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STABILITY OF SET-VALUED PEXIDER FUNCTIONAL EQUATIONS

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Abstract. In this paper, we investigate a set-valued solution of the following Pexider functional equation

$$F(ax+by) = \alpha G(x) + \beta H(y)$$

with three unknown functions F, G and H, where a, b, α, β are positive real scalars.

1. Introduction and preliminaries

Assume that Y is a topological vector space satisfying the T_0 separation axiom. For real numbers s, t and sets $A, B \subset Y$ we put $sA + tB := \{y \in Y; y = sa + tb, a \in A, b \in B\}$. Suppose that the space 2^Y of all subsets of Y is endowed with the Hausdorff topology (see [4]). A set-valued function $F : X \to 2^Y$ is said to be additive if it satisfies the Cauchy functional equation $F(x_1 + x_2) = F(x_1) + F(x_2), x_1, x_2 \in X$. The family of all closed and convex subsets of Y will be denoted by $CC(Y)$, and the sets of all real, rational and positive integer numbers are denoted by $\mathbb{R}, \mathbb{Q}, \mathbb{N}$, respectively.

Lemma 1.1. [1] *Let λ and μ be real numbers. If A and B are nonempty subsets of a real vector space X, then*

$$\lambda(A+B) = \lambda A + \lambda B,$$
$$(\lambda+\mu)A \subseteq \lambda A + \mu B.$$

Moreover, if A is a convex set and $\lambda, \mu \geq 0$, then we have

$$(\lambda+\mu)A = \lambda A + \mu A.$$

Lemma 1.2. [3] *Let A, B be subsets of Y and assume that B is closed and convex. If there exists a bounded and nonempty set $C \subset Y$ such that $A + C \subset B + C$, then $A \subset B$.*

Lemma 1.3. *If $(A_n)_{n\in\mathbf{N}}$ and $(B_n)_{n\in\mathbf{N}}$ are decreasing sequences of compact subsets of Y, then $\bigcap_{n\in\mathbb{N}}(A_n + B_n) = \bigcap_{n\in\mathbb{N}} A_n + \bigcap_{n\in\mathbb{N}} B_n$.*

Lemma 1.4. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of compact subsets of Y, then $A_n \to \bigcap_{n\in\mathbb{N}} A_n$.*

Lemma 1.5. *If A is a bounded subset of Y and $(s_n)_{n\in\mathbb{N}}$ is a real sequence converging to an $s \in \mathbb{R}$, then $s_n A \to sA$.*

Lemma 1.6. *If $A_n \to A$ and $B_n \to B$, then $A_n + B_n \to A + B$.*

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Lemma 1.7. *If $A_n \to A$ and $A_n \to B$, then $clA = clB$.*

Lemma 1.3–1.7 are rather known and can be easily verified. The proofs of them can be found in [1, 2].

2. Set-valued solution of the Pexider functional equation

In the section, we give the solution of the Pexider functional equation.

Theorem 2.1. *Assume that $(X, +)$ is a vector space and Y is a T_0 topological vector space. If set-valued functions $F : X \to CC(Y), G : X \to CC(Y)$ and $H : X \to CC(Y)$ satisfy the functional equation*

$$F(ax + by) = \alpha G(x) + \beta H(y) \tag{2.1}$$

for all $x, y \in X$, where a, b, α and β are positive real numbers, then there exist an additive set-valued function $F_0 : X \to CC(Y)$ and sets $A, B \in CC(Y)$ such that

$$F(x) = \alpha F_0(x) + \alpha A + \beta B, \quad G(x) = F_0(ax) + A \quad \text{and} \quad H(x) = F_0(bx) + B$$

for all $x \in X$.

Proof. First, assume that $0 \in G(0)$ and $0 \in H(0)$. Then, for all $x, y \in X$, we have

$$\begin{aligned} F(x + y) &= F\left(a\frac{x}{a} + b\frac{y}{b}\right) = \alpha G\left(\frac{x}{a}\right) + \beta H\left(\frac{y}{b}\right) \\ &\subset \alpha G\left(\frac{x}{a}\right) + \beta H(0) + \alpha G(0) + \beta H\left(\frac{y}{b}\right) \\ &= F(x) + F(y). \end{aligned}$$

Letting $x = y$ in the above equation, we get $F(2x) \subset 2F(x)$, which implies that the sequence $(2^{-n}F(2^n x))_{n\in N}$ is decreasing. Put $F_0(x) := \bigcap_{n\in\mathbb{N}} 2^{-n}F(2^n x), x \in X$. It is clear that $F_0(x) \in CC(Y)$ for all $x \in X$. Similarly, we get

$$\begin{aligned} \alpha G(2x) + \beta H(0) &= F(2ax) = F\left(ax + b\left(\frac{ax}{b}\right)\right) \\ &= \alpha G(x) + \beta H\left(\frac{ax}{b}\right) \subset \alpha G(x) + \alpha G(0) + \beta H\left(\frac{ax}{b}\right) \\ &= \alpha G(x) + F(ax) = \alpha G(x) + \alpha G(x) + \beta H(0) = 2\alpha G(x) + \beta H(0). \end{aligned}$$

In view of Lemma 1.2, we obtain that $G(2x) \subset 2G(x)$, and consequently the sequence $(2^{-n}G(2^n x))_{n\in\mathbb{N}}$ is decreasing. Applying Lemma1.3 and this equality $F(a2^n x) = \alpha G(2^n x) + \beta H(0), n \in \mathbb{N}$, we obtain

$$F_0(ax) = \bigcap_{n\in\mathbb{N}} 2^{-n}F(a2^n x) = \alpha \bigcap_{n\in\mathbb{N}} 2^{-n}G(2^n x) + \beta \bigcap_{n\in\mathbb{N}} 2^{-n}H(0).$$

But $\bigcap_{n\in\mathbb{N}} 2^{-n}H(0) = \{0\}$, since the set $H(0)$ is bounded. Therefore $F_0(ax) = \alpha \bigcap_{n\in\mathbb{N}} 2^{-n}G(2^n x)$ for all $x \in X$. In an analogous way we show that the sequence $(2^{-n}H(2^n x))_{n\in\mathbb{N}}$ is decreasing and $F_0(bx) = \beta \bigcap_{n\in\mathbb{N}} 2^{-n}H(2^n x)$ for all $x \in X$. Hence,using once more Lemma 1.3, we get

$$\begin{aligned} F_0(x_1+x_2) &= \bigcap_{n\in\mathbb{N}} 2^{-n}F(2^n x_1 + 2^n x_2) = \bigcap_{n\in\mathbb{N}} 2^{-n}\left(\alpha G\left(\frac{2^n x_1}{a}\right) + \beta H\left(\frac{2^n x_2}{b}\right)\right) \\ &= \bigcap_{n\in\mathbb{N}} 2^{-n}\alpha G\left(\frac{2^n x_1}{a}\right) + \bigcap_{n\in\mathbb{N}} 2^{-n}\beta H\left(\frac{2^n x_2}{b}\right) \\ &= F_0(x_1) + F_0(x_2), x_1, x_2 \in X, \end{aligned}$$

which means that the set-valued function F_0 is additive.

Now observe that

$$F(nbx) + (n-1)\beta H(0) = F(bx) + (n-1)\beta H(x) \tag{2.2}$$

for all $x \in X$ and $n \in \mathbb{N}$. Indeed, for $n = 1$ the equality is trivial. Assume that it holds for a natural number k. Then, in virtue of (2.1), we obtain

$$\begin{aligned} F((k+1)bx) + k\beta H(0) &= \alpha G\left(\frac{kbx}{a}\right) + \beta H(x) + k\beta H(0) = F(kbx) + \beta H(x) + (k\beta - \beta)H(0) \\ &= F(bx) + (k-1)\beta H(x) + \beta H(x) = F(x) + k\beta H(x). \end{aligned}$$

which proves that (2.2) holds for $n = k + 1$. Thus, by induction, it holds for all $n \in \mathbb{N}$. In particular, we have

$$F(2^n x) + (2^n - 1)H(0) = F(x) + (2^n - 1)H\left(\frac{x}{b}\right),$$

and so

$$2^{-n}F(2^n x) + (1 - 2^{-n})H(0) = 2^{-n}F(x) + (1 - 2^{-n})H\left(\frac{x}{b}\right)$$

for all $x \in X$. By Lemma 1.4, $2^{-n}F(2^n x) \to \bigcap_{n\in\mathbb{N}} 2^{-n}F(2^n x) = F_0(x)$.

On the other hand, by Lemma 1.5, $1-2^{-n}H(0) \to H(0), 2^{-n}F(x) \to \{0\}$ and $(1-2^{-n})H\left(\frac{x}{b}\right) \to H\left(\frac{x}{b}\right)$. Thus, using Lemmas 1.6 and 1.7, we get $cl[F_0(x) + H(0)] = clH\left(\frac{x}{b}\right)$, whence $H\left(\frac{x}{b}\right) = F_0(x) + H(0)$ for all $x \in X$. Similarly, we can obtain $G\left(\frac{x}{a}\right) = F_0(x) + G(0), x \in X$. Let $A := G(0)$ and $B := H(0)$. Then $G(x) = F_0(ax) + A$ and $H(x) = F_0(bx) + B$ for all $x \in X$. Moreover $F(x) = \alpha F_0(x) + \alpha A + \beta B, x \in X$. This finishes our proof in the case that $0 \in G(0)$ and $0 \in H(0)$.

In the opposite case, fix arbitrarily points $a \in G(0)$ and $b \in H(0)$, and consider the set-valued functions $F_1, G-1, H_1 : X \to CC(Y)$ defined by $F_1(x) := F(x) - \alpha a - \beta b, G_1(x) := G(x) - a$ and $H_1 := H(x) - b, x \in X$. These set-valued functions satisfy the equation (2.1) and moreover $0 \in G_1(0)$ and $0 \in H_1(0)$. Therefore, by what we have discussed previously, we can get the same result. This completes the proof. □

In [2], Nikodem proved that a set-valued function $F_0 : [0, \infty) \to CC(Y)$, where Y is a locally convex Hausdorff space, is additive if and only if there exists an additive function $f : [0, \infty) \to Y$ and a set $K \in CC(Y)$ such that $F_0(x) = f(x) + xK, x \in [0, \infty)$. Thus we can get the following.

Theorem 2.2. *Let Y be a locally convex Hausdorff space. The set-valued functions $F : [0, \infty) \to CC(Y), G : [0, \infty) \to CC(Y)$ and $H : [0, \infty) \to CC(Y)$ satisfy the functional equation (2.1) if and only if there exist an additive function $f : [0, \infty) \to Y$ and sets $K, A, B \in CC(Y)$ such that*

$$F(x) = \alpha f(x) + \alpha Kx + \alpha A + \beta B, G(x) = f(ax) + akx + A \quad \textit{and} \quad H(x) = f(bx) + bkx + B$$

for all $x \in [0, \infty)$.

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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