On the possibility of acoustic field theory formulation

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Abstract

Acoustic field equations, as well as the conservation of energy and momentum are determined using quaternionic momentum-eigen value equation. The pressure and vector velocity are shown to be connected to the scalar and vector wave functions, respectively. When the pressure and velocity are determined to satisfy a wave equation traveling at c, the source and force are chosen. Maxwell's electrodynamics with a vanishing magnetic field and a nonzero scalar (longitudinal) wave is demonstrated to be comparable to a sound wave with external force and source. However, the velocity and pressure satisfy the Klein–Gordon equation under specific conditions. This condition can be found in London's superconductivity. A horn wave that obeys the Klein-Gordon equation connects the horn's cross-sectional area to an external source and force densities.

Key words: Quaternionic Acoustic Field Theory, Klein-Gordon Equation, Energy-Momentum Tensor, Quantum-Analogous Sound Waves

1 Introduction

Burns *et al.* offer an acoustic field theory that describes the local vector features of longitudinal(curl-free) acoustic fields [1]. They believe their method explains the recently discovered nonzero spin angular momentum density in inhomogeneous sound fields in fluids or gases. They also statedthat the typical acoustic Lagrangian formulation with a scalar potential is incapable of effectively describing such vector features of acoustic fields. By incorporating a displacement vector potential equivalent to the electromagnetic vector potential [2], a scalar field theory is obtained. Note thatthe Dirac and Klein-Gordon fields have spin $-\frac{1}{2}$ and 0, respectively. The spin of the particle (field) determines its statistical behavior. In the realm of quantum mechanics, it is argued that spin is a relativistic behavior of the field.

The Dirac and Klein-Gordon equations describe spinor and scalar fields, respectively. These equations are generalizations to the nonrelativistic Schrodinger equation. It was argued by Maxwell that the electromagnetic field bears a fluid aspect. This can be realized if we express the energy- momentum tensor of the electromagnetic field analogously to that of a fluid. The electric current and voltage in the transmission line are found to have this pattern in addition to the quantum field as represented by the telegraph equation [3]. The obtained system of equations can be used to describe the quantum mechanics of massive bosons. An energy-momentum tensor is also associated with the scalar field described by the Klein-Gordon equation. Acoustic waves can have a similar behavior. Tothis aim, we replace the wave function in this equation with the velocity field. They are also related to Euler's equation (conservation of momentum density) and the continuity equation [4]. The pressure, velocity, force, and source satisfy the Klein-Gordon equation when sound is subjected to external force and source. Interestingly, electromagnetic, quantum, and sound waves all have analogous properties. In field theory, a Lagrangian formalism is adopted from which the field equations and the energy-momentum tensors are derived. This tensor satisfies energy and momentum conservation equations.

We will employ here quaternions to develop acoustic field theory, which was obtained differently by Burns *et al*. Quaternions are generalization to complex number endowed with interesting algebraic merits. Quaternions are a non-abelian group of order 8 [5]. They also have intriguing geometrical aspects that let them have wide applications. Their salient property is that they are noncommutative, and this make them excellent candidates to describe quantum operators. A single quaternionic equation yields four equations. This helps unifies several physical laws emerging from a single entity. Maxwell equations were originally written in terms of quaternions. However, the quantum electrodynamics is formulated in terms of tensors. We would like here to avail quaternionic algebraic properties to formulate acoustics as a field theory rather than a wave theory. We need field parameters to achieve this. The velocity and pressure are linked to space and time derivatives of the scalar potential describing the acoustic wave. The acoustic wave is depicted as moving like a fluidwith an energy density and a stress tensor. Recently, it is found that the Klein-Gordon equation canbe associated with acoustical phenomena relating acoustic phenomena to quantum ones [6]. Earlier,it is shown that the Webster equation can be expressed in a Schrodinger-like form bring acoustic to quantum effect [7].

2 Vector quantum mechanics

A new formulation of quantum mechanics that copes with transmission lines is introduced [8]. Here the wave behavior of the electrons due to their mass is the same as that due to their charge that was characterized by electromagnetism. We express the quaternionic momentum eigen-value equation by [3]

$$\tilde{P}^*\tilde{\Psi} = mc\tilde{\Psi}, \qquad \tilde{\Psi} = \left(\frac{i}{c}\psi_0, \vec{\psi}\right), \qquad \tilde{P} = \left(\frac{i}{c}\hat{E}, \hat{p}\right), \tag{1}$$

where ψ_0 and $\vec{\psi}$ describe the wave functions of the particle. These wave functions describe the matter wave associated with the particle mass. They are analogous to the scalar and vector potentials in electrodynamics associated with the charge's electromagnetic field. Expanding the quaternion product

in eq.(1) and writing $\hat{E} = i\hbar\frac{\partial}{\partial t}$ and $\hat{p} = -i\hbar\vec{\nabla}$, yields

$$\frac{1}{c^2}\frac{\partial\psi_0}{\partial t}+\vec{\nabla}\cdot\vec{\psi}+\frac{m}{\hbar}\psi_0=0,\qquad \frac{\partial\vec{\psi}}{\partial t}+\vec{\nabla}\psi_0+\frac{mc^2}{\hbar}\vec{\psi}=0,\qquad \vec{\nabla}\times\vec{\psi}=0 \tag{2}$$

which yields the two-wave equations

$$\frac{1}{c^2}\frac{\partial^2\vec{\psi}}{\partial t^2}-\nabla^2\vec{\psi}+\frac{2m}{\hbar}\frac{\partial\vec{\psi}}{\partial t}+\left(\frac{mc}{\hbar}\right)^2\vec{\psi}=0, \tag{3}$$

and

$$\frac{1}{c^2}\frac{\partial^2\psi_0}{\partial t^2}-\nabla^2\psi_0+\frac{2m}{\hbar}\frac{\partial\psi_0}{\partial t}+\left(\frac{mc}{\hbar}\right)^2\psi_0=0, \tag{4}$$

where m is the particle mass, the ψ_0 and $\vec{\psi}$ are analogous to V and I, are the particle wave functions, respectively. As in Dirac's theory of an electron, the group velocity of the telegraph wave is equal tothe speed of light in a vacuum. However velocity of the classical telegraph wave depends on the wire properties, *viz.*, $v = 1/\sqrt{LC}$, where L and C are inductance and capacitance per unit length of the wire. The telegraph equation describes the propagation of electric signals in transmission lines.They are expressed as [15]

$$\frac{1}{v^2}\frac{\partial^2 I}{\partial t^2}-\frac{\partial^2 I}{\partial x^2}+(RC+GL)\frac{\partial I}{\partial t}+\mathrm{RG}I=0, \tag{3a}$$

and

$$\frac{1}{v^2}\frac{\partial^2 \mathrm{V}}{\partial t^2}-\frac{\partial^2 \mathrm{V}}{\partial x^2}+(RC+GL)\frac{\partial V}{\partial t}+\mathrm{RGV}=0, \tag{4a}$$

where G is the conductance per unit length of the wire. These are longitudinal waves analogous to acoustic waves.

Equations (3) and (4) can be called dissipative Klein-Gordon equation. It is also known as the undistorted telegraph equation that preserves the identity of the traveling particle. In electromag- netism, V and I are analogous to the electric and magnetic fields, respectively. However, ψ_0 and $\vec{\psi}$ iare analogous to the scalar and vector potentials, φ and $\vec{A}$, respectively. In quantum mechanics, the wave functions ψ and $\vec{\psi}$ represent an undistorted traveling wave packet. Sound wave also travels in a medium due to pressure and velocity field variations, and therefore we would expect their intensityto decrease as time goes on.

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[1] For two quaternions, $\widetilde{A}=(a_0\,,\vec{a}) and \widetilde{B}=(b_0\,,\vec{b})$,one has $\widetilde{A}\widetilde{B}=(a_0b_0-\vec{a}\cdot\vec{b}\,,a_0\vec{b}+\vec{a}b_0+\vec{a}\times\vec{b})$.

The energy and momentum conservation equations of the matter wave described by eq.(2) are

$$\frac{\partial u_M}{\partial t} + \vec{\nabla} \cdot \vec{S}_M = -\frac{u_M}{\tau}, \tag{5}$$

and

$$\left(\frac{\partial \vec{g}_M}{\partial t}\right) + \partial_j \sigma_{ij}^M = -\frac{{g_i}^M}{\tau}, \quad \sigma_{ij}^M = \psi_i \psi_j - \frac{\delta_{ij}}{2}\left(\psi^2 - \frac{{\psi_0}^2}{c^2}\right), \tag{6}$$

Where

$$u_M = \frac{\psi^2}{2} + \frac{{\psi_0}^2}{2c^2}, \quad \vec{S}_M = \psi_0 \vec{\psi}, \quad \tau = \frac{\hbar}{2mc^2}, \quad \vec{g}^M = \frac{\vec{S}_M}{c^2}. \tag{7}$$

One can now combine $\vec{S}_M$ and ${\sigma_{ij}}^M$ in a single tensor, the energy-momentum tensor, as

$$T_{\mu\nu} = \psi_\mu \psi_\nu - \frac{2\eta_{\mu\upsilon}}{2} \psi_\lambda \psi^\lambda, \qquad \psi^\mu = \left(\frac{1}{c}\psi_0, \vec{\psi}\right). \tag{8}$$

Intriguingly, eq.(8) does not involve the particle mass but its field only though the particle has mass. This is unlike the situation with Klein-Gordon and Dirac equations. The electromagnetic field that Maxwell envisaged as a fluid has stress tensor [2]

$$\sigma_{ij} = \varepsilon_0 E_i E_j + {\mu_0}^{-1} B_i B_j - \frac{\delta_{ij}}{2}(\varepsilon_0 E^2 + {\mu_0}^{-1} B^2). \tag{9}$$

The energy-momentum tensor of a scalar field is given by

$$\sigma_{ij}^\phi = (\partial_i \phi)(\partial_j \phi) - \frac{1}{2} g_{ij} (\partial_\lambda \phi)(\partial^\lambda \phi). \tag{10}$$

One now defines the energy and momentum densities of the matter-wave as u_M and $\vec{g}_M$, respectively. The energy and momentum tensor, eq.(6) for matter waves, are analogous to that of the electromagnetic and scalar fields, eqs.(9) and (10).

Burnt et al. described the acoustic dynamics in the absence of external forces and sources by [1]

$$\rho \frac{\partial \vec{v}}{\partial t} + \vec{\nabla} p = 0, \qquad \beta \frac{\partial p}{\partial t} + \vec{\nabla} \cdot \vec{v} = 0, \qquad \vec{\nabla} \times \vec{v} = 0, \tag{11}$$

Where $c_s = \frac{1}{\sqrt{\beta\rho}}$. They employ a scalar potential ϕ and define

$$\sqrt{\rho}\, \vec{v} = \vec{\nabla}\phi, \qquad \sqrt{\beta}\, p = -\frac{1}{c_s}\frac{\partial \phi}{\partial t}. \tag{12}$$

The free scalar field Lagrangian is defined by $\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)=\frac{1}{2}\left(-\left|\vec{\nabla}\phi\right|^2+\left(\frac{\partial\phi}{\partial t}\right)^2\right)$ which can be written as

$$\mathcal{L}=\frac{1}{2}(\beta p^2-\rho v^2)\,, \tag{13}$$

for sound, employing eq.(12). Setting up the sound Lagrangian, one can easily derive the field equation of motion and the energy and momentum conservation equations.

Comparing eq.(2) and (12) reveals that

$$\psi_0=\frac{p}{\rho}\,,\qquad\qquad \vec{\psi}=\vec{v}. \tag{14}$$

One can similarly reduce ψ_0 and $\vec{\psi}$ to one scalar using the definition

$$\psi^\mu=-\,\partial^\mu\phi\qquad\Longrightarrow\qquad\vec{\psi}=\vec{\nabla}\phi\,,\quad \psi_0=-\frac{1}{c_s}\frac{\partial\phi}{\partial t}\,. \tag{15}$$

This choice would require $m=0$, as evident from eq.(2). Therefore, the right-hand side of eq.(5) vanishes. Hence, the system becomes classical ($\hbar$ disappears). Therefore, the acoustic wave we are dealing with has no quantum character.

Taking the divergence of the first equation in eq.(11) and using the second one, yields

$$\frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p=0,\qquad \frac{1}{c_s^2}\frac{\partial^2\vec{v}}{\partial t^2}-\nabla^2\vec{v}=0. \tag{16}$$

Therefore, $\vec{v}$ and p satisfy a wave equation. Manipulation of Eq.(11) yields the "Poynting" vector, energy density, and stress tensor of the acoustic wave as

$$\frac{\partial u^A}{\partial t}+\vec{\nabla}\cdot\vec{S}^A=0,\qquad\qquad \frac{\partial\vec{g}^A}{\partial t}+\partial_j\sigma_{ij}^A=0, \tag{17}$$

Where

$$\vec{S}^A=p\vec{v},\qquad \vec{g}^A=\frac{\vec{S}^A}{c_s^2},\qquad u^A=\frac{1}{2}(\rho v^2+\beta p^2),\quad \sigma_{ij}^A=\rho v_i v_j+\frac{\delta_{ij}}{2}u^A. \tag{18}$$

3 Sound wave under external force and source

A wave can move under external force and a source. Let us now consider the acoustic theory withforce and source as expressed by [2] [1]

$$\rho\frac{\partial\vec{v}}{\partial t}+\vec{\nabla}p=-\vec{g},\qquad \beta\frac{\partial p}{\partial t}+\vec{\nabla}\cdot\vec{v}=g_0,\qquad \vec{\nabla}\times\vec{v}=0. \tag{19}$$

[2]We define $-\vec{g}$.instead of $\vec{g}$.

Taking the divergence of the first equation in eq.(19) and the time-derivative of the second equation,yields

$$\frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p=\frac{\partial}{\partial t}(\rho g_0)+\vec{\nabla}\cdot\vec{g}. \tag{20}$$

and

$$\frac{1}{c_s^2}\frac{\partial^2\vec{v}}{\partial t^2}-\nabla^2\vec{v}=-\left(\vec{\nabla}g_0+\beta\frac{\partial\vec{g}}{\partial t}\right). \tag{21}$$

If the pressure and velocity satisfy a wave equation traveling at speed c, then

$$\frac{\partial}{\partial t}(\rho g_0)+\vec{\nabla}\cdot\vec{g}=0,\qquad \vec{\nabla}g_0+\beta\frac{\partial\vec{g}}{\partial t}=0. \tag{22}$$

Hence, eq.(22) reveals that the source and force satisfy a wave equation too. Equation (22) shows that the force and source satisfy the same equation as that of p and $\vec{v}$. The first equation in eq.(22) represents the mass-like conservation and the second one Euler-like (momentum) equation. Equation (20) can be expressed as a continuity-like equation when expressed in the form

$$\frac{\partial}{\partial t}\left(\rho g_0-\beta\rho\frac{\partial p}{\partial t}\right)+\vec{\nabla}\cdot\left(\vec{\nabla}p+\vec{g}\right)=0. \tag{23}$$

One can thus consider new transformations

$$g_0{}'=g_0-\beta\frac{\partial p}{\partial t},\qquad \vec{g}'=\vec{g}+\vec{\nabla}p, \tag{24}$$

that are similar to the gauge transformations of the vector and scalar potentials of electromagnetism [2]. Therefore, eq.(19) is invariant under the transformations in eq.(24). Now the pressure and velocity in eqs.(20) and (21) can be made to satisfy the Klein-Gordon equation upon choosing

$$-k^2 p=\frac{\partial}{\partial t}(\rho g_0)+\vec{\nabla}\cdot\vec{g}, \tag{25}$$

and

$$k^2\vec{v}=\vec{\nabla}g_0+\beta\frac{\partial\vec{g}}{\partial t}, \tag{26}$$

where k is a constant having dimension of length. This situation is analogous to London's equationsof superconductivity [9]. Now eqs.(25) and (26) reveal that the force and source satisfy the Klein- Gordon equation, *viz.,*

$$\frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p+k^2 p=0, \qquad \frac{1}{c_s^2}\frac{\partial^2\vec{v}}{\partial t^2}-\nabla^2\vec{v}+k^2\vec{v}=0. \tag{27}$$

Manipulations of eqs.(25) and (26) show that

$$\frac{1}{c_s^2}\frac{\partial^2 g_0}{\partial t^2}-\nabla^2 p+k^2 g_0=0, \qquad \frac{1}{c_s^2}\frac{\partial^2\vec{g}}{\partial t^2}-\nabla^2\vec{g}+k^2\vec{g}=0. \tag{27a}$$

Dotting the first equation in eq.(19) by $\vec{v}$ and use the second one yields

$$\frac{\partial u^A}{\partial t}+\vec{\nabla}\cdot\vec{S}^A=pg_0-\vec{v}\cdot\vec{g}, \tag{28}$$

and multiplying the first equation by p and the second one by $\vec{v}$, yield

$$\frac{\partial\vec{g}^A}{\partial t}+\partial_j\sigma_{ij}=\rho g_0\vec{v}-\beta p\vec{g}. \tag{29}$$

Equation (28) is the energy equation of sound in the presence of a source and force. Here $\vec{S}^A$ represents the energy flux carried by the wave which follows that of Umov [10]. The right-hand side
on eq.(28) represents the power density lost and gained by the fluid (air), and the right-hand side on
eq.(29) represents the force density acting on the fluid (air). Thus, a sound wave has a characteristic
of a wave and a particle at the same time. This is reminiscent of Einstein's mass-energy equivalence.
Therefore, sound exhibits wave and particle nature. This phenomenon is found to be reflected by
quantum particles. Therefore, the above formalism of sound exhibits a quantum nature of sound
waves.
Following eq.(13) and (19), the sound Lagrangian including external source and force can be expressed as

$$\mathcal{L}=\frac{1}{2}(\beta p^2-\rho v^2)+p-{g_0}^{-1}\vec{v}\cdot\vec{g}\,. \tag{30}$$

4 Flow of electric charge in electromagnetism

We are interested in electromagnetism where the medium is filled with electric charges and currents. The electromagnetic field satisfies a wave equation traveling at the speed of light in a vacuum. However, not only the electromagnetic field travels in free space at the speed of light but in any medium governed by [11]

$$\vec{\nabla}\cdot J+\frac{\partial\rho}{\partial t}=0,\qquad \vec{\nabla}\rho+\frac{1}{c^2}\frac{\partial\vec{J}}{\partial t}=0,\qquad \vec{\nabla}\times\vec{J}=0. \tag{31}$$

These equations imply that any change in ρ and $\vec{J}$ is transmitted as a wave traveling at the speed oflight. Manipulations of the above equation yield the wave equations

$$\frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2}-\nabla^2\rho=0\,,\qquad \frac{1}{c^2}\frac{\partial^2\vec{J}}{\partial t^2}-\nabla^2\vec{J}=0. \tag{32}$$

Thus, interestingly the electric current travels at the speed of light in a vacuum. The flow of currentconserves energy and momentum governed by

$$\vec{\nabla}\cdot\vec{S}_J+\frac{\partial u_J}{\partial t}=0,\qquad \frac{\partial g_i}{\partial t}+\partial_j\sigma_{ij}^J=0, \tag{33}$$

where

$$\vec{S}_J=\rho\vec{J},\qquad u_J=\frac{\rho^2}{2}+\frac{J^2}{2c^2},\qquad \sigma_{ij}^J=c^{-2}J_iJ_j+\frac{1}{2}\left(\rho^2-\frac{J^2}{c^2}\right),\qquad \vec{g}=\frac{\vec{S}_J}{c^2}. \tag{34}$$

5 Webster horn equation

An interesting acoustic (sound) equation is that due to Webster [12]. It describes the propagation ofsound waves in 1-dimension inside horns with a variable cross-section, $S(x)$. It is given by

$$\frac{\partial p}{\partial x}=-\frac{\rho}{S(x)}\frac{\partial u}{\partial t},\qquad \frac{\partial v}{\partial x}=-\frac{S}{\rho c_s^2}\frac{\partial p}{\partial t}, \tag{35}$$

Which when combined yield

$$\frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2}=\frac{1}{S(x)}\frac{\partial}{\partial x}\left(S(x)\frac{\partial p}{\partial x}\right), \tag{36}$$

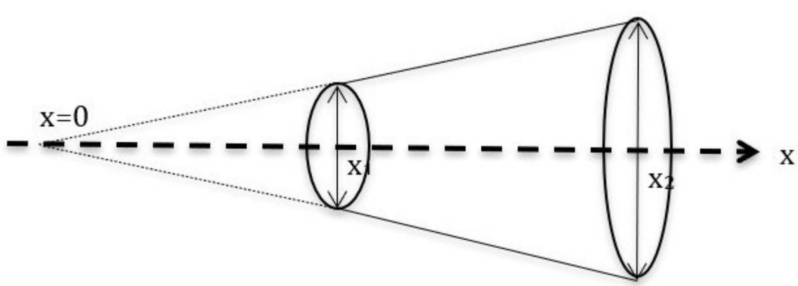
where u and p are the velocity field and pressure. It reduces to the Klein-Gordon equation

$$\frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p + k_x^2 p = 0, \tag{37}$$

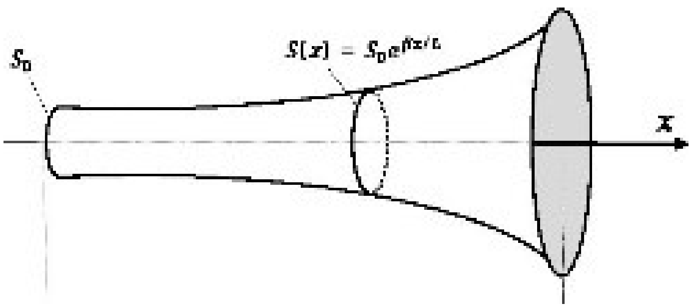
With

$$k_x^2 = \frac{1}{\sqrt{S(x)}}\frac{d^2\sqrt{s(x)}}{dx^2}. \tag{38}$$

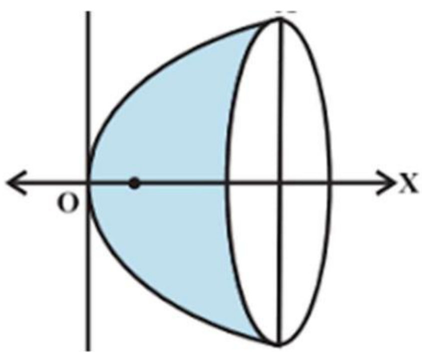
When eqs.(25) and (26) are combined with eq.(38), the cross-sectional area $S(x)$ can be connected with the source and force densities, g_0 and $\vec{g}$. Three interesting types of horns are those having parabolic, conical, and exponential shapes (see the figure) [13]. The effective mass (k_x) of the traveling wave across the horn will depend on the form of the horn, as evident from eq.(38). The horn is characterized by its impedance which is defined as the complex ratio between the sound pressure p and particle velocity v, *i.e.*, $Z = p/v$. The horn exhibits resonance characteristics due to the large change in acoustic impedance waves in passing from the mouth to the free atmosphere(air) [14].



A horn with a conical cross-section



A horn with an exponential cross-section area (S)



A horn with a parabolic cross-section

Figure 1: Horn different cross-sections

6 Concluding remarks

We used quaternions to develop acoustic field theory, which was obtained differently by Burns etal. The velocity and pressure are linked to a scalar potential describing the acoustic wave. The acoustic wave is seen as moving like a fluid with an energy density and a stress tensor. The energy and momentum conservation equations for sound are determined with and without a source and an external force. The sound field equations are invariant under new transformations for the force and the source. Sound and quantum waves are discovered to have comparable properties. The pressure and velocity of the sound wave satisfy the Klein-Gordon equation under particular conditions. This ensures that sound is a mass-carrying field. Electromagnetic, quantum, and acoustic waves all share several characteristics. This helps us treat acoustic waves as a field endowed with energy and momentum. The energy-momentum tensor mimics that of an electromagnetic field, scalar, Dirac, and vector quantum waves. The source and force densities of a sound wave traveling inside a horn are related to the cross-sectional area of a horn. The horn's nonuniform cross-sectional area exerts a force on the air inside it.

References

[1] Burns, L., Bliokh, K. Y., Nori, F., and Dressel, J., *Acoustic versus electromagnetic field theory: scalar, vector, spinor representations and the emergence of acoustic spin*, New J. Phys. 22, 053050(2020).

[2] Jackson, J. D., *Classical Electrodynamics,* Wiley (1998).

[3] Arbab, A. I., *Quantum Telegraph equation: New matter-wave equation*, Optik, 140, 1010 (2017).

[4] Allen, J., *The reactive radiation impedance of acoustic horns*, https://jontalle.web.engr.illinois.edu/uploads/403/Horns.pdf

[5] Dean, R., *A Rational Polynomial whose Group is the Quaternions*, The Am. Math. Month.88, 42 (1981).

[6] Barbara, J. F., Pike, E. R. and Harp, D. B., *The acoustical Klein-Gordon equation: The wave- mechanical step and barrier potential functions*, J. Acoust. Soc. Am. 114, 1291 (2003).

[7] David, B., Julius, and O. Smith, III., *On the use of Schrodinger's equation in the analytic deter-mination of horn reflectance*, J. Acoust. Soc. Am. 96, 3300 (1994).

[8] Heaviside, O., *The Transmission of Electromagnetic Waves along Wires without Distortion,* Elec- trical Papers, V.2, 307 (1894).