# Generated Fuzzy Ideals in Semirings

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**Abstract.** This study offers a comprehensive characterization of fuzzy left ideals, fuzzy right ideals, fuzzy ideals, fuzzy bi-ideals, fuzzy interior ideals, and fuzzy quasi-ideals within the context of semirings. These fuzzy structures are analyzed using level subsets and strong level subsets. Additionally, we introduce the generated fuzzy left ideal, fuzzy right ideal, fuzzy ideal, fuzzy bi-ideal, fuzzy interior ideal, and fuzzy quasi-ideals by a fuzzy set in semirings, with or without a multiplicative identity. Furthermore, the work presents an expression for generating fuzzy left ideal, fuzzy right ideal, fuzzy ideal, fuzzy bi-ideal, fuzzy interior ideal, and fuzzy quasi-ideals in terms of left ideal, right ideal, ideal, bi-ideal, interior ideal, and quasi-ideal generated by level subsets and strong level subsets of the given fuzzy set.
**Keywords:** Semirings; arbitrary intersection; level subsets; strong level subsets; generated fuzzy ideals.

## 1. Introduction

A semiring is an algebraic structure that generalizes notions of both rings and distributive lattices, a concept first introduced by Vandiver in 1934 [14]. Semirings occupy a structural position that bridges rings and semigroups, integrating properties from both frameworks. Iseki [4] introduced the notion of quasi-ideals within the framework of semirings, while Ahsan and Saifullah [1] explored the theory of fuzzy semirings. Donges [2] contributed several foundational statements regarding quasi-ideals in semirings. Neggers [10] provided a characterization of L-fuzzy ideals in semirings, while Mandal [6, 7] conducted detailed investigations into fuzzy ideals, fuzzy bi-ideals, fuzzy interior ideals, and fuzzy quasi-ideals in ordered semirings. Shabi et.al. [12] investigated quasi-deals in 2004 while Munir and Mustafa [8] further characterized semirings through the study of bi-ideals and quasi-ideals in 2016.

Srivastava and Sarma [13] defined the concept of fuzzy quasi-ideals in semirings and derived point wise definition of fuzzy quasi-ideal. More recently, Munir and Shafiq [9] extended the theory to define generated left, right, and two-sided ideals in semirings, both with and without a multiplicative identity.

In this study, we investigate the generated fuzzy ideals including left, right, two-sided, and interior ideals along with fuzzy bi-ideals and quasi-ideals, which are generated by a fuzzy set within a semiring structure. These findings mirror their classical algebraic equivalents. We provide a comprehensive analysis of these fuzzy ideals through their level subsets and strong level subsets, offering explicit formulations of these generated fuzzy ideals, following the framework established by R. Kumar [5]. In Section 3, we delve into precise characterizations of fuzzy (left, right, two-sided, interior) ideals, as well as fuzzy bi-ideals and quasi-ideals, utilizing level and strong level subsets as key tools. Section 4 focuses on the generated fuzzy ideals within semirings that possess a multiplicative identity, whereas Section 5 extends the scope of the analysis to include semirings that lack such an identity.

## 2. Preliminaries

A semiring $S$ is a non-empty set $S$ equipped with two binary operations '+' and '$\circ$' such that $(S,+)$ and $(S,\ \circ)$ forms semigourps. Additionally, these operations are related by the distributive property, i.e., $a.(b+c)=a.b+a.c\ \forall\ a,b,c\in S$. A semiring may also possess a multiplicative identity, 1, characterized by $a\circ 1=1\circ a=a$. A non-empty subset $A$ of a semirings is termed as subsemiring if $A$ is closed under addition and multiplication. A non-empty subset $I$ of a semiring is called a left (right) ideal of $S$ if $I$ forms a subsemigroup of $S$ under addition and for $a\in I$ and $s\in S, sa(as)\in I$. An ideal of $S$ is a set that is both a left ideal and a right ideal. A subsemigroup $B$ of semiring $S$ is termed a bi-ideal (interior) ideal of $S$ if $BSB(SBS)\subseteq B$. Similarly, a subset $Q$ of semiring $S$ is called a quasi-ideal of $S$ if Q is a subsemigroup under addition and $QS\cap SQ\subseteq Q$.

We recall the definition of fuzzy set given by Zadeh [15]. A fuzzy set $\gamma$ on a non-set $X$ is defined to be a mapping $\gamma: X\to[0,1]$.

We recall the following definitions for subsequent use.

**Definition 2.1 ([13]).** A fuzzy set $\gamma$ of a semiring $S$ is called a fuzzy subsemiring of $S$ if $\gamma(x+y)\geq\min\{\gamma(x),\gamma(y)\}$ and $\gamma(xy)\geq\min\{\gamma(x),\gamma(y)\}\ \forall\ x,y\in S$.

**Definition 2.2 ([13]).** A fuzzy set $\gamma$ of a semiring $S$ is called a fuzzy left (right) ideal of $S$ if $\gamma(x+y)\geq\min\{\gamma(x),\gamma(y)\}$ and $\gamma(xy)\geq\gamma(y)(\gamma(xy)\geq\gamma(x))$.

**Definition 2.3 ([13]).** A fuzzy set $\gamma$ of a semiring S is called a fuzzy bi-ideal of $S$ if $\gamma(x+y) \geq \min\{\gamma(x), \gamma(y)\}, \gamma(xy) \geq \min\{\gamma(x), \gamma(y)\}$ and $\gamma(xyz) \geq \min\{\gamma(x), \gamma(z)\}\ \forall\ x, y, z \in S$.

**Definition 2.4 ([13]).** A fuzzy set $\gamma$ of a semiring S is called a fuzzy interior ideal of $S$ if $\gamma(x+y) \geq \min\{\gamma(x), \gamma(y)\}, \gamma(xy) \geq \min\{\gamma(x), \gamma(y)\}$ and $\gamma(xyz) \geq \gamma(y)\ \forall\ x, y, z \in S$.

**Definition 2.5 ([13]).** A fuzzy set $\gamma$ of a semiring $S$ ia called a fuzzy quasi-ideal of $S$ if $\gamma(x+y) \geq \min\{\gamma(x), \gamma(y)\}$ and $\gamma(z) \geq \min\{\sup_{z=xy} \gamma(x), \sup_{z=xy} \gamma(y)\}\ \forall z \in S$.

The generated fuzzy left (right, two-sided, interior, bi-ideal, quasi-) ideal within a semiring is ensured by the principle that the non-empty intersection of any collection of these respective fuzzy structures results in the same type of fuzzy structure. Specifically, if one takes an arbitrary family of fuzzy left ideals, their intersection will yield another fuzzy left ideal; similarly, this holds true for fuzzy right (two-sided, interior, bi-ideals, quasi-) ideal. This property has been rigorously established in the work of Mandal $[6, 7]$, who demonstrated that in ordered semirings, the non-empty intersection of any collection of fuzzy left (right, two-sided, interior, bi-ideal, quasi-) ideals will also produce a structure of the same kind. We define the similar result for semirings as:

**Lemma 2.6.** The intersection of any non-empty collection of fuzzy left ideals, fuzzy right ideals, fuzzy ideals, fuzzy bi-ideals, and fuzzy interior ideals in a semiring is itself a fuzzy left ideal, fuzzy right ideal, fuzzy ideal, fuzzy bi-ideal, and fuzzy interior ideal.

**Lemma 2.7 ([13]).** The non-empty intersection of an arbitrary family of fuzzy quasi-ideals of a semiring is a fuzzy quasi-ideal.

## 3. Level subsets and strong level subsets of a fuzzy set in a semiring

**Definition 3.1.** The level subset $\gamma_r$, and strong level subset $\gamma_r^>$ of a fuzzy set $\gamma$ on a non-empty set X are defined as

$$\gamma_r = \{x \in X \mid \gamma(x) \geq r\} \text{ and } \gamma_r^> = \{x \in X \mid \gamma(x) > r\}.$$

**Lemma 3.2.** Let $\gamma, \lambda \in F(S)$. Then, $\forall\ t \in [0,1)$

(1) $(\gamma \circ \lambda)_t^> = \gamma_t^> \lambda_t^>$.

(2) $(\gamma \cap \lambda)_t^> = \gamma_t^> \cap \lambda_t^>$.

(3) $(\gamma \cup \lambda)_t^> = \gamma_t^> \cup \lambda_t^>$.

**Theorem 3.3.** Following assertions are equivalent in a semiring $S$ :

(1) $\gamma$ is a fuzzy left ideal of $S$.
(2) Each $\emptyset \neq \gamma_t$ is a left ideal of $S$.
(3) Each $\emptyset \neq \gamma_t^{>}$is a left ideal of $S$.

**Proof.** To established $(\mathbf{1}) \Rightarrow (3)$, let $\gamma$ be a fuzzy left ideal of $S$. We claim that $S\gamma_t^{>} \subseteq \gamma_t^{>}$. Let $x \in S\gamma_t^{>}$. Then $x = sy$ for some $s \in S$ and $y \in \gamma_t^{>}$. Now $y \in \gamma_t^{>}$ implies that $\gamma(y) > t$. Also $\gamma$ is a fuzzy left ideal, therefore, $\gamma(sy) \geq \gamma(y) > t$. Thus $x = sy \in \gamma_t^{>}$. Hence $S\gamma_t^{>} \subseteq \gamma_t^{>}$.
$(\mathbf{3}) \Rightarrow (\mathbf{2})$. Let $\emptyset \neq \gamma_t$ be a level subset of $f$. Then, Then, $\gamma_t = \bigcap_{r<t} \gamma_r^{>}$ . Since $\emptyset \neq \gamma_t$, $\emptyset \neq \gamma_r^{>}$ for each $r < t$. By (3) $\gamma_r^{>}$is a left ideal of $S$. As the non-empty intersection of an arbitrary family of left ideals of $S$ is a left ideal of $S, \gamma_t$ is a left ideal.
$(\mathbf{2}) \Rightarrow (\mathbf{1})$. Assume that $\gamma$ is not a fuzzy left ideal of $S$. Since $\gamma_t$ is a left ideal of $S$, it is a subsemiring and hence $\gamma$ is a fuzzy subsemiring of $S$. Now $\gamma(sy) < \gamma(y)$ some $s, y \in S$ since $\gamma$ is not a fuzzy left ideal of $S$. That is $\gamma(z) < \gamma(y)$ where $z = sy$. Let $t$ be a real number such that $\gamma(z) < t < \gamma(y)$. Then $z \notin \gamma_t$ and $y \in \gamma_t$. Therefore, $z = sy \in S\gamma_t$, but $z \notin \gamma_t$. Hence $S\gamma_t \nsubseteq \gamma_t$, which is a contradicts as $\gamma_t$ is a left ideal of by (2).

Similarly, we can prove:
**Theorem 3.4.** Following assertions are equivalent in a semiring $S$ :
(1) $\gamma$ is a fuzzy rihgt (two sided) ideal of $S$.
(2) Each $\emptyset \neq \gamma_t$ is a right (two sided) ideal of $S$.
(3) Each $\emptyset \neq \gamma_t^{>}$is a right (two sided) ideal of $S$.

**Theorem 3.5.** Following assertions are equivalent in a semiring $S$ :
(1) $\gamma$ is a fuzzy interior ideal of $S$.
(2) Each $\emptyset \neq \gamma_t$ is an interior ideal of $S$.
(3) Each $\emptyset \neq \gamma_t^{>}$is an interior ideal of $S$.

**Proof.** $(\mathbf{1}) \Rightarrow (\mathbf{3})$. Let $\gamma$ be a fuzzy interior ideal of $S$. To establish that $\gamma_t^{>}$ is an interior ideal of $S$. That is to show that $S\gamma_t^{>}S \subseteq \gamma_t^{>}$. Let $x \in S\gamma_t^{>}S$. Then $x = s_1 y s_2$ for some $s_1, s_2 \in S$ and $y \in \gamma_t^{>}$. Now $y \in \gamma_t^{>}$implies $\gamma(y) > t$. Since $\gamma$ is a fuzzy interior ideal, $\gamma(s_1 y s_2) \geq \gamma(y) > t$. Thus $x = s_1 y s_2 \in \gamma_t^{>}$. Hence $S\gamma_t^{>}S \subseteq \gamma_t^{>}$.

$(\mathbf{3}) \Rightarrow (\mathbf{2})$. Follows similar to Theorem 3.3.

Now to establish $(\mathbf{2}) \Rightarrow (\mathbf{1})$, let $\gamma$ is not a fuzzy interior ideal of $S$. Since $\gamma_t$ is an interior ideal of $S$, it is a subsemiring and hence $\gamma$ is a fuzzy subsemiring of $S. \gamma(s_1 y s_2) < \gamma(y)$ some $s_1, s_2, y \in S$ as we have assumed that $\gamma$ is not a fuzzy interior ideal of $S$. This implies $\gamma(z) < \gamma(y)$ where $z = s_1 y s_2$. Select a real number $t$ such that $\gamma(z) < t < \gamma(y)$. Then $z \notin \gamma_t$ and $\gamma \in \gamma_t$. Therefore, $z = s_1 y s_2 \in S\gamma_t S$, but $z \notin \gamma_t$. Hence $S\gamma_t S \nsubseteq \gamma_t$, which is a contradicts as $\gamma_t$ is an interior ideal by (2).

**Theorem 3.6.** Following assertions are equivalent in a semiring $S$ :

(1) $\gamma$ is a fuzzy bi-ideal of $S$.

(2) Each $\emptyset \neq \gamma_t$ is a bi-ideal of $S$.

(3) Each $\emptyset \neq \gamma_t^{>}$is a bi-ideal of $S$.

**Proof.** $(\mathbf{1}) \Rightarrow (\mathbf{3})$. Let $\gamma$ be a bi-ideal of $S$. To establish that $\gamma_t^{>}$is a bi-ideal of $S$.That is to show that $\gamma_t^{>} S \gamma_t^{>} \subseteq \gamma_t^{>}$. Let $x \in \gamma_t^{>} S \gamma_t^{>}$. Then $x = y_1 s y_2$ for some $s \in S$ and $y_1, y_2 \in \gamma_t^{>}$. Now $y_1, y_2 \in \gamma_t^{>}$implies $\gamma(y_1) > t < \gamma(y_2)$. Since $\gamma$ is a fuzzy bi-ideal, $\gamma(y_1, y_2) \geq \min\{\gamma(y_1), \gamma(y_2)\} > t$. Thus $x = y_1 s y_2 \in \gamma_t^{>}$. Hence $\gamma_t^{>} S \gamma_t^{>} \subseteq \gamma_t^{>}$.

$(\mathbf{3}) \Rightarrow (\mathbf{2})$. Follows similar to Theorem 3.3.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$. Since $\gamma_t$ is a bi-ideal of $S$, it is a subsemiring and hence $\gamma$ is a fuzzy subsemiring of $S$. Let $\gamma$ is not a fuzzy bi-ideal of $S$, therefore $\gamma(x'z'y') < \min\{\gamma(x'), \gamma(y')\}$ for some $x', y', z' \in S$. This implies $\gamma(k) < \gamma(x')$ and $\gamma(k) < \gamma(y')$, where $k = x'z'y'$. Select a real number $t$ such that $\gamma(k) < t < \min\{\gamma(x'), \gamma(y')\}$. Then $k \notin \gamma_t$ and $x', y' \in \gamma_t$. Therefore, $k = x'z'y' \in \gamma_t S \gamma_t$, but $k \notin \gamma_t$. Hence $\gamma_t S \gamma_t \nsubseteq \gamma_t$, which is a contradicts as $\gamma_t$ is a bi-ideal by (2).

**Theorem 3.7 ([13]).** Following assertions are equivalent in a semiring $S$ :

(1) $\gamma$ is a fuzzy quasi-ideal of $S$.

(2) Each $\emptyset \neq \gamma_t$ is a quasi-ideal of $S$.

(3) Each $\emptyset \neq \gamma_t^{>}$is a quasi-ideal of $S$.

## 4. Generated fuzzy ideals in a semiring with multiplicative identity

**Definition 4.1.** For a fuzzy set γ of a semiring S, we define:

(1) the smallest fuzzy left ideal generated by γ is $\langle \overset{l}{\gamma} \rangle = \bigcap_{\mu \in FL(S)} \{\mu \mid \gamma \subseteq \mu\}$

(2) the smallest fuzzy right ideal generated by γ is $\langle \overset{r}{\gamma} \rangle = \bigcap_{\mu \in FR(S)} \{\mu \mid \gamma \subseteq \mu\}$

(3) the smallest fuzzy two sided ideal generated by γ is $\langle \overset{t}{\gamma} \rangle = \bigcap_{\mu \in FT(S)} \{\mu \mid \gamma \subseteq \mu\}$

(4) the smallest fuzzy interior ideal generated by γ is $\langle \overset{i}{\gamma} \rangle = \bigcap_{\mu \in FI(S)} \{\mu \mid \gamma \subseteq \mu\}$

(5) the smallest fuzzy bi-ideal generated by γ is $\langle \overset{b}{\gamma} \rangle = \bigcap_{\mu \in FB(S)} \{\mu \mid \gamma \subseteq \mu\}$

(6) the smallest fuzzy quasi-ideal generated by γ is $\langle \overset{q}{\gamma} \rangle = \bigcap_{\mu \in FQ(S)} \{\mu \mid \gamma \subseteq \mu\}$

Where $FL(S)(FT(S))$ is the set of all fuzzy left (two sided) ideals of $S$ etc.

The following results are well known in classical algebra in a semiring with multiplicative identity

**Lemma 4.2.** If A is a non-empty subset of a semiring $S$ with multiplicative identity, then

$$\overset{l}{\langle A\rangle} = SA \qquad \overset{r}{\langle A\rangle} = AS \qquad \overset{t}{\langle A\rangle} = SAS$$
$$\overset{r}{\langle A\rangle} = A^2 \cup SAS \qquad \overset{q}{\langle A\rangle} = A \cup ASA \qquad \overset{q}{\langle A\rangle} = AS \cap SA$$

To obtain a similar result in the fuzzy setting, we proceed in the following way. While doing this, we obtain several expressions for a generated fuzzy ideal in the process.

**Definition 4.3.** For a fuzzy set $\gamma$ of a semiring S, we define the fuzzy sets $\overset{l}{\gamma}$, $\overset{ll}{\gamma}$, $\overset{r}{\gamma}$, $\overset{rr}{\gamma}$, $\overset{t}{\gamma}$, $\overset{tt}{\gamma}$, $\overset{i}{\gamma}$, $\overset{ii}{\gamma}$, $\overset{b}{\gamma}$, $\overset{bb}{\gamma}$, $\overset{q}{\gamma}$ and $\overset{qq}{\gamma}$ on $S$ as follows:

$$\overset{l}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{l}{\gamma_t}\right\rangle\right\} \qquad \overset{ll}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{l}{\gamma_t^{>}}\right\rangle\right\}$$

$$\overset{r}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{r}{\gamma_t}\right\rangle\right\} \qquad \overset{rr}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{r}{\gamma_t^{>}}\right\rangle\right\}$$

$$\overset{t}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{t}{\gamma_t}\right\rangle\right\} \qquad \overset{tt}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{t}{\gamma_t^{>}}\right\rangle\right\}$$

$$\overset{i}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{i}{\gamma_t}\right\rangle\right\} \qquad \overset{ii}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{i}{\gamma_t^{>}}\right\rangle\right\}$$

$$\overset{b}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{b}{\gamma_t}\right\rangle\right\} \qquad \overset{bb}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{b}{\gamma_t^{>}}\right\rangle\right\}$$

$$\overset{q}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{q}{\gamma_t}\right\rangle\right\} \qquad \overset{qq}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{q}{\gamma_t^{>}}\right\rangle\right\}$$

where $\left\langle \overset{b}{\gamma_t}\right\rangle \left(\left\langle \overset{b}{\gamma_t^{>}}\right\rangle\right)$ is a bi-ideal generated by level (strong level) subset $\gamma_t(\gamma_t^{>})$.

**Lemma 4.4.** Let $\gamma$ be a fuzzy set in a semiring $S$ with multiplicative identity and $t \in [0,1)$. Then

(i) $\left(\overset{rr}{\gamma}\right)_t^{>} = \left\langle \overset{r}{\gamma_t^{>}}\right\rangle$ (ii) $\left(\overset{ll}{\gamma}\right)_t^{>} = \left\langle \overset{l}{\gamma_t^{>}}\right\rangle$ (iii) $\left(\overset{tt}{\gamma}\right)_t^{>} = \left\langle \overset{t}{\gamma_t^{>}}\right\rangle$

(iv) $\left(\overset{ii}{\gamma}\right)_t^{>} = \left\langle \overset{i}{\gamma_t^{>}}\right\rangle$ (v) $\left(\overset{bb}{\gamma}\right)_t^{>} = \left\langle \overset{b}{\gamma_t^{>}}\right\rangle$ (vi) $\left(\overset{qq}{\gamma}\right)_t^{>} = \left\langle \overset{q}{\gamma_t^{>}}\right\rangle$

**Proof.** (i) Consider $z \in \left(\overset{rr}{\gamma}\right)_t^>$. Therefore, $\overset{rr}{\gamma}(z) > t$. This implies that $\sup\left\{t_i \mid z \in \left\langle \overset{r}{\gamma_{t_i}^>}\right\rangle\right\} > t$. Consequently, there exists $t_0 > t$ and $z \in \gamma_{t_0}^> S \subseteq \gamma_t^> S$. Hence $z \in \left\langle \overset{r}{\gamma_t^>}\right\rangle$.

Conversely, let $z \in \left\langle \overset{r}{\gamma_t^>}\right\rangle$. Then $z \in \gamma_t^> S$. Therefore $z = as$ for some $a \in \gamma_t^>$ and $s \in S$. Now $a \in \gamma_t^>$ implies $\gamma(a) > t$. Choosing $t_1$ such that $\gamma(a) > t_1 > t$, we get $a \in \gamma_{t_1}^>$. Then $z = as \in \gamma_{t_1}^> S$. Therefore $z = as \in \gamma_{t_1}^> S \subseteq \langle \gamma_{t_1}^> \rangle$, where $t_1 > t$. Thus, $z \in \left\langle \overset{r}{\gamma_{t_i}^>}\right\rangle$ for some $t_i > t$ and as a result, we get, $\sup\left\{t_i \mid z \in \left\langle \overset{r}{\gamma_{t_i}^>}\right\rangle\right\} > t$. That is $z \in \left(\overset{rr}{\gamma}\right)_t^>$.

(ii) and (iii) follows similarly.

(iv) Consider $z \in \left(\overset{ii}{\gamma}\right)_t^>$. Therefore $\overset{ii}{\gamma}(z) > t$. This implies that $\sup\left\{t_i \mid z \in \left\langle \overset{r}{\gamma_{t_i}^>}\right\rangle\right\} > t$. Consequently, there exists $t_0 > t$ and $z \in \gamma_{t_0}^> \gamma_{t_0}^> \cup S\gamma_{t_0}^> S \subseteq \gamma_t^> \gamma_t^> \cup S\gamma_t^> S$. Hence $z \in \left\langle \overset{r}{\gamma_t^>}\right\rangle$.

Conversely, let $z \in \left\langle \overset{r}{\gamma_t^>}\right\rangle$. Then $z \in \gamma_t^> \gamma_t^> \cup S\gamma_t^> S$. Therefore, either $z \in \gamma_t^> \gamma_t^>$ or $z \in S\gamma_t^> S$.

If $z \in \gamma_t^> \gamma_t^>$, then $z = a_1 a_2$ for some $a_1, a_2 \in \gamma_t^>$. Therefore, $\gamma(a_1) > t < \gamma(a_2)$. Choosing $t_1, t_2$ such that $\gamma(a_1) > t_1 > t < t_2 < \gamma(a_2)$, we get $a_1 \in \gamma_{t_1}^>$ and $a_2 \in \gamma_{t_2}^>$. Write $t_3 = \min\{t_1, t_2\}$. Then $a_1, a_2 \in \gamma_{t_3}^>$. Therefore, $z = a_1 a_2 \in \gamma_{t_3}^> \gamma_{t_3}^> \subseteq \left\langle \overset{i}{\gamma_{t_3}^>}\right\rangle$, where $t_3 > t$.

If $z \in S\gamma_t^> S$, then $z = s_1\ a\ s_2$ for some $a \in \gamma_t^>$ and $s_1, s_2 \in S$. Now $a \in \gamma_t^>$ implies $\gamma(a) > t$. Choosing $t_0$ such that $\gamma(a) > t_0 > t$. Then $z \in \gamma_{t_0}^> \subseteq \left\langle \overset{i}{\gamma_{t_0}^>}\right\rangle$. Therefore $z = s_1\ a\ s_2 \in S\gamma_{t_0}^> S \subseteq \left\langle \overset{i}{\gamma_{t_0}^>}\right\rangle$, where $t_0 > t$. Hence, under all circumstance, we have, $z \in \left\langle \overset{i}{\gamma_{t_i}^>}\right\rangle$ for some $t_i > t$ and as a result, we get, $\sup\left\{t_i \mid z \in \left\langle \overset{i}{\gamma_{t_i}^>}\right\rangle\right\} > t$. That is $z \in \left(\overset{ii}{\gamma}\right)_t^>$.

(v) Consider $z \in \left(\overset{bb}{\gamma}\right)_t^{>}$. Therefore $\overset{bb}{\gamma}(z) > t$. This implies that $\sup\{t_i \mid z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle\} > t$. Consequently, there exists $t_0 > t$ and $z \in \gamma_{t_0}^{>} \cup \gamma_{t_0}^{>} S \gamma_{t_0}^{>} \subseteq \gamma_t^{>} \cup \gamma_t^{>} S \gamma_t^{>}$. Hence, $z \in \left\langle \overset{b}{\gamma_t^{>}} \right\rangle$.

Conversely, let $z \in \left\langle \overset{b}{\gamma_t^{>}} \right\rangle$. Then $z \in \gamma_t^{>} \cup \gamma_t^{>} S \gamma_t^{>}$. Therefore, either $z \in \gamma_t^{>}$or $z \in \gamma_t^{>} S \gamma_t^{>}$. Suppose $z \in \gamma_t^{>}$. Then $\gamma(z) > t$. Select $t_0$ such that $\gamma(z) > t_0 > t$. Then $z \in \gamma_{t_0}^{>} \subseteq \left\langle \overset{b}{\gamma_{t_0}^{>}} \right\rangle$, where $t_0 > t$. Hence, under all circumstance, we have, $z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle$ for some $t_i > t$ and as a result, we get, $\sup\left\{t_i \mid z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. That is $z \in \left(\overset{bb}{\gamma}\right)_t^{>}$.

If $z \in \gamma_t^{>} S \gamma_t^{>}$, then $z = a_1\ sa_2$ for some $a_1, a_2 \in \gamma_t^{>}$and $s \in S$. Choosing $t_1, t_2$ such that $\gamma(a_1) > t_1 > t < t_2 < \gamma(a_2)$, we get, $a_1 \in \gamma_{t_1}^{>}$and $a_2 \in \gamma_{t_2}^{>}$. Write $t_3 = \min\{t_1, t_2\}$. Therefore, $z = s_1$ a $s_2 \in \gamma_{t_3}^{>} S \gamma_{t_3}^{>} \subseteq \left\langle \overset{b}{\gamma_{t_3}^{>}} \right\rangle$, where $t_3 > t$.

Hence, under all circumstances we have, $z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle$ for some $t_i > t$ and as a result, we get, $\sup\left\{t_i \mid z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. That is $z \in \left(\overset{bb}{\gamma}\right)_t^{>}$.

(vi) Consider $z \in \left(\overset{qq}{\gamma}\right)_t^{>}$. Therefore $\overset{qq}{\gamma}(z) > t$. This implies that $\sup\left\{t_i \mid z \in \left\langle \overset{q}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. Consequently, there exists $t_0 > t$ and $z \in \gamma_{t_0}^{>} S \cap S \gamma_{t_0}^{>} \subseteq \gamma_t^{>} S \cap S \gamma_t^{>}$. Hence, $z \in \left\langle \overset{q}{\gamma_t^{>}} \right\rangle$.

Conversely, let $z \in \left\langle \overset{q}{\gamma_t^{>}} \right\rangle$. Then $z \in \gamma_t^{>} S \cap S \gamma_t^{>}$. Thus, $z = a_1 s_1 = s_2 a_2$ for some $a_1, a_2 \in \gamma_t^{>}$ and $s_1, s_2 \in S$. Choosing $t_1, t_2$ such that $\gamma(a_1) > t_1 > t < t_2 < \gamma(a_2)$, we get, $a_1 \in \gamma_{t_1}^{>}$ and $a_2 \in \gamma_{t_2}^{>}$. Write $t_3 = \min\{t_1, t_2\}$. Then $a_1, a_2 \in \gamma_{t_3}^{>}$.

Therefore, $z = a_1 s_1 = s_2 a_2 \in \gamma^{>}_{t_3} S \cap S\gamma^{>}_{t_3} S \subseteq \left\langle \overset{q}{\gamma^{>}_{t_3}} \right\rangle$, where $t_3 > t$. Thus, $z \in \left\langle \overset{q}{\gamma^{>}_{t_i}} \right\rangle$ for some $t_i > t$ and as a result $\sup\left\{ t_i \mid z \in \left\langle \overset{q}{\gamma^{>}_{i}} \right\rangle \right\} > t$. That is $z \in \left(\overset{qq}{\gamma}\right)^{>}_{t}$.

**Theorem 4.5.** Let $\gamma$ be a fuzzy set in a semiring $S$ with multiplicative identity. Then, $\forall\ t \in [0,1)$, we have,

(i) $\left\langle \overset{l}{\gamma} \right\rangle = \overset{ll}{\gamma}$ (ii) $\left\langle \overset{r}{\gamma} \right\rangle = \overset{rr}{\gamma}$ (iii) $\left\langle \overset{t}{\gamma} \right\rangle = \overset{tt}{\gamma}$

(iv) $\left\langle \overset{i}{\gamma} \right\rangle = \overset{ii}{\gamma}$ (v) $\left\langle \overset{b}{\gamma} \right\rangle = \overset{bb}{\gamma}$ (vi) $\left\langle \overset{q}{\gamma} \right\rangle = \overset{qq}{\gamma}$

**Proof.** We establish (iv), others follow similarly. Since $\left\langle \overset{i}{\gamma^{>}_{t}} \right\rangle$ is the interior ideal generated by $\gamma^{>}_{t}$ and $\left(\overset{ii}{\gamma}\right)^{>}_{t} = \left\langle \overset{i}{\gamma^{>}_{t}} \right\rangle\ \forall\ t \in [0,1)$ by Lemma 4.4, therefore $\left(\overset{ii}{\gamma}\right)^{>}_{t}$ is an interior ideal of $S\ \forall\ t \in [0,1)$. Consequently, $\overset{ii}{\gamma}$ is a fuzzy interior ideal of $S$ by Theorem 3.5. Now we establish that $\overset{ii}{\gamma}$ is the fuzzy interior ideal of $S$ generated by $\gamma$. To achieve this, we show that $\gamma \subseteq \overset{ii}{\gamma}$. On contrary, let $\gamma \nsubseteq \overset{ii}{\gamma}$. Then for some $z_0 \in S,\ \gamma(z_0) > \overset{ii}{\gamma}(z_0) = t_0$ (say). Thus, $z_0 \in \gamma^{>}_{t_0} \subseteq \left\langle \overset{i}{\gamma^{>}_{t_0}} \right\rangle = \left(\overset{ii}{\gamma}\right)^{>}_{t_0}$ by Lemma 4.4. This implies $\overset{ii}{\gamma}(z_0) > t_0$, which is a contradiction. Finally, we show that $\overset{ii}{\gamma}$ is the smallest fuzzy interior ideal of $S$ which containing $\gamma$. Let $\delta$ be fuzzy interior ideal of $S$ containing $\gamma$. Then, $\gamma^{>}_{t} \subseteq \delta^{>}_{t}$. Thus $\left\langle \overset{i}{\gamma^{>}_{t}} \right\rangle \subseteq \delta^{>}_{t}$. Therefore, $\left(\overset{ii}{\gamma}\right)^{>}_{t} \subseteq \delta^{>}_{t}$ for all $t \in [0,1)$. Hence $\overset{ii}{\gamma} \subseteq \delta$.

**Lemma 4.6.** For a fuzzy set $\gamma$ in a semiring $S$ with multiplicative identity. Then, we have,

(i) $\overset{ll}{\gamma} = \overset{l}{\gamma}$ (ii) $\overset{rr}{\gamma} = \overset{r}{\gamma}$ (iii) $\overset{tt}{\gamma} = \overset{t}{\gamma}$

(iv) $\overset{ii}{\gamma} = \overset{i}{\gamma}$ (v) $\overset{bb}{\gamma} = \overset{b}{\gamma}$ (vi) $\overset{qq}{\gamma} = \overset{q}{\gamma}$

**Proof.** We establish (iv) i.e. $\overset{ii}{\gamma} = \overset{i}{\gamma}$, others follow similarly. Clearly by definition, $\overset{ii}{\gamma} \subseteq \overset{i}{\gamma}$. We begin by demonstrating that $\left(\overset{i}{\gamma}\right)^{>}_{t} \subseteq \left\langle \overset{i}{\gamma^{>}_{t}} \right\rangle\ \forall\, t \in [0,1)$ to establish the reverse inclusion. Now consider $t \in [0,1)$ and $x \in \left(\overset{i}{\gamma}\right)_t^{>}$. This implies that $\sup\left\{t_i \mid x \in \left\langle \overset{i}{\gamma_{t_i}} \right\rangle\right\} > t$. Therefore, there exists $t_0 > t$ such that $x \in \left\langle \overset{i}{\gamma_{t_0}^{>}} \right\rangle$. Now $\gamma_{t_0} \subseteq \gamma_t^{>}$ implies $\left\langle \overset{i}{\gamma_{t_0}} \right\rangle \subseteq \left\langle \overset{i}{\gamma_{t_0}^{>}} \right\rangle$. Thus $x \in \left\langle \overset{i}{\gamma_{t_0}} \right\rangle \subseteq \left\langle \overset{i}{\gamma_t^{>}} \right\rangle$. Hence $\left(\overset{i}{\gamma}\right)_t^{>} \subseteq \left\langle \overset{i}{\gamma_t^{>}} \right\rangle$, $\forall t \in \left[0,1\right[$. Also $\left\langle \overset{i}{\gamma_t^{>}} \right\rangle = \left(\overset{ii}{\gamma}\right)_t^{>}$ follows from Lemma 4.4. Therefore, $\forall t \in [0,1)$, $\left(\overset{i}{\gamma}\right)_t^{>} \subseteq \left(\overset{ii}{\gamma}\right)_t^{>}$. Consequently, $\overset{i}{\gamma} \subseteq \overset{ii}{\gamma}$, leading to the conclusion that $\overset{ii}{\gamma} = \overset{i}{\gamma}$.

The subsequent theorem is a direct consequence of Theorem 4.5 when combined with Lemma 4.6.

**Theorem 4.7.** For a fuzzy set $\gamma$ in a semiring $S$ with multiplicative identity. Then,

(i) $\left\langle \overset{l}{\gamma} \right\rangle(z) = \sup\left\{t \mid z \in \left\langle \overset{l}{\gamma_t^{>}} \right\rangle\right\}$ (ii) $\left\langle \overset{r}{\gamma} \right\rangle(z) = \sup\left\{t \mid z \in \left\langle \overset{r}{\gamma_t^{>}} \right\rangle\right\}$

(iii) $\left\langle \overset{t}{\gamma} \right\rangle(z) = \sup\left\{t \mid z \in \left\langle \overset{t}{\gamma_t^{>}} \right\rangle\right\}$ (iv) $\left\langle \overset{i}{\gamma} \right\rangle(z) = \sup\left\{t \mid z \in \left\langle \overset{i}{\gamma_t^{>}} \right\rangle\right\}$

(v) $\left\langle \overset{b}{\gamma} \right\rangle(z) = \sup\left\{t \mid z \in \left\langle \overset{b}{\gamma_t^{>}} \right\rangle\right\}$ (vi) $\left\langle \overset{q}{\gamma} \right\rangle(z) = \sup\left\{t \mid z \in \left\langle \overset{q}{\gamma_t^{>}} \right\rangle\right\}$

**Lemma 4.8.** For a fuzzy set $\gamma$ in a semiring $S$ with multiplicative identity and $\forall t \in [0,1)$,

(i) $\left\langle \overset{l}{\gamma} \right\rangle_t^{>} = \left\langle \overset{l}{\gamma_t^{>}} \right\rangle$ (ii) $\left\langle \overset{r}{\gamma} \right\rangle_t^{>} = \left\langle \overset{r}{\gamma_t^{>}} \right\rangle$ (iii) $\left\langle \overset{t}{\gamma} \right\rangle_t^{>} = \left\langle \overset{t}{\gamma_t^{>}} \right\rangle$

(iv) $\left\langle \overset{i}{\gamma} \right\rangle_t^{>} = \left\langle \overset{i}{\gamma_t^{>}} \right\rangle$ (v) $\left\langle \overset{b}{\gamma} \right\rangle_t^{>} = \left\langle \overset{b}{\gamma_t^{>}} \right\rangle$ (vi) ) $\left\langle \overset{q}{\gamma} \right\rangle_t^{>} = \left\langle \overset{q}{\gamma_t^{>}} \right\rangle$

**Proof.** We establish (iv), others follow similarly. Let $t \in [0,1)$. By Lemma 4.6 and 4.4, $\overset{i}{\gamma} = \overset{ii}{\gamma}$ and $\left(\overset{ii}{\gamma}\right)_t^{>} = \left\langle \overset{i}{\gamma_t^{>}} \right\rangle$, therefore, $\left(\overset{i}{\gamma}\right)_t^{>} = \left\langle \overset{i}{\gamma_t^{>}} \right\rangle$. By Definition 4.3, $\overset{i}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{i}{\gamma_t^{>}} \right\rangle\right\}$ and $\overset{ii}{\gamma}(z) = \sup\left\{t \mid z \in \left\langle \overset{i}{\gamma_t^{>}} \right\rangle\right\}$. Also, by Theorem 4.7, $\left\langle \overset{i}{\gamma} \right\rangle(z) =$

$\sup\left\{t \mid z \in \left\langle \overset{i}{\gamma_t^>} \right\rangle\right\}$. Thus $\overset{ii}{\gamma} = \left\langle \overset{i}{\gamma} \right\rangle$. Therefore, by Lemma 4.6, $\overset{i}{\gamma} = \left\langle \overset{i}{\gamma} \right\rangle$. This implies $\left(\overset{i}{\gamma}\right)_t^> = \left\langle \overset{i}{\gamma} \right\rangle_t^>$. But $\left(\overset{i}{\gamma}\right)_t^> = \left\langle \overset{i}{\gamma_t^>} \right\rangle$. Thus $\left\langle \overset{i}{\gamma} \right\rangle_t^> = \left\langle \overset{i}{\gamma_t^>} \right\rangle$.

**Theorem 4.9.** For a fuzzy set $\gamma$ in a semiring $S$ with multiplicative identity, we have

(i) $\left\langle \overset{l}{\gamma} \right\rangle = S \circ \gamma$.

(ii) $\left\langle \overset{r}{\gamma} \right\rangle = \gamma \circ S$.

(iii) $\left\langle \overset{t}{\gamma} \right\rangle = S \circ \gamma \circ S$.

(iv) $\left\langle \overset{i}{\gamma} \right\rangle = \gamma \circ \gamma \cup S \circ \gamma \circ S$.

(v) $\left\langle \overset{b}{\gamma} \right\rangle = \gamma \cup \gamma \circ S \circ \gamma$.

(vi) $\left\langle \overset{q}{\gamma} \right\rangle = \gamma \circ S \cap S \circ \gamma$.

**Proof.** (i) Let $\overset{l}{\alpha} = S \circ \gamma$. Then, for any $t \in [0,1[$,

$$\begin{aligned}\left(\overset{l}{\alpha}\right)_t^> &= (S \circ \gamma)_t^> \\ &= Sf_t^> \quad \text{by Lemma 3.2} \\ &= \left\langle \overset{l}{\gamma_t^>} \right\rangle \\ &= \left\langle \overset{l}{\gamma} \right\rangle_t^> \quad \text{by Lemma 4.8}\end{aligned}$$

Thus $\overset{l}{\alpha} = \langle \overset{l}{\gamma} \rangle$.

(ii) and (iii) follows similarly.

(iv) Let $\overset{i}{\alpha} = \gamma \circ \gamma \cup S \circ \gamma \circ S$. Then, for any $t \in [0,1[$,

$$\begin{aligned}\left(\overset{i}{\alpha}\right)_t^> &= (\gamma \circ \gamma \cup S \circ \gamma \circ S)_t^> \\ &= (\gamma \circ \gamma)_t^> \cup (S \circ \gamma \circ S)_t^> \quad \text{by Lemma 3.2} \\ &= \gamma_t^> \gamma_t^> \cup S\gamma_t^> S \quad \text{by Lemma 3.2} \\ &= \left\langle \overset{i}{\gamma_t^>} \right\rangle \\ &= \left\langle \overset{i}{\gamma} \right\rangle_t^> \quad \text{by Lemma 4.8}\end{aligned}$$

Thus $\overset{i}{\alpha} = \langle \overset{i}{\gamma} \rangle$.

(v) Let $\overset{b}{\alpha} = \gamma \cup \gamma \circ S \circ \gamma$. Then, for any $t \in [0,1[$,

$$\left(\overset{b}{\alpha}\right)_t^{>} = (\gamma \cup \gamma \circ S \circ \gamma)_t^{>}$$
$$= \gamma_t^{>} \cup (\gamma \circ S \circ \gamma)_t^{>} \quad \text{by Lemma 3.2}$$
$$= \gamma_t^{>} \cup \gamma_t^{>} S \gamma_t^{>} \quad \text{by Lemma 3.2}$$
$$= \left\langle \overset{b}{\gamma_t^{>}} \right\rangle$$
$$= \left\langle \overset{b}{\gamma} \right\rangle_t^{>} \quad \text{by Lemma 4.8}$$

Thus $\overset{b}{\alpha} = \langle \overset{b}{\gamma} \rangle$.

(vi) Let $\overset{q}{\alpha} = \gamma \circ S \cap S \circ \gamma$. Then, for any $t \in [0,1[$,

$$\left(\overset{q}{\alpha}\right)_t^{>} = (\gamma \circ S \cap S \circ \gamma)_t^{>}$$
$$= (\gamma \circ S)_t^{>} \cap (S \circ \gamma)_t^{>} \quad \text{by Lemma 3.2}$$
$$= \gamma_t^{>} S \cap S \gamma_t^{>} \text{by Lemma 3.2}$$
$$= \left\langle \overset{q}{\gamma_t^{>}} \right\rangle$$
$$= \left\langle \overset{q}{\gamma} \right\rangle_t^{>} \text{ by Lemma 4.8.}$$

Thus $\overset{q}{\alpha} = \left\langle \overset{b}{\gamma} \right\rangle$.

## 5 Generated fuzzy ideals in a semiring without multiplicative identity

The following results are well known in classical algebra in a semiring without multiplicative identity:

**Lemma 5.1.** Let A be a non-empty subset of a semiring S without multiplicative identity. Then

(1) $\langle \overset{l}{A} \rangle = A \cup SA$ (2) $\langle \overset{r}{A} \rangle = A \cup AS$

(3) $\left\langle \overset{t}{A} \right\rangle = A \cup SA \cup AS \cup SAS$ (4) $\left\langle \overset{i}{A} \right\rangle = A \cup A^2 \cup SAS$

(5) $\langle \overset{b}{A} \rangle = A \cup A^2 \cup ASA$ (6) $\langle \overset{q}{A} \rangle = (A \cup SA) \cap (A \cup AS)$

**Lemma 5.2.** For a fuzzy set $\gamma$ in a semiring $S$ without multiplicative identity and $t \in [0,1)$, we have

(i) $\left(\overset{ll}{\gamma}\right)_t^{>} = \left\langle \overset{l}{\gamma_t^{>}} \right\rangle$ (ii) $\left(\overset{rr}{\gamma}\right)_t^{>} = \left\langle \overset{r}{\gamma_t^{>}} \right\rangle$ (iii) $\left(\overset{tt}{\gamma}\right)_t^{>} = \left\langle \overset{t}{\gamma_t^{>}} \right\rangle$

(iv) $\left(\overset{ii}{\gamma}\right)_t^{>} = \left\langle \overset{i}{\gamma_t^{>}} \right\rangle$ (v) $\left(\overset{bb}{\gamma}\right)_t^{>} = \left\langle \overset{b}{\gamma_t^{>}} \right\rangle$ (vi) $\left(\overset{qq}{\gamma}\right)_t^{>} = \left\langle \overset{q}{\gamma_t^{>}} \right\rangle$

Proof. (i) Consider $z \in \left(\overset{ll}{\gamma}\right)_t$. Therefore $\overset{ll}{\gamma}(z) > t$. This implies that $\sup\left\{t_i \mid z \in \left\langle \overset{l}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. Consequently, there exists $t_0 > t$ and $z \in \gamma_{t_0}^{>} \cup S\gamma_{t_0}^{>} \subseteq \gamma_t^{>} \cup S\gamma_t^{>}$. Hence $z \in \left\langle \overset{l}{\gamma_t^{>}} \right\rangle$.

Conversely, let $z \in \left\langle \overset{l}{\gamma_t^{>}} \right\rangle$. Then $z \in \gamma_t^{>} \cup S\gamma_t^{>}$. Therefore, either $z \in \gamma_t^{>}$ or $z \in S\gamma_t^{>}$.

Suppose $z \in \gamma_t^{>}$. Then $\gamma(z) > t$. Choose $t_0$ such that $\gamma(z) > t_0 > t$. Then $z \in \gamma_{t_0} \subseteq \left\langle \overset{l}{\gamma_{t_0}^{>}} \right\rangle$ for some $t_0 > t$. Thus, $z \in \left\langle \overset{l}{\gamma_{t_i}^{>}} \right\rangle$ for some $t_i > t$ and as a result, we get, $\sup\left\{t_i \mid z \in \left\langle \overset{l}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. That is $z \in \left(\overset{ll}{\gamma}\right)_t^{>}$.

Suppose $z \in S\gamma_t^{>}$, then $z = sa$ for some $a \in \gamma_t^{>}$ and $s \in S$. Choosing $t_1$ such that $\gamma(a) > t_1 > t$, we get $a \in \gamma_{t_1}$. Then $z = sa \in S\gamma_{t_1}^{>}$. Therefore $z = sa \in S\gamma_{t_1}^{>} \subseteq \left\langle \overset{l}{\gamma_{t_1}^{>}} \right\rangle$, where $t_1 > t$. Thus $z \in \left\langle \overset{l}{\gamma_{t_i}^{>}} \right\rangle$ for some $t_i > t$ and as a result $\sup\left\{t_i \mid z \in \left\langle \overset{l}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. Hence, $z \in \left(\overset{ll}{\gamma}\right)_t^{>}$.

(ii) and (iii) follows similarly.

(iv) Consider $z \in \left(\overset{ii}{\gamma}\right)_t$. Therefore $\overset{ii}{\gamma}(z) > t$. This implies that $\sup\{t_i \mid z \in \left\langle \overset{i}{\gamma_{t_i}^{>}} \right\rangle\} > t$. Consequently. there exists $t_0 > t$ and $z \in \gamma_{t_0}^{>} \cup \gamma_{t_0}^{>}\gamma_{t_0}^{>} \cup S\gamma_{t_0}^{>}S \subseteq \gamma_t^{>} \cup \gamma_t^{>}\gamma_t^{>} \cup S\gamma_t^{>}S$. Hence $z \in \left\langle \overset{i}{\gamma_t^{>}} \right\rangle$.

Conversely, let $z \in \left\langle \overset{i}{\gamma_t^{>}} \right\rangle$. Then $z \in \gamma_t^{>} \cup \gamma_t^{>}\gamma_t^{>} \cup S\gamma_t^{>}S$. Therefore, either $z \in \gamma_t^{>}$ or $z \in \gamma_t^{>}\gamma_t^{>}$ or $z \in S\gamma_t^{>}S$. Suppose $z \in \gamma_t^{>}$. Similar to part (i), it can be shown easily that $\sup\left\{t_i \mid z \in \left\langle \overset{l}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$.

If $z \in \gamma_t^{>}\gamma_t^{>}$, then $z = a_1a_2$ for some $a_1, a_2 \in \gamma_t^{>}$. Choosing $t_1, t_2$ such that $\gamma(a_1) > t_1 > t < t_2 < \gamma(a_2)$, we get $a_1 \in \gamma_{t_1}^{>}$ and $a_2 \in \gamma_{t_2}^{>}$. Write $t_3 = \min\{t_1, t_2\}$. Then $a_1, a_2 \in \gamma_{t_3}^{>}$. Therefore, $z = a_1a_2 \in \gamma_{t_3}^{>}\gamma_{t_3}^{>} \subseteq \left\langle \overset{i}{\gamma_{t_3}^{>}} \right\rangle$, where $t_3 > t$.

If $z \in S\gamma_t^{>}S$, then $z = s_1$ a $s_2$ for some $a \in \gamma_t^{>}$ and $s_1, s_2 \in S$. Choosing $t_4$ such that $\gamma(a_1) > t_4 > t$. Then, $z \in \gamma_{t_4}^{>} \subseteq \left\langle \overset{i}{\gamma_{t_4}^{>}} \right\rangle$. Therefore, $z = s_1$ a $s_2 \in S\gamma_{t_4}^{>}S \subseteq \left\langle \overset{i}{\gamma_{t_4}^{>}} \right\rangle$, where $t_4 > t$. Hence, under all circumstances we have, $z \in \left\langle \overset{i}{\gamma_{t_i}^{>}} \right\rangle$ for some $t_i > t$ and as a result $\sup\left\{t_i \mid z \in \left\langle \overset{i}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. That is $z \in \left(\overset{ii}{\gamma}\right)_t^{>}$.

(v) Consider $z \in \left(\overset{bb}{\gamma}\right)_t^{>}$. Therefroe $\overset{bb}{\gamma}(z) > t$. This implies that $\sup\left\{t_i \mid z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. Consequently, there exists $t_0 > t$ and $z \in \gamma_{t_0}^{>} \cup \gamma_{t_0}^{>}\gamma_{t_0}^{>} \cup \gamma_{t_0}^{>}S\gamma_{t_0}^{>} \subseteq \gamma_t^{>} \cup \gamma_t^{>}\gamma_t^{>} \cup \gamma_t^{>}S\gamma_t^{>}$. Hence, $z \in \left\langle \overset{b}{\gamma_t^{>}} \right\rangle$.

Conversely, let $z \in \left\langle \overset{b}{\gamma_t^{>}} \right\rangle$. Then $z \in \gamma_t^{>} \cup \gamma_t^{>}\gamma_t^{>} \cup \gamma_t^{>}S\gamma_t^{>}$. Therefore either $z \in \gamma_t^{>}$ or $z \in \gamma_t^{>}\gamma_t^{>}$ or $z \in \gamma_t^{>}S\gamma_t^{>}$. Suppose $z \in \gamma_t^{>}$. Similar to part (i), it can be shown easily that $\sup\left\{t_i \mid z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$.

If $z \in \gamma_t^{>}\gamma_t^{>}$, similar to part (iv), it can be shown easily that $\sup\left\{t_i \mid z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$.

If $z \in \gamma_t^{>}S\gamma_t^{>}$, then $z = a_1sa_2$ for some $a_1, a_2 \in \gamma_t^{>}$ and $s \in S$. Choosing $t_1', t_2'$ such that such that $\gamma(a_1) > t_1' > t < t_2' < \gamma(a_2)$, we get $a_1 \in \gamma_{t_1'}^{>}$ and $a_2 \in \gamma_{t_2'}^{>}$. Write $t_3' = \min\{t_1', t_2'\}$. Then, $a_1, a_2 \in \gamma_{t_3'}^{>}$. Therefore, $z = a_1$ s $a_2 \in \gamma_{t_3'}^{>}S\gamma_{t_3'}^{>} \subseteq \left\langle \overset{b}{\gamma_{t_3'}^{>}} \right\rangle$, where $t_3' > t$. Hence, under all circumstances, we have, $z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle$ for some $t_i > t$ and as a result $\sup\left\{t_i \mid z \in \left\langle \overset{b}{\gamma_{t_i}^{>}} \right\rangle\right\} > t$. That is $z \in \left(\overset{bb}{\gamma}\right)_t^{>}$.

(vi) Follows from [13, Lemma 3.4].

**Note:** Since the Theorem 4.5, Lemma 4.7, Theorem 4.7, Lemma 4.8 also holds when $S$ is a semiring with multiplicative identity, therefore, we omit the proofs.

**Theorem 5.3.** For a fuzzy set $\gamma$ in a semiring $S$ without multiplicative identity, we have.
(i) $\langle \overset{r}{\gamma} \rangle = \gamma \cup \gamma \circ S$.
(ii) $\langle \overset{l}{\gamma} \rangle = \gamma \cup S \circ \gamma$.
(iii) $\langle \overset{t}{\gamma} \rangle = \gamma \cup \gamma \circ S \cup S \circ \gamma \cup S \circ \gamma \circ S$.
(iv) $\langle \overset{i}{\gamma} \rangle = \gamma \cup \gamma \circ \gamma \cup S \circ \gamma \circ S$.
(v) $\langle \overset{b}{\gamma} \rangle = \gamma \cup \gamma \circ \gamma \cup \gamma \circ S \circ \gamma$.
(vi) $\langle \overset{q}{\gamma} \rangle = (\gamma \cup S \circ \gamma) \cap (\gamma \cup \gamma \circ S)$.

**Proof.** (i) Let $\overset{r}{\alpha} = \gamma \cup \gamma \circ S$. Then, for any $t \in [0,1[$,

$$\begin{aligned}
\left(\overset{r}{\alpha}\right)_t^{>} &= (\gamma \cup \gamma \circ S)_t^{>} \\
&= \gamma_t^{>} \cup (\gamma \circ S)_t^{>} \quad \text{by Lemma 3.2} \\
&= \gamma_t^{>} \cup \gamma_t^{>} S \quad \text{by Lemma 3.2} \\
&= \left\langle \overset{r}{\gamma_t^{>}} \right\rangle \\
&= \langle \overset{r}{\gamma} \rangle_t^{>} \quad \text{by Lemma 4.8}
\end{aligned}$$

Thus $\overset{r}{\alpha} = \langle \overset{r}{\gamma} \rangle$.

(ii) and (iii) follows similarly.

(iv) Let $\overset{in}{\alpha} = \gamma \cup \gamma \circ \gamma \cup S \circ \gamma \circ S$. Then, for any $t \in [0,1[$,

$$\begin{aligned}\left(\overset{in}{\alpha}\right)_t^{>} &= (\gamma \cup \gamma \circ \gamma \cup S \circ \gamma \circ S)_t^{>} \\ &= \gamma_t^{>} \cup (\gamma \circ \gamma)_t^{>} \cup (S \circ \gamma \circ S)_t^{>} \text{ by Lemma 3.2} \\ &= \gamma_t^{>} \cup (\gamma_t^{>}\gamma_t^{>}) \cup (S\gamma_t^{>}S) \text{ by Lemma 3.2} \\ &= \left\langle \overset{i}{\gamma_t^{>}} \right\rangle \\ &= \overset{i}{\langle\gamma\rangle}_t^{>} \text{by Lemma 4.8.}\end{aligned}$$

Thus $\overset{in}{\alpha} = \overset{i}{\langle\gamma\rangle}$.

(v) Let $\overset{b}{\alpha} = \gamma \cup \gamma \circ \gamma \cup \gamma \circ S \circ \gamma$. Then, for any $t \in [0,1[$,

$$\begin{aligned}\left(\overset{b}{\alpha}\right)_t^{>} &= (\gamma \cup \gamma \circ \gamma \cup \gamma \circ S \circ \gamma)_t^{>} \\ &= \gamma_t^{>} \cup (\gamma \circ \gamma)_t^{>} \cup (\gamma \circ S \circ \gamma)_t^{>} \quad \text{by Lemma 3.2} \\ &= \gamma_t^{>} \cup (\gamma_t^{>}\gamma_t^{>}) \cup (\gamma_t^{>}S\gamma_t^{>}) \quad \text{by Lemma 3.2} \\ &= \left\langle \overset{b}{\gamma_t^{>}} \right\rangle \\ &= \overset{b}{\langle\gamma\rangle}_t^{>} \quad \text{by Lemma 4.8}\end{aligned}$$

Thus $\overset{b}{\alpha} = \overset{b}{\langle\gamma\rangle}$.

(vi) Follows from [19, Theorem 3.9]

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