Simulation and analysis of fractional model of Diffusion process and wave propagation via Caputo operator with Natural transform

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Abstract: This work focuses on the solution and analysis of the fractional model of Diffusion process and wave propagation. This model is used to study the diffusion process, anomalous diffusive system, wave propagation and various physical phenomena. The Natural transform decomposition method is applied for getting the numerical solution. This method perfectly combines the Natural transforms and an adomian polynomial based technique. The existence and uniqueness is analysed by the aid of the fixed point theorem. The accuracy of the presented method is shown by calculating errors and comparing the exact and approximate solution graphically.

Keywords: Fractional model of Diffusion process and wave propagation; Caputo operator; Natural transform decomposition method; Existence and Uniqueness analysis.

1. Introduction

In the recent years, the fractional calculus has been developed into a significant area of applied mathematics. When modeling real-world occurrences, fractional derivatives and fractional integrals yield better results than classical derivatives. The Caputo operator [1] is the most accurate derivative among all fractional derivatives. There are several intriguing applications for modeling physical processes, including dynamical systems, biology, chemistry, electronics, signal processing, viscoelasticity, and finance. Various researchers are working on advances of fractional calculus and studied various models like the nonlinear Zakharov–Kuznetsov equation with fractional order [2], the fractional cancer-tumor-immune systems [3], the coupled nonlinear Schrodinger equation [4], the fractional three dimensional Zakharov–Kuznetsov equation [5], the fractional systems of Impulse Control [6], the Nemati and Torres model the respiratory syncytial virus infection [7], the fractional lur'e systems [8], fractional systems of state dependent Delayed Impulses [9], the novel coronavirus transmission with optimal control [10], the networks model with time-varying delays [11], the acute and chronic hepatitis B with optimal control [12], the fractional model of diabetes with restraining and time delay [13], the fractional COVID-19 model [14], fractional control problem [15], fractional Cattaneo model [16], fractional Brain tumor model [17], fractional BBM-Burger equation [18], fractional model of oil pollution [19], heat conduction equation [20] and the time-fractional Tricomi equation [21].

For most scientists and academics, fractional calculus is an important field of study due to its exciting applications, and many professionals are interested in the analysis of fractional differential equations. However, finding exact solutions to fractional differential equations is difficult. It can therefore be handled numerically and through the use of approximation techniques. Numerous efficient techniques to solve the fractional differential equations have been investigated and encouraged such as homotopy analysis transform approach [22], operational matrix process [23], fractional differential transform technique [24], tanh-sech

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approach [25], residual power series technique [26], Sumudu generalized Kudryashov approach [27], discrete-time neural networks [28], novel cryptosystem [29], q-homotopy analysis transform method [30] and homotopy analysis Sumudu transform technique [31].

Fractional random walks may be explained, anomalous diffusive systems can be simulated, and the diffusion and wave propagation processes can be united thanks to the recent extensive use of anomalous diffusion model [32]. The basic diffusion and wave equation can be used to create the fractional diffusion-wave model [33]. Generally speaking, a wave equation represents a procedure in which a disturbance propagates at a constant speed, while a diffusion model represents a procedure in which a disruption spreads infinitely quickly. The fractional model of Diffusion process and wave propagation, in a way, interpolates between these two distinct behaviours in light of their responses to a localised disruption [34]. Schneider and Wyss [35] also provided a description of the diffusion and wave equations. Agrawal [36] used the infinite sine and Laplace transform to create a general solution described in a bounded domain. Next, he examined the fractional model of Diffusion process and wave propagation [37] in the presence of a nonhomogeneous environment that may be either deterministic or stochastic. Luchko, et al. [38] looked at the fundamental Cauchy problem solution for the fractional model of Diffusion process and wave propagation and identified the greatest location of the solution as well as other important characteristics. Luchko [39, 40] also analysed the given model with $\alpha, 1 \le \alpha \le 2,$ in space and in time.

We consider the fractional model of Diffusion process and wave propagation as given below [40]

$$\frac{1}{M}\,{}_{0}^{C}D_{t}^{\alpha}u(x,t) = \Delta u(x,t) + \frac{1}{K}f(x,t), \qquad (x,t) \in (0,L) \times (0,T], T > 0, \tag{1.1}$$

with initial condition $u(x,0) = h(x)$.

Where M and K are constants, ${}_{0}^{C}D_{t}^{\alpha}$ is the Caputo operator of order α $(1 < \alpha \le 2)$, $f(x,t)$ is source term and $h(x)$ is smooth function of x.

The motive of this research is to analyse and simulate the fractional model of Diffusion process and wave propagation via Caputo operator. The existence and uniqueness is analysed by aid of fixed point theorem. The Natural transform decomposition technique (NTDT) is used to obtain the numerical solution. It perfectly combines the Natural transforms (NT) and an adomian polynomial based technique. The main contribution of this research is to provide an effective numerical method for obtaining solution of the given model. The results of this work will be helpful in the study of wave propagation phenomena, anomalous diffusive systems, the diffusion process, and many other real-world phenomena. Also, two examples are solved for demonstrating the efficiency of the suggested technique NTDT.

2. Preliminaries

Definition 2.1. [1] The Caputo derivative of $y(t)$ is given by

$$D^{\alpha}y(t) = I^{m-\alpha}D^{m}y(t) = \frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}(t-f)^{m-\alpha-1}y^{m}(f)df,$$

where $m - 1 < \alpha \le m$.

Definition 2.2. [41] The NT of $y(t) > 0$ is given by

$$NT[y(t)] = \int_0^\infty e^{-st} y(vt)dt,\ \ s > 0, v > 0$$

Definition 2.3. [41] The inverse NT of $NT[y(t)]$ is $y(t)$ and is given by

$$NT^{-1}[NT[y(t)]] = y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{st}{v}} NT[y(t)]ds,\ \ s > 0, v > 0$$

Theorem 2.1. [1] The unique solution of ${}_0^cD_t^\alpha y(t) = e(t)$ is the given by

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t - f)^{\alpha-1} e(f)df,$$

here $0 < \alpha \leq 1$.

3. Existence and Uniqueness Analysis

The fractional model of Diffusion process and wave propagation (1.1) can be written in the form

$${}_0^cD_t^\alpha u(x,t) = \varphi(x,t,u), \tag{3.1}$$

where $$\varphi(x,t,u) = M\Delta u(x,t) + \frac{M}{K} f(x,t),$$

Using theorem (2.1), the equation (3.1) can be transform to the Voltera equation as:

$$u(x,t) - u(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}\varphi(x,t,u)ds. \tag{3.2}$$

Next, we have to prove that $\varphi(x,t,u)$ satisfy Lipschitz condition.

Theorem 3.1. The function, $\varphi(x,t,u)$ in the given Voltera equation satisfy the Lipschitz condition and contraction if $0 < \eta \leq 1$, where $\eta = M\delta^2$.

Proof. Let us assume that $u(x,t)$ is a bounded function. So, we have

$$\|\varphi(x,t,u) - \varphi(x,t,p)\|$$

$$= \|M\Delta u(x,t) + \frac{M}{K} f(x,t) - M\Delta p(x,t) - \frac{M}{K} f(x,t)\|,$$

$$= \|M\Delta\{u(x,t) - p(x,t)\}\|,$$

$$\leq M\delta^2\|u(x,t) - p(x,t)\|.$$

Now by letting $\eta = M\delta^2$, we get

$$\|\varphi(x,t,u) - \varphi(x,t,p)\| \leq \eta\|u(x,t) - p(x,t)\|. \tag{3.3}$$

Thus, $\varphi(x,t,u)$ satisfy the Lipschitz condition and contraction if $0 < \eta \leq 1$.

We take the following iterative formula for the existence of the solution

$$u_{n+1}(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}\varphi(x,t,u_n)ds, \tag{3.4}$$

with initial condition as $u(x,0) = u(x,t_0)$.

The two consecutive terms are differ by

$$\varphi_n(x,t) = u_n(x,t) - u_{n-1}(x,t),$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{\varphi(x,t,u_{n-1}) - \varphi(x,t,u_{n-2})\}ds. \quad (3.5)$$

It can be observed that

$$u_n(x,t) = \sum_{i=0}^{n} \varphi_i(x,t), \quad (3.6)$$

so, from equation (3.5), we have

$$\|\varphi_n(x,t)\| = \|u_n(x,t) - u_{n-1}(x,t)\|. \quad (3.7)$$

Applying triangular inequality on equation (3.4), we have

$$\|\varphi_n(x,t)\| \le \frac{1}{\Gamma(\alpha)}\eta \,\|\int_0^t (t-s)^{\alpha-1}\varphi_{n-1}(x,s)ds\|. \quad (3.8)$$

Theorem 3.2. The solution of the fractional model of Diffusion process and wave propagation exist if $\exists$, t_0, satisfying

$$\frac{1}{\Gamma(\alpha)}\eta t_0^\alpha \le 1.$$

Proof. Let $u(x,t)$ is a bounded function which satisfy the Lipschitz condition. Utilising equation (3.8), we get

$$\|\varphi_n(x,t)\| \le \|u_n(x,t)\| \left[\frac{1}{\Gamma(\alpha)}\eta . t^\alpha\right]^n. \quad (3.9)$$

Hence, the existence and continuousness of the obtained solution is proved.

$$u(x,t) - u(x,0) = u_n(x,t) - \chi_n(x,t). \quad (3.10)$$

Here, we consider that

$$\|\chi_n(x,t)\| = \|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{\varphi(x,t,u_n) - \varphi(x,t,u_{n-1})\}ds\|,$$

$$\le \frac{1}{\Gamma(\alpha)}\,\|\int_0^t (t-s)^{\alpha-1}\{\varphi(x,t,u_n) - \varphi(x,t,u_{n-1})\}ds\|,$$

$$\le \frac{1}{\Gamma(\alpha)}\,\eta\|u_n(x,t) - u_{n-1}(x,t)\|t.$$

In the same way at t_0, we obtain

$$\|\chi_n(x,t)\| \leq \left[\frac{1}{\Gamma(\alpha)} t_0^{\alpha}\right]^{n+1} \eta^{n+1} N, \tag{3.11}$$

as $n \to \infty$, we can clearly see that $\|\chi_n(x,t)\| \to 0$.

Theorem 3.3. The solution of the fractional model of Diffusion process and wave propagation is unique if the following condition satisfied

$$\left(1 - \frac{1}{\Gamma(\alpha)} \eta t^{\alpha}\right) > 0.$$

Proof. Suppose $u^*(x,t)$ is another solution of the fractional model of Diffusion process and wave propagation, then

$$\|u(x,t) - u^*(x,t)\| = \left\|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \{\varphi(x,t,u) - \varphi(x,t,u^*)\} ds\right\|,$$

$$\leq \frac{1}{\Gamma(\alpha)} \eta \|u(x,t) - u^*(x,t)\|. \tag{3.12}$$

Now, on simplifying above equation, we get

$$\|u(x,t) - u^*(x,t)\| \left(1 - \frac{1}{\Gamma(\alpha)} \eta t^{\alpha}\right) \leq 0,$$

hence, if

$$\left(1 - \frac{1}{\Gamma(\alpha)} \eta t^{\alpha}\right) > 0, \tag{3.15}$$

so, $u(x,t) = u^*(x,t)$.

Hence, the existence and uniqueness of the solution of the fractional model of Diffusion process and wave propagation is proved.

4. Proposed Numerical Technique

In this segment, we present the steps to obtain solution of the fractional model of Diffusion process and wave propagation by the proposed technique, NTDT.

The fractional model of Diffusion process and wave propagation is given by

$${}^{C}_{0}D^{\alpha}_{t} u(x,t) = \Delta u(x,t) + f(x,t), \qquad (x,t) \in (0,L) \times (0,T], T > 0, \tag{4.1}$$

with

$$u(x,0) = h(x), \tag{4.2}$$

taking NT of equation (4.1), we get

$$NT[{}^{c}_{0}D^{\alpha}_{t} u(x,t)] = NT[\Delta u(x,t) + f(x,t)].$$

Using differential property of NT and initial condition, we get

$$\frac{s^{\alpha}}{v^{\alpha}} NT[u(x,t)] - \frac{s^{\alpha-1}}{v^{\alpha}} h(x) = NT[\Delta u(x,t) + f(x,t)],$$

$$NT[u(x,t)] = \frac{1}{s} h(x) + \left(\frac{v}{s}\right)^{\alpha} NT[\Delta u(x,t) + f(x,t)],$$

taking inverse NT of above equation, we get

$$u(x,t) = h(x) + NT^{-1} \left\{\left(\frac{v}{s}\right)^{\alpha} NT[\Delta u(x,t) + f(x,t)]\right\}. \tag{4.3}$$

Applying NTDT to above equation. So, we put

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{4.4}$$

putting the values of $u(x,t)$ form equation (4.4) in equation (4.3), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = h(x) + pNT^{-1} \left\{\left(\frac{v}{s}\right)^{\alpha} NT\left[\sum_{n=0}^{\infty} p^n \Delta u_n(x,t) + f(x,t)\right]\right\}. \tag{4.5}$$

Comparing the coefficients of like power of p, we get

$$u_0(x,t) = h(x),$$

$$u_1(x,t) = NT^{-1} \left\{\left(\frac{v}{s}\right)^{\alpha} NT[\Delta u_0(x,t) + f(x,t)]\right\},$$

$$u_2(x,t) = NT^{-1} \left\{\left(\frac{v}{s}\right)^{\alpha} NT[\Delta u_1(x,t) + f(x,t)]\right\},$$

$$\vdots$$

$$u_n(x,t) = NT^{-1} \left\{\left(\frac{v}{s}\right)^{\alpha} NT[\Delta u_{n-1}(x,t) + f(x,t)]\right\},$$

The final solution is

$$u(x,t) = \lim_{k\to\infty} \sum_{n=0}^{k} u_n(x,t). \tag{4.6}$$

5. Simulation

In this segment, we apply NTDT for obtaining the solution of two examples of fractional model of Diffusion process and wave propagation. To prove the efficiency of the NTDT, absolute errors are calculated and the approximate solution is compared to the exact solution graphically.

Example 5.1. Let us assume the following fractional model of Diffusion process and wave propagation

$${}_{0}^{C}D_{t}^{\alpha}\, u(x,t) - \Delta u(x,t) = f(x,t), 1 < \alpha \leq 2, \quad (5.1)$$

with the source term $f(x,t) = e^{x}[\Gamma(\alpha+1) - t^{\alpha} - t - 1]$ and the exact solution of (5.1), for $\alpha = 2$, is $u(x,t) = e^{x}[t^{\alpha} + t + 1]$.

Solution. From the exact solution we have, $u(x,0) = e^{x}$. Applying NTDT to equation (5.1), so we get

$$u_0(x,t) = u(x,0) = e^{x}, \quad (5.2)$$

$$u_1(x,t) = NT^{-1}\{\frac{v^{\alpha}}{s^{\alpha}} NT[\Delta u_0(x,t) + f(x,t)]\}$$

$$= e^{x}[\frac{\Gamma(\alpha+1)t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}], \quad (5.3)$$

$$u_2(x,t) = NT^{-1}\{\frac{v^{\alpha}}{s^{\alpha}} NT[\Delta u_1(x,t) + f(x,t)]\}$$

$$= -e^{x}[\frac{\Gamma(\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (\Gamma(\alpha+1)+1)\frac{t^{\alpha}}{\Gamma(\alpha+1)}], \quad (5.4)$$

$$u_3(x,t) = NT^{-1}\{\frac{v^{\alpha}}{s^{\alpha}} NT[\Delta u_2(x,t) + f(x,t)]\}$$

$$= -e^{x}[\frac{\Gamma(\alpha+1)t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (1-\Gamma(\alpha+1))\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (2\Gamma(\alpha+1)+1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}], \quad (5.5)$$

and the final solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \quad (5.6)$$

Table 1. Absolute error (AE) at $t = 0.001$, and different values of α, for Ex. 5.1.

x	**AE at $\alpha = 1.90$**	**AE at $\alpha = 1.95$**	**AE at $\alpha = 2$**
0.1	$1.10758e^{-03}$	$1.10680e^{-03}$	$1.10627e^{-03}$
0.2	$1.22407e^{-03}$	$1.22320e^{-03}$	$1.22262e^{-03}$
0.3	$1.35280e^{-03}$	$1.35185e^{-03}$	$1.35120e^{-03}$
0.4	$1.49508e^{-03}$	$1.49403e^{-03}$	$1.49331e^{-03}$
0.5	$1.65232e^{-03}$	$1.65116e^{-03}$	$1.65037e^{-03}$
0.6	$1.82609e^{-03}$	$1.82481e^{-03}$	$1.82394e^{-03}$
0.7	$2.01815e^{-03}$	$2.01673e^{-03}$	$2.01576e^{-03}$
0.8	$2.23040e^{-03}$	$2.22883e^{-03}$	$2.22776e^{-03}$
0.9	$2.46497e^{-03}$	$2.46324e^{-03}$	$2.46206e^{-03}$
1.0	$2.72422e^{-03}$	$2.72230e^{-03}$	$2.72100e^{-03}$

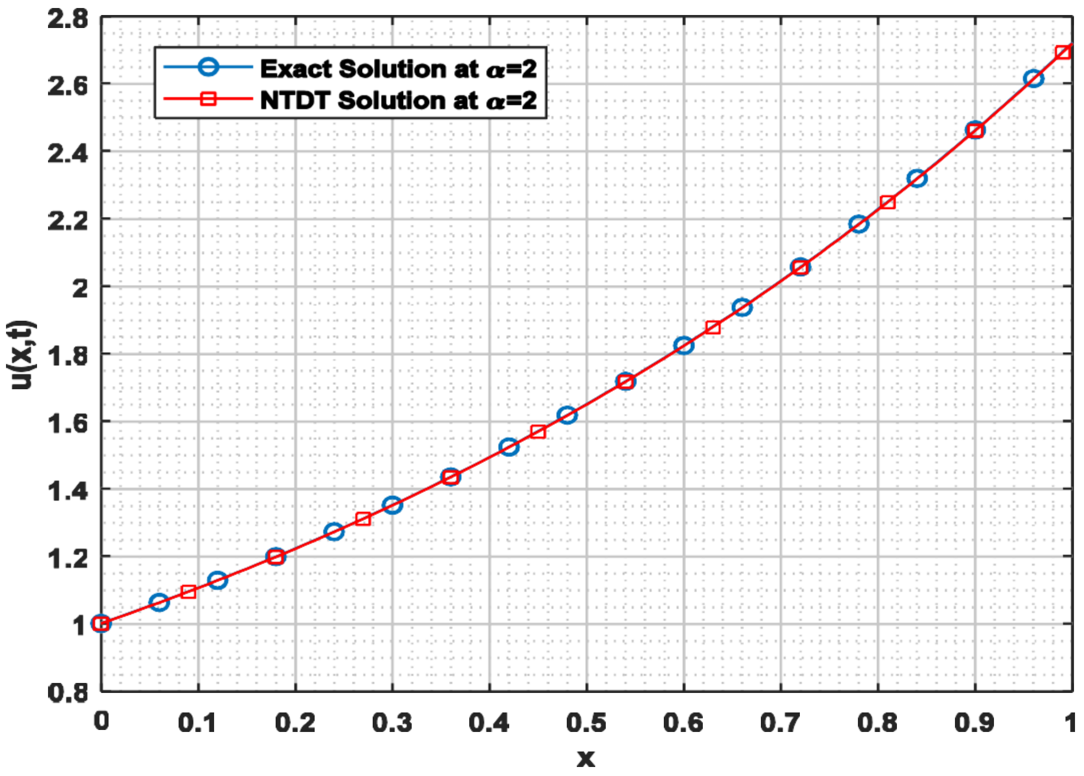

Figure 1. The exact and NTDT solutions at $t = 0.01$, for Ex. 5.1.

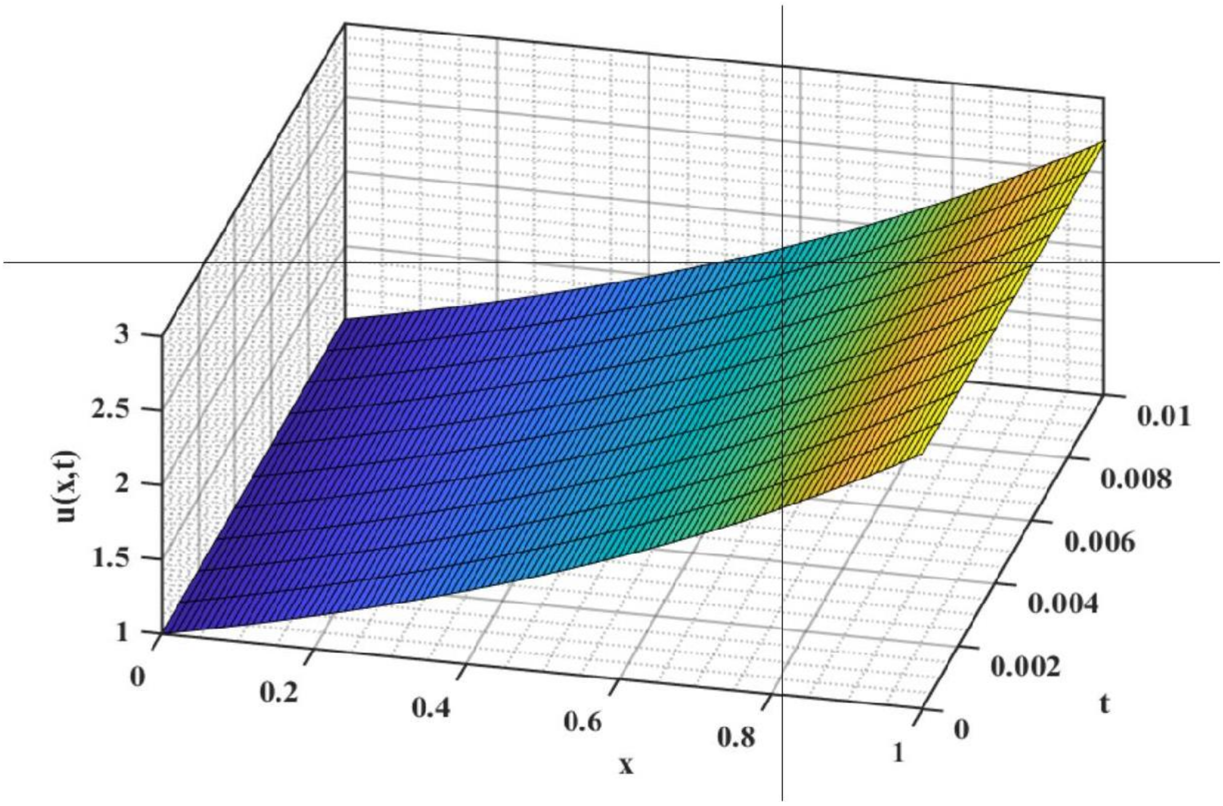

Figure 2. The NTDT solution for $\alpha = 2$, for Ex. 5.2.

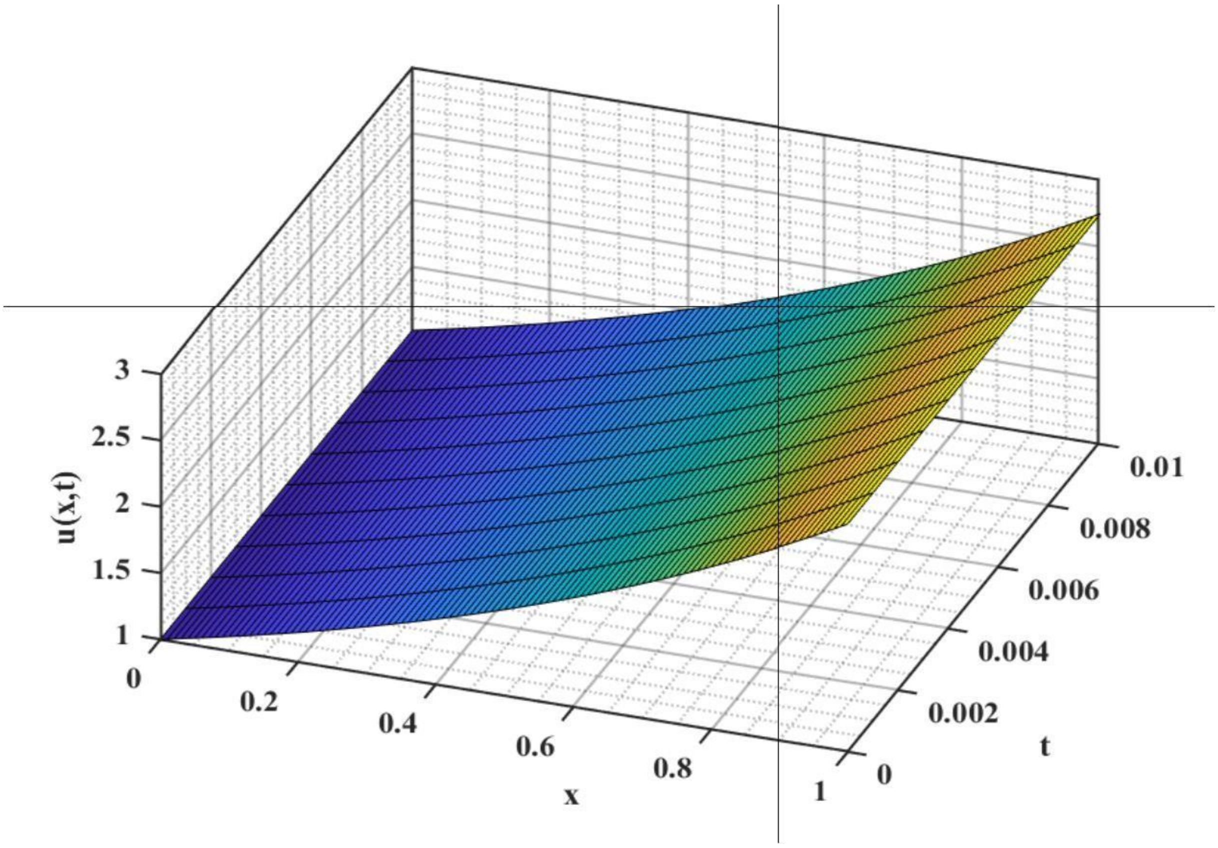

Figure 3. The exact solution for $\alpha = 2$, for Ex.

Example 5.2. Let us assume the following fractional model of Diffusion process and wave propagation

$${}_{0}^{C}D_{t}^{\alpha} u(x,t) - \Delta u(x,t) = f(x,t), 1 < \alpha \leq 2, \tag{5.7}$$

with the source term $f(x,t) = sin(\pi x)\, t^{\frac{5}{2}} \left(\frac{t^{-\alpha}\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-\alpha)} - \pi^2\right)$ and the exact solution of (5.7), for $\alpha = 2$, is $u(x,t) = sin(\pi x)\, t^{5/2}$.

Solution. From the exact solution we have, $u(x,0) = 0$. Applying NTDT to equation (5.7), so we get

$$u_0(x,t) = u(x,0) = 0, \tag{5.8}$$

$$u_1(x,t) = NT^{-1}\{\frac{v^{\alpha}}{s^{\alpha}} LT[\Delta u_0(x,t) + f(x,t)]\}$$

$$= \Gamma\left(\frac{7}{2}\right) sin(\pi x)\left[\frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)} - \frac{\pi^2 t^{\alpha+5/2}}{\Gamma(\alpha + 7/2)}\right], \tag{5.9}$$

$$u_2(x,t) = NT^{-1}\{\frac{v^{\alpha}}{s^{\alpha}} NT[\Delta u_1(x,t) + f(x,t)]\}$$

$$= \Gamma\left(\frac{7}{2}\right) sin(\pi x)\left[\frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)} - \frac{2\pi^2 t^{\alpha+\frac{5}{2}}}{\Gamma\left(\alpha + \frac{7}{2}\right)} + \frac{\pi^4 t^{2\alpha+5/2}}{\Gamma(2\alpha + 7/2)}\right], \tag{5.10}$$

$$u_3(x,t) = NT^{-1}\{\frac{v^\alpha}{s^\alpha} NT[\Delta u_2(x,t) + f(x,t)]\}$$

$$= \Gamma\left(\frac{7}{2}\right) sin(\pi x)\left[\frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)} - \frac{2\pi^2 t^{\alpha+\frac{5}{2}}}{\Gamma\left(\alpha+\frac{7}{2}\right)} + \frac{2\pi^4 t^{2\alpha+5/2}}{\Gamma(2\alpha+7/2)} - \frac{\pi^6 t^{3\alpha+\frac{5}{2}}}{\Gamma\left(3\alpha+\frac{7}{2}\right)}\right], \quad (5.11)$$

and the final solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \quad (5.12)$$

Table 2. Absolute error (AE) at $x = 15$, and various values of α, for Ex. 6.2.

t	AE at $\alpha = 1.90$	AE at $\alpha = 1.95$	AE at $\alpha = 2$
0.01	$1.2565\ e^{-23}$	$9.2152\ e^{-24}$	$6.7550\ e^{-24}$
0.02	$2.6532\ e^{-22}$	$2.0144\ e^{-22}$	$1.5286\ e^{-22}$
0.03	$1.5801\ e^{-21}$	$1.2241\ e^{-21}$	$9.4791\ e^{-22}$
0.04	$5.6046\ e^{-21}$	$4.4046\ e^{-21}$	$3.4600\ e^{-21}$
0.05	$1.4967\ e^{-20}$	$1.1893\ e^{-20}$	$9.4466\ e^{-21}$
0.06	$3.3399\ e^{-20}$	$2.6780\ e^{-20}$	$2.1465\ e^{-20}$
0.07	$6.5847\ e^{-20}$	$5.3202\ e^{-20}$	$4.2968\ e^{-20}$
0.08	$1.1857\ e^{-19}$	$9.6430\ e^{-20}$	$7.8394\ e^{-19}$
0.09	$1.9923\ e^{-19}$	$1.6297\ e^{-19}$	$1.3325\ e^{-19}$
0.10	$3.1699\ e^{-19}$	$2.6061\ e^{-19}$	$2.1420\ e^{-19}$

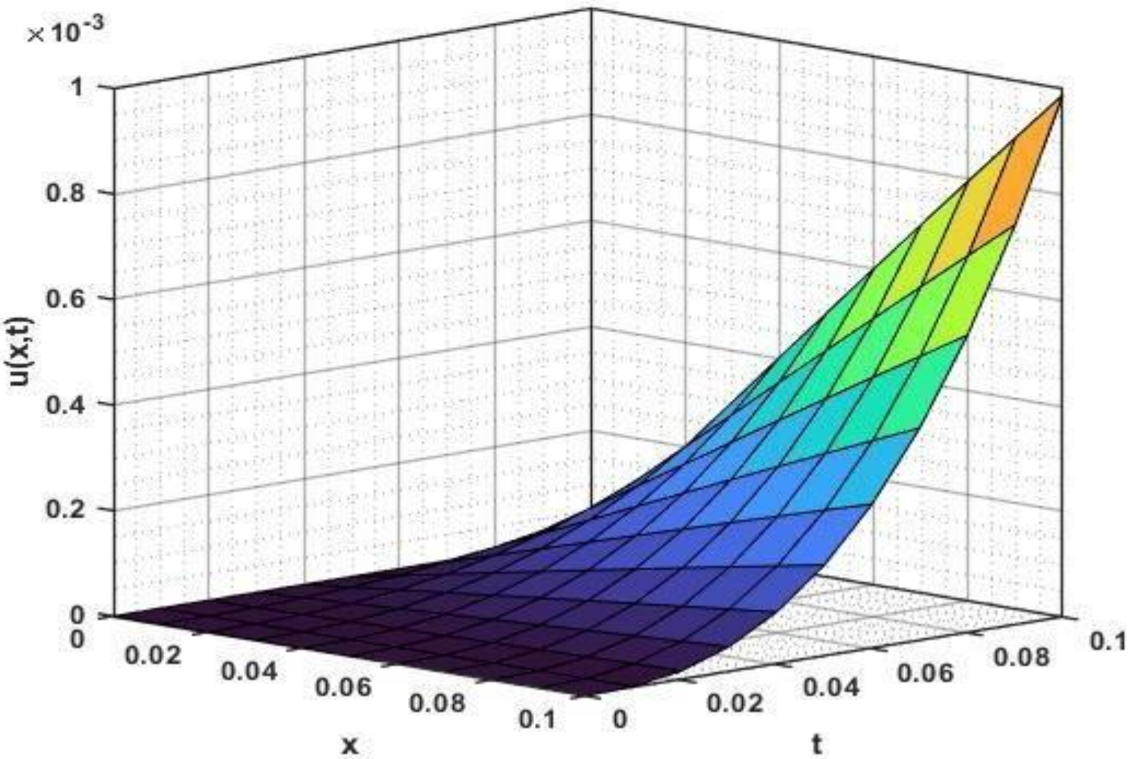

Figure 4. The NTDT solution for $\alpha = 2$, for Ex. 5.2.

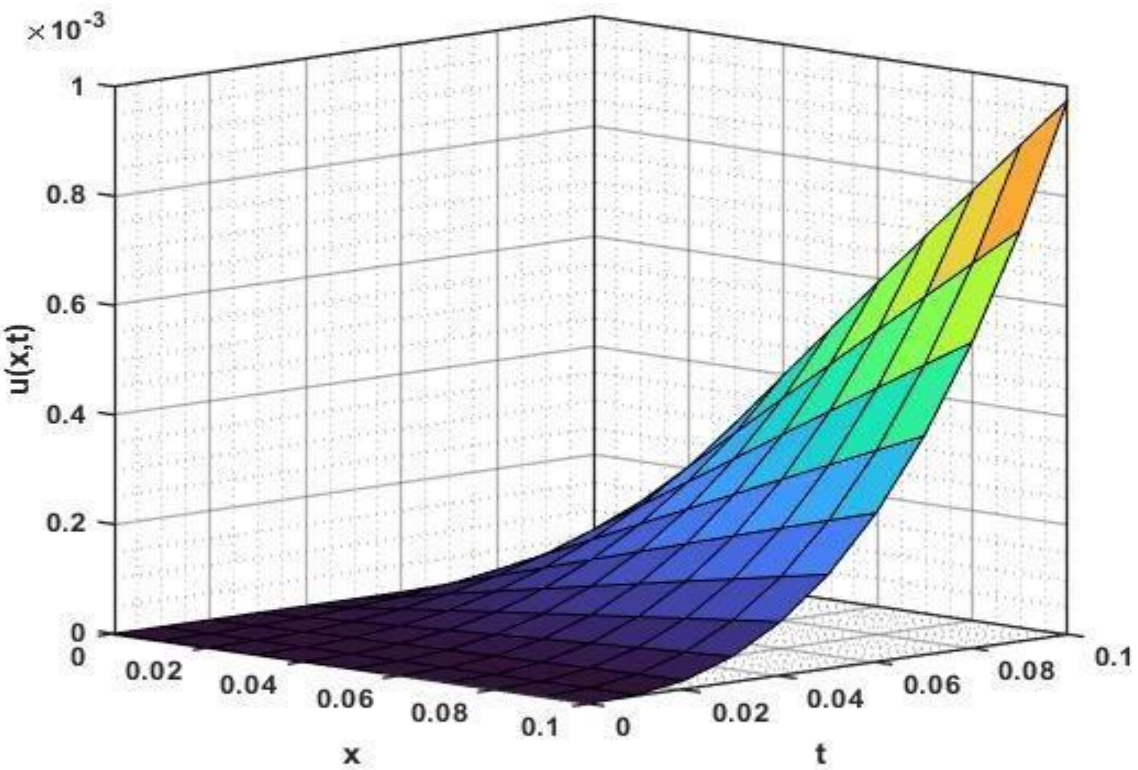

Figure 5. The exact solution for $\alpha = 2$, for Ex.

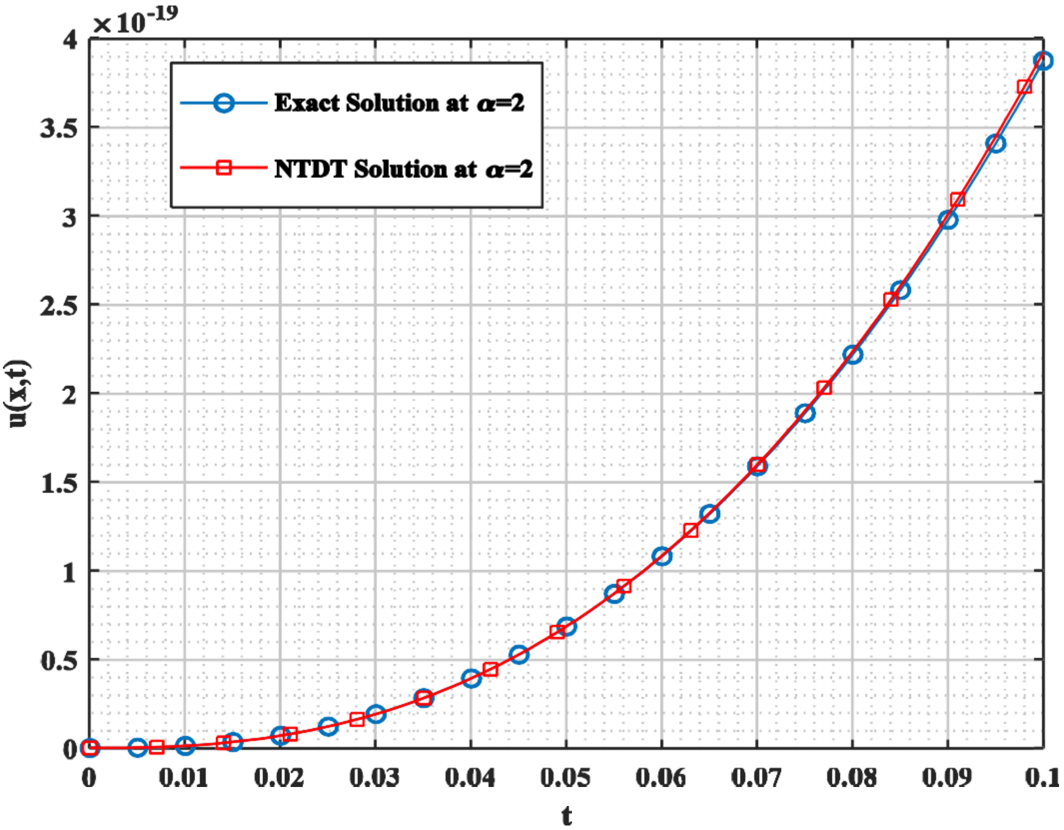

Figure 6. The exact and NTDT solutions for $x = 1$, for Ex. 5.2.

6. RESULT DISCUSSION

The fractional model of Diffusion process and wave propagation is investigated in this article via Natural transform. The NTDT is used to get the numerical solution of fractional model of Diffusion process and wave propagation. The absolute error is calculated at $t = 0.001$ and different values of α and the results are given in table 1, for Ex. 5.1. The absolute errors are computed at $x = 15$ and various values of α, for Ex. 5.2 and given in table 2. From, the tables it noticed that the NTDT is very accurate as errors are very less for the proposed technique NTDT. In Fig. 1, the 2-D graphs of exact and NTDT solution at $t = 0.01$ are presented for Ex. 5.1. Here, we can clearly see that both solutions overlap each other. The 3-D graphs of exact and NTDT solutions shown through Fig. $(2 - 3)$ and Fig. $(4 - 5)$ at $\alpha = 2$, for Ex. 5.1 and Ex. 5.2, respectively. It is observed that both solution surface shows exactly same behaviour. Fig. 6 demonstrate the two dimensional graph of exact and approximate solutions at $x = 1$ and again they overlap each other. Hence, from the results of tables and graphs it can be said that the NTDT is an effective technique to solve the fractional models.

7. CONCLUSION

The fractional model of Diffusion process and wave propagation is investigated in this work by a semi analytical technique. The NTDT, which is combination of Natural transform and Adomian decomposition technique, is used to solve the fractional model of Diffusion process and wave propagation. The existence and uniqueness of the solution is investigated by aid of fixed point theorem. We solve two different examples, and the numerical results show that the suggested technique, NTDT, provides a solution that converges to the correct answer with relatively little absolute error. Studying anomalous diffusive systems, diffusion theory, wave propagation phenomena, and a variety of real-world phenomena will be greatly aided by this work.

Thus, it can be said that the suggested technique, NTDT, is a very effective method that manages non-linearity and other constraints with ease and is applicable to a variety of real life physical processes.

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