Solution of Gaussian Hypergeometric Differential Equation Through Fixed Point Theorems

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Abstract

The fixed-point theorem is a fundamental result in mathematics that establishes the existence of a fixed point for certain types of functions. A fixed point of a function is a point in the domain of the function that maps to itself under the given function. The most well-known and widely used fixed-point theorem is the Banach fixed-point theorem. Brouwer's theorem and Katutani fixed point theorems are the extension of the Banach fixed point theorems. Similarly, Hypergeometric differential equations are a class of differential equations that arise in mathematics and physics. Hypergeometric differential equations are of the form: z(1-z)y'' + [c - (a+b+1)z]y' - aby = 0, The hypergeometric differential equation is a second-order linear ordinary differential equation. This work is to study the existence and uniqueness solution of the second order hypergeometric differential equation. The coefficients of a hypergeometric series are the solutions of second order ordinary or partial differential equations. Here we will study the solution of the hypergeometric differential equation by Fixed Point theorem.

Keywords:- Hypergeometric function, Hypergeometric differential equation, Banach Fixed Point Theorem.

1. Introduction

1.1 Fixed Point Theorems

Fixed point theorems are fundamental results that establish the existence and properties of fixed points in certain types of functions or mappings. A fixed point of a function is a point in the domain of the function that maps to itself under the defined function. In other words, if we have a function f and an element x in its domain, then x is a fixed point of f if $f(x) = x$. Huang and

Zhang [6] introduced cone metric spaces which are generalizations of metric spaces, and they extended Banach's contraction principle to such spaces, others studied fixed point theorems in cone metric spaces.

1.1.1 Banach fixed point theorem

Banach fixed point theorem is one of the most well-known fixed point theorems. It is also known as the contraction mapping principle. It states that if a complete metric space (a set equipped with a distance function that satisfies certain properties) contains a contraction mapping (a function that contracts the distance between points), then the mapping has a unique fixed point. It states the sufficient conditions for the existence and uniqueness of a fixed point, which is seen as a point that is mapped to itself. The theorem also gives an iterative process by which we can obtain approximations to the fixed point along with error bounds.[8]

Definition 1.

A fixed point to a function T a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped on to itself, that is $Tx = x$

Definition 2.

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction on X if there exists a positive constant $K < 1$ such that $d(T(x),T(y)) \leq Kd(x, y)$ for all $x, y \in X$.

Theorem 2 (Banach's Fixed Point Theorem).

Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x \in X$ such that $T(x) = x$.

This theorem is widely used in the analysis of iterative algorithms and provides a rigorous foundation for many numerical methods.

1.1.2 Brouwer's Fixed Point Theorem[5]

Another important fixed point theorem is Brouwer's fixed point theorem, which guarantees the existence of at least one fixed point for any continuous function defined on a closed and bounded subset of Euclidean space. Brouwer's theorem has profound implications in topology and has applications in areas such as game theory and economics.

Brouwer’s theorem states that for any continuous function $f : B^n \to B^n$, there exists $x \in B^n$ such that $f(x) = x$. Brouwer’s theorem also has applications in various other fields. For example, it is related to existence theorems in differential equations and plays a role in proving the existence of the Nash equilibrium in game theory.

Suppose any continuous function $g : X \to X$ has a fixed point. Let $f : X \to Y$ be a homeomorphism and let $h : Y \to Y$ be continuous. Then h has a fixed point.

1.1.3 Kakutani Fixed Point Theorem[14]

Kakutani fixed point theorem, establishes the existence of a fixed point for correspondences (set-valued mappings) with certain properties, and the Schauder fixed point theorem, provides conditions for the existence of fixed points in infinite-dimensional spaces.

Kakutani’s fixed point theorem generalizes Brouwer’s fixed point theorem in two aspects. A point-to-point mapping is generalized to point-to-set mapping, and continuous mapping is generalized to upper semi-continuous mapping.

Definition 1.

A point-to-set map is a relation where every input is associated with at least one output in which at least one input is associated with two or more outputs.

Definition 2.

A point-to-set mapping $\Phi : x \to \Phi(x) \in \Omega(S)$ of S into $\Omega(S)$ is called upper semi-continuous if $x_n \to x_0$, $y_n \in \Phi(x_n)$ and $y_n \to y_0$ imply $y_0 \in \Phi(x_0)$.

Fixed point theorems not only have theoretical significance but also find practical applications in various fields. They are utilized in optimization algorithms, the study of dynamical systems, the analysis of equilibrium points in mathematical models, and the design of algorithms for solving equations or finding solutions to various problems.

1.2 Hypergeometric Function

Hypergeometric functions are one of the oldest transcendental functions and are the extensions of the geometric series [9]. Generally exponential functions are generalized in terms of hypergeometric functions. They can be analytically manipulated as well [1].The hypergeometric series plays the significant role in the number system, partition theory, graph theory, Lie algebra, etc. [2]. According to Rao [13], John Wallis (1616-1703) extended the ordinary geometric series

$$1+a^2+a^3+a^4+a^5+a^6+.... \quad \dots(1.1)$$

to the hypergeometric series of the form

$$1+a+a(a+b)+a(a+b)(a+2b)+a(a+b)(a+2b)(a+3b)+.... \quad \dots(1.2)$$

At b =1, the representation at (2) is written can be written in the form

$$=1+a+a(a+1)+a(a+1)(a+2)+a(a+b)(a+2)(a+3)+....$$

Whose last term is given by

$$a_n=\prod_{k=1}^{n}(a+k-1)$$

$$=\frac{\Gamma(a+k)}{\Gamma(a)} \quad \dots(1.3)$$

For n is a non-negative integer. The equation is called the Pohhammer function [11][13]. In 1707-83 Leonhard Euler introduced the power series expansion of the form[9]

$$a+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+......... \quad \dots(1.4)$$

Which can be expressed as

$$\text{F(a, b; c; z)} = {}_2F_1\begin{bmatrix} a & b; & \\ & c; & z \end{bmatrix}=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad \dots(1.5)$$

or equivalently[4],

$$ {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix} ;1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \ldots(1.6) $$

Hypergeometric functions are not only expressed as the Euler Hypergeometric functions but are also the solutions of the second order differential equation

$$ z(1-z)\frac{d^2y}{dz^2} + [c-(a+b+1)z]\frac{dy}{dz} - aby = 0 \quad \ldots(1.7) $$

In term of operator the above equation can be expressed as

$$ [\theta(\theta+c-1) - z(\theta+a)(\theta+b)]w = 0 \quad \ldots(1.8) $$

The series is convergent if |z|<1 and divergent for |z| >1 and z =1 for *R*(c-a-b)>0 The equation (1.3.9) has a regular singularity at $z = 0,1$ and infinity [10]

The integral representation [4] [10] of (1.7) is

$$ F(t) = \frac{\Gamma(b)}{\Gamma(c)\Gamma(b-c)}\int_a^b t^{-a}(1-t)^{c-b-1}(1-st)^{-a}\,ds $$

Or, $$ F(a,b;c;s) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_a^b t^{b-1}(1-t)^{c-b-1}(1-st)^{-a}\,dt \quad \ldots(1.9) $$

This can be written in the form of Freadholm second type of integral equation,[4]

Or, $$ X(t) = \frac{\Gamma(b)}{\Gamma(c)\Gamma(b-c)}\int_a^b k(t,s,s(t))u(s)ds \quad \ldots(1.10) $$

Or, $$ X(t) = \lambda\int_a^b (t,s,s(t))u(s)ds \quad \ldots(1.11) $$

Where $\lambda = \frac{\Gamma(b)}{\Gamma(c)\Gamma(b-c)}$

2. Results

Let us define the exponential function

$$e_{q(s,a)} = e^{\int_a^t q(t)dt} \quad \ldots(2.1)$$

for $s \in S$ and let S is the non-empty closed subset of R. Also, Let c >0 a constant and $\|.\|$ denotes the Euclidean norm on R^n and define the metric,[3]

$$d_c(x, y) = \sup_{s \in (a,b)S} \frac{\|X(t) - Y(t)\|}{ec_{(s,a)}} \quad \ldots(2.2)$$

Where $d_0(x, y) = \sup_{s \in (a,b)S} \|x(t) - y(t)\|$ such that

$$\|x\|_s = \sup_{s \in (a,b)S} \frac{\|X(s)\|}{ec_{(s,a)}}$$

and $\|x\|_0 = \sup_{s \in (a,b)S} \|X(s)\|$ $\quad \ldots(2.3)$

be a continuous map and $A : C[\{a,b\}_S : R^n] \to C[\{a,b\}_S : R^n]$ then (1.7) has the unique solution[7]. Now we have from (2.2)

$$d_c(x, y) = \sup_{s \in (a,b)S} \frac{\|X(s) - Y(s)\|}{ec_{(s,a)}}, \forall (x, y) \in C, \quad \ldots(2.4)$$

Now $$d_t(Ax, Ay) = \sup_{t \in (a,b)T} \frac{\|(Ax)(t) - (Ay)(t)\|}{e\beta_{(t,a)}}$$

$$= \sup_{s \in (a,b)S} \frac{\gamma}{ec_{(s,a)}} \left\{ \int_a^b \|[k(s,t,x(t))x(t) - k(s,t,y(t))y(t)]\| dt \right\}$$

$$\leq L \sup_{s \in (a,b)T} \frac{\gamma}{ec_{(s,a)}} \left\{ \int_a^b \|x(t) - y(t)\| dt \right\}$$

$$\leq \gamma L d_s(x,y) \sup_{t \in (a,b)} \frac{e_c(b,a)}{e_c(s,a)} \text{ Where } L = \sup_{s \in (a,b) S} \|[K(s,t,x(t))x(t)]\|$$

$$\leq \gamma \frac{L}{\lambda} d_c(x,y) \text{ where } \frac{1}{\lambda} = \sup_{s \in (a,b)} \frac{e_c(b,a)}{e_c(s,a)}$$

$$\leq \gamma \frac{L}{\lambda} d_t(x,y) \qquad \ldots(2.5)$$

$$\therefore d_t(Ax, Ay) \leq \gamma \frac{L}{\lambda} d_t(x,y)$$

3. Conclusion

Fixed point theorems have applications in various branches of mathematics and other fields, including analysis, topology, economics, computer science, and physics. They provide powerful tools for proving the existence of solutions to equations, establishing the convergence of iterative algorithms, and studying the properties of dynamic systems. In summary, fixed point theorems are powerful mathematical tools that establish the existence and properties of fixed points in functions or mappings. Likewise Hypergeometric differential equations are a class of ordinary differential equations that arise in various areas of mathematics and physics. In this paper we have established the existence of unique solution of the hypergeometric differential equation by using the Banach Hypergeometric theorem. These unique proofs are peculiar in mathematics and provide the unique solution through alternative approach. This approach is applicable in mathematics and branches of applied sciences.

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