n-1} \frac{1 + bx_{-4}x_{-2}s}{1 + bx_{-2}x_0s},$$

$$x_{4n-3} = x_{-5}^n x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{1 + bx_{-3}x_{-1}s}{1 + bx_{-5}x_{-3}(s+1)},$$

$$x_{4n-2} = x_{-4}^n x_0^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{1 + bx_{-2}x_0s}{1 + bx_{-4}x_{-2}(s+1)}.$$

5 The case a_n, b_n are 2-periodic

In this case, we have $\{a_n\}_{n=0}^\infty = a_0, a_1, a_0, a_1, \dots$, and similarly $\{b_n\}_{n=0}^\infty = b_0, b_1, b_0, b_1, \dots$ where $a_0 \neq a_1$, and $b_0 \neq b_1$. Then the solution is given by

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^n \prod_{s=0}^{n-1} \frac{a_0^s + b_0 x_{-5} x_{-3} \sum_{l=0}^{s-1} a_0^l}{a_0^s + b_0 x_{-3} x_{-1} \sum_{l=0}^{s-1} a_0^l},$$

$$x_{4n-4} = x_{-4}^{1-n} x_0^n \prod_{s=0}^{n-1} \frac{a_1^s + b_1 x_{-4} x_{-2} \sum_{l=0}^{s-1} a_1^l}{a_1^s + b_1 x_{-2} x_0 \sum_{l=0}^{s-1} a_1^l},$$

$$x_{4n-3} = x_{-5}^n x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{a_0^s + b_0 x_{-3} x_{-1} \sum_{l=0}^{s-1} a_0^l}{a_0^{s+1} + b_0 x_{-5} x_{-3} \sum_{l=0}^s a_0^l}$$

and

$$x_{4n-2} = x_{-4}^n x_0^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{a_1^s + b_1 x_{-2} x_0 \sum_{l=0}^{s-1} a_1^l}{a_1^{s+1} + b_1 x_{-4} x_{-2} \sum_{l=0}^s a_1^l}$$

as long as any of the denominators does not vanish.

6 The case a_n, b_n are 4-periodic

We assume that $\{a_n\} = a_0, a_1, a_2, a_3, a_0, a_1, a_2, a_3, \dots$ and $\{b_n\} = b_0, b_1, b_2, b_3, b_0, b_1, b_2, b_3, \dots$. The solution is given by

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^n \prod_{s=0}^{n-1} \frac{a_0^s + b_0 x_{-5} x_{-3} \sum_{l=0}^{s-1} a_0^l}{a_2^s + b_2 x_{-3} x_{-1} \sum_{l=0}^{s-1} a_2^l}, \quad (43)$$

$$x_{4n-4} = x_{-4}^{1-n} x_0^n \prod_{s=0}^{n-1} \frac{a_1^s + b_1 x_{-4} x_{-2} \sum_{l=0}^{s-1} a_1^l}{a_3^s + b_3 x_{-2} x_0 \sum_{l=0}^{s-1} a_3^l}, \quad (44)$$

$$x_{4n-3} = x_{-5}^n x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{a_2^s + b_2 x_{-3} x_{-1} \sum_{l=0}^{s-1} a_2^l}{a_0^{s+1} + b_0 x_{-5} x_{-3} \sum_{l=0}^s a_0^l} \quad (45)$$

and

$$x_{4n-2} = x_{-4}^n x_0^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{a_3^s + b_3 x_{-2} x_0 \sum_{l=0}^{s-1} a_3^l}{a_1^{s+1} + b_1 x_{-4} x_{-2} \sum_{l=0}^s a_1^l} \quad (46)$$

as long as any of the denominators does not vanish.

7 Conclusion

In this paper, non-trivial symmetries for difference equations of the form (1) were found. Consequently, the results were used to find formulas for the solutions of the equations. Specific cases of the solutions were also discussed.

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