Arbitrary Hilbert Transformer Design Using DST Methods

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Received: 23.06.2024 Revised: 13.08.2024 Accepted: 01.09.2024

Abstract: - In recent years, various generalized arbitrary order Hilbert transformer have been developed for controlling the phase response. This paper presents the designing approach of Hilbert transform of arbitrary order using discrete sine transform-II. First, the theory of arbitrary order frequency response of Hilbert transformers is reviewed. Then, with the help of the DST-II method determine the filter coefficient. Next, for optimal design values some numerical problems are discussed for the frequency response of window and non-window techniques.
Keywords- Arbitrary order Hilbert transformer (AHT), Frequency-Response (F-R), DST-II, Lanczos window.

1. Introduction:- Hilbert Transformer used to design a fractional order differentiator, which works to determine position and edges of images. Theory of discrete sine transform (DST) is discussed and it is used successfully in the application of speech processing [7], signal interpolation [8] etc. Definition and properties of arbitrary order Hilbert transform frequency response is described in section II. In section III, design method of DST-II and transfer function is determined by using the Lancozs window. In section IV, various design examples are discussed. Finally, conclusion is made.

2. Arbitrary Order Hilbert Transformer:

Frequency response of AHT is defined as [2][3].

$$H_\alpha(\omega) = \begin{cases} e^{i\left(\frac{\pi}{2}\right)} & -\infty<\omega<0 \\ 0 & \omega = 0 \\ e^{-i\left(\frac{\pi}{2}\right)} & 0 < \omega < \infty \end{cases}$$

Following are some important properties of AHT frequency response:

i. Additive property: $H_{\alpha_1+\alpha_2}(\omega) = H_\alpha(\omega)$ [1][2].

ii. If x is the period of Hilbert transformer, then frequency response $H_{\alpha+x}(\omega) = H_\alpha(\omega)$

iii. AHT's of some well-known signals are

$$AHT[cos\omega t + \phi] = \left[cos\omega t + \phi - \frac{\alpha\pi}{2}\right]$$

$$AHT[sin\omega t + \phi] = \left[sin\omega t + \phi - \frac{\alpha\pi}{2}\right]$$

3. Design Method using DST-II

A given data signal i.e., continuous signal z(t) is sampled as z (0), z (1)….., z (L-1). The desired data is obtained from the discrete time sequence by using the interpolation function of DST-II and it can be express as:

$$Z(q) = \sqrt{\frac{2}{L}}\sum_{p=0}^{L-1} c_q z(p)\, sin\left(\frac{(p+0.5)(q+1)\pi}{L}\right) \quad (1)$$

$$z(p) = \sqrt{\frac{2}{L}}\sum_{q=0}^{L-1} c_q\, Z(q)\, sin\left(\frac{(p+0.5)(q+1)\pi}{L}\right) \quad (2)$$

$$c_q = \frac{1}{\sqrt{2}} \text{ if} q = L-1$$

Put the value of eqn. (1), in to the eqn. (2), we get;

$$z(p) = \sum_{r=0}^{L-1} z(r)\left\{\frac{2}{L}\sum_{q=0}^{L-1} c_q^2\, sin\left(\frac{(r+0.5)(q+1)\pi}{L}\right) sin\left(\frac{(p+0.5)(q+1)\pi}{L}\right)\right\}$$

Replace t in place of p then

$$z(t) = \sum_{r=0}^{L-1} z(r)\, K_{1r}(t) \quad (3)$$

Where $K_{1r(t)}$ is define as

$$K_{1r}(t) = \frac{2}{L}\sum_{q=0}^{L-1} c_q^2\, sin\left(\frac{(r+0.5)(q+1)\pi}{L}\right) sin\left(\frac{(t+0.5)(q+1)\pi}{L}\right) \quad (4)$$

Apply Hilbert transform of arbitrary order on equation (3) then the system output would be

$$AHT[z(t)] = \sum_{r=0}^{L-1} z(r)\, AHT[K_{1r}(t)] \quad (5)$$

Where

$$AHT[K_{1r}(t)] = \frac{2}{L}\sum_{q=0}^{L-1} sin\left(\frac{(r+0.5)(q+1)\pi}{L}\right) sin\left(\frac{(t+0.5)(q+1)\pi}{L} - \frac{\alpha\pi}{2}\right) \quad (6)$$

Substitute the eq. (6) in to (5) then we get

$$AHT[z(t)] = \sum_{r=0}^{L-1} z(r)\, G_r(t) \quad (7)$$

Where

$$G_r(t) = \frac{2}{L}\sum_{q=0}^{L-1} sin\left(\frac{(r+0.5)(q+1)\pi}{L}\right) sin\left(\frac{(t+0.5)(q+1)\pi}{L} - \frac{\alpha\pi}{2}\right) \quad (8)$$

Following is the expression of transfer function for DST-II method (Ideally)

$$H_{d_*}(\omega) = F_\beta(\omega) e^{-i\omega I} \quad (9)$$

Following is the expression of FIR filter in terms of transfer function

$$H_*(z) = \sum_{u=0}^{L-1} h(u)\, z^{-u} \quad (10)$$

If an input signal, $s_1(p)$ is applied on FIR filter, then output $s_1(p), s_1(p-1), s_1(p-2), \ldots, s_1(p-L+1)$ integer delay sample values is obtained, then the output of the system is

$$z(p) = \sum_{u=0}^{L-1} h(u)\, s_1(p-u) \quad (11)$$

Now our main problem is to determine filter coefficient $h(u)$ from the eq. (10) let us assume that eq. (11) approximately equal to the differentiation of arbitrary order i.e. $D^\alpha s_1(p-I)$, then

$$z(p) \approx AHT[s_1(p-I)] \quad (12)$$

With the help of index mapping technique, we get $z(t) = s_1(p-(L-1)+t)$ and put this value into equation (7), then

$$AHT[s_1(p-(L-1)+t)] = \sum_{r=0}^{L-1} s_1(p-(L-1)+r)\, G_r(t) \quad (13)$$

$$h(u) = G_{(L-1-u)}(L-1-I) \quad (14)$$

After compared between eq. (8) and (14), the value of filter coefficient will be

$$h(u) = \frac{2}{L}\sum_{q=0}^{L-1} sin\left(\frac{(L-u-0.5)(q+1)\pi}{L}\right) sin\left(\frac{(L-I-0.5)(q+1)\pi}{L} - \frac{\alpha\pi}{2}\right) \quad (15)$$

Lanczos window is used to modify the filter coefficient and its transfer function is as follows

$$\omega(u) = sinc\left(\frac{2u}{L-1} - 1\right) \tag{16}$$

Modified filter coefficient is defined as

$$h_*(u) = h(u)\omega(u) \tag{17}$$

Arbitrary order Hilbert Transformer based transfer function performance can be calculated using following integral square error formula.

$$\sqrt{\int_{0.1\pi}^{0.9\pi} \left|H_*(e^{i\omega}) - H_{d_*}(\omega)\right|^2 d\omega} \tag{18}$$

3.1 Design Examples

Example 1: Fig. 1 shows a comparison result between window design (dark line) and non-window design (dotted line) methods with the help of an error versus arbitrary order plot for varying values of arbitrary order. The optimal design values are chosen as sample size L = 60, delay value I = 40, arbitrary order α = (−2, 2). In Fig. 1 window design shows the smaller size of error for as compared to non-window design for varying values of arbitrary order. So, the window design method performs better as compared to non-window design methods for the above given optimal design values. Therefore, window design methods are well suited for the arbitrary order Hilbert differentiator design.

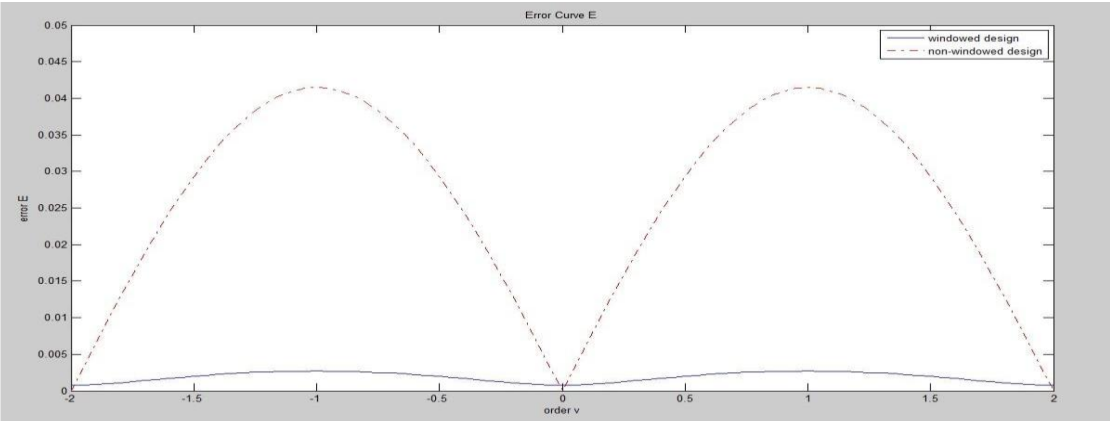

Fig.1: Arbitrary order Hilbert transformer comparison result for window and non-window techniques using DST-II

Example 2: Fig. 2 represents a comparative result between magnitude and phase response for window and non-window techniques. Dark line represents the frequency response of window design and dotted line represent the frequency response of non- window design. Frequency response of the non-window design is slightly different at both ends as compared to the frequency response of the window design method.

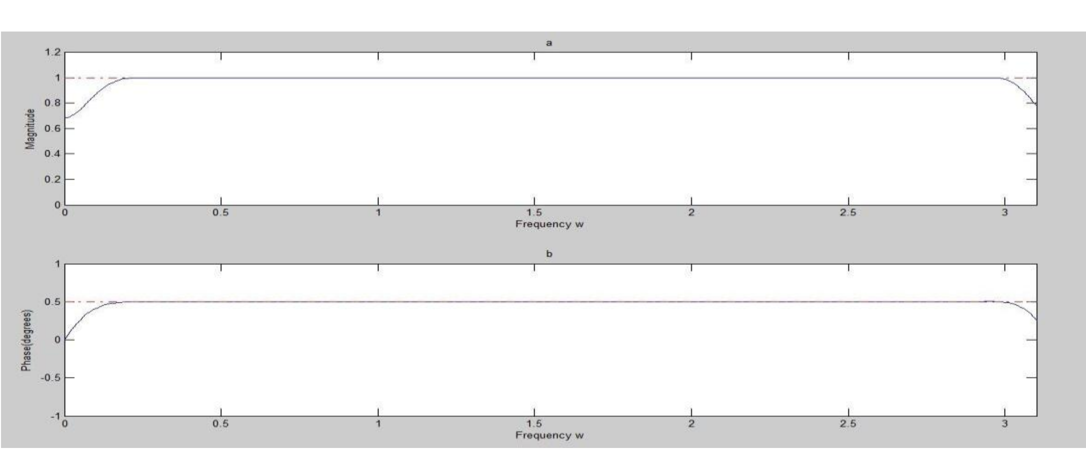

Fig. 2: Comparison between magnitude and phase response for window and non-window technique

Example 3: Fig. 3 shows the comparison between arbitrary order Hilbert transformer frequency response and ideal frequency response for window and non-window design methods. Dark line shows the ideal response and the dotted line shows the F-R of proposed method. F-R of proposed method using window techniques overlaps with F-R of ideal design method. Therefore, the proposed window design method is well suited with the ideal method. Fig. 3(a) and 3(b) show F-R of ideal design method while 3(c) and 3(d) show F-R of proposed method. On the basis of figures for given optimal design values Hilbert transformer method (proposed method) using DST-II performs better as compared with ideal method.

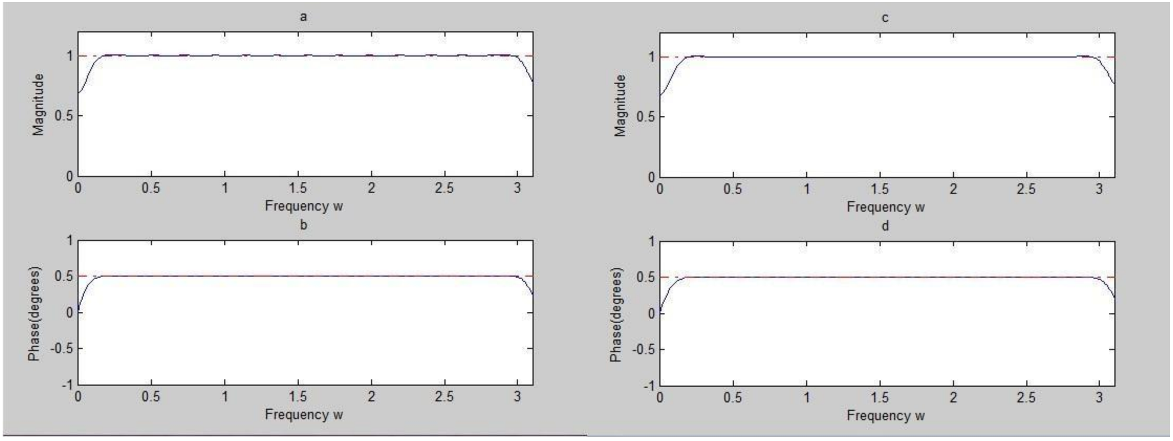

Fig.3: Comparison between F-R of the ideal design and the proposed method

3.2 Design using DST-I

$$Z(q) = \sqrt{\frac{2}{L+1}} \sum_{p=0}^{L-1} z(p) \sin\left(\frac{(p+1)(q+1)\pi}{L+1}\right) \tag{19}$$

Inverse function of $Z(q)$ can be written as

$$z(p) = \sqrt{\frac{2}{L+1}} \sum_{q=0}^{L-1} Z(q) \sin\left(\frac{(p+1)(q+1)\pi}{L+1}\right) \tag{20}$$

For DST-I method Hilbert transform filter co-efficient is

$$h(u) = \frac{2}{L+1} \sum_{q=0}^{L-1} \sin\left(\frac{(L-u-1)(q+1)\pi}{L+1}\right) \sin\left(\frac{(L-I-1)(q+1)\pi}{L+1} - \frac{\alpha\pi}{2}\right) \tag{21}$$

3.3 Design using DST-III

$$Z(q) = \sqrt{\frac{2}{L}} \sum_{p=0}^{L-1} c_q\, z(p) \sin\left(\frac{(p+1)(2q+1)\pi}{2L}\right) \tag{22}$$

Inverse function of $z(q)$ can be written as

$$Z(p) = \sqrt{\frac{2}{L}} \sum_{q=0}^{L-1} c_p\, z(q) \sin\left(\frac{(p+1)(2q+1)\pi}{2L}\right) \tag{23}$$

$$c_q = c_p = \begin{cases} \frac{1}{\sqrt{2}} & if\ 0 \le q \le L-1 \\ 1 & otherwise \end{cases}$$

For DST-III method Hilbert filter co-efficient is

$$h(u) = \frac{2}{L} \sum_{q=0}^{L-1} \sin\left(\frac{(L-u)(2q+1)\pi}{2L}\right) \sin\left(\frac{(L-I)(2q+1)\pi}{2L} - \frac{\alpha\pi}{2}\right) \tag{24}$$

3. 4 Design using DST-IV

$$Z(q) = \sqrt{\frac{2}{L}} \sum_{p=0}^{L-1} z(p) \sin\left(\frac{(2p+1)(2q+1)\pi}{4L}\right) \tag{25}$$

Inverse function of $Z(q)$ can be defined as

$$z(p) = \sqrt{\frac{2}{L}} \sum_{q=0}^{L-1} Z(q) \sin\left(\frac{(2p+1)(2q+1)\pi}{4L}\right) \tag{26}$$

For DST-IV method Hilbert transform filter co-efficient is

$$h(u) = \frac{2}{L}\sum_{q=0}^{L-1} sin\left(\frac{(2L-2u-1)(2q+1)\pi}{4L}\right) sin\left(\frac{(2L-2I-1)(2q+1)\pi}{4L} - \frac{\alpha\pi}{2}\right) \quad (27)$$

4. Design Examples

The following figures 4,5,6, and 7 shows the frequency response among the various DST method using Hilbert transformer for different arbitrary order i.e., for 0.3, 0.5 0.7 and 0.9. The following figures are drawn with help of integral square formula of error (eq. 3.1.18), for given optimal design values i.e., sample size L = 60, delay value I = 40, arbitrary order α = 0.3, 0.5 0.7 and 0.9. The red line shows the ideal magnitude response for various arbitrary order while the blue line shows the magnitude response of different DST methods of arbitrary order.

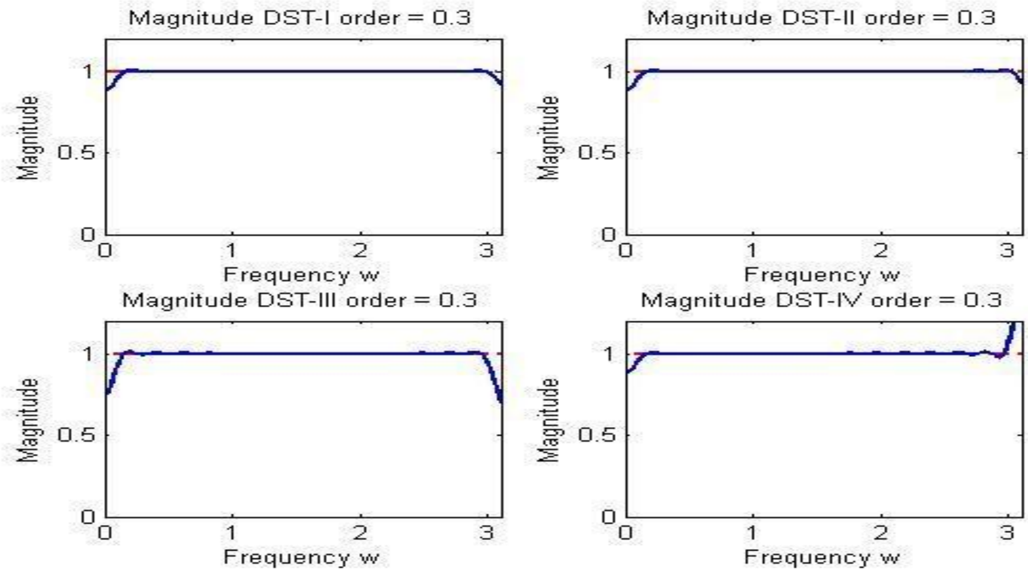

Fig.4: Magnitude response for $\alpha = 0.3$

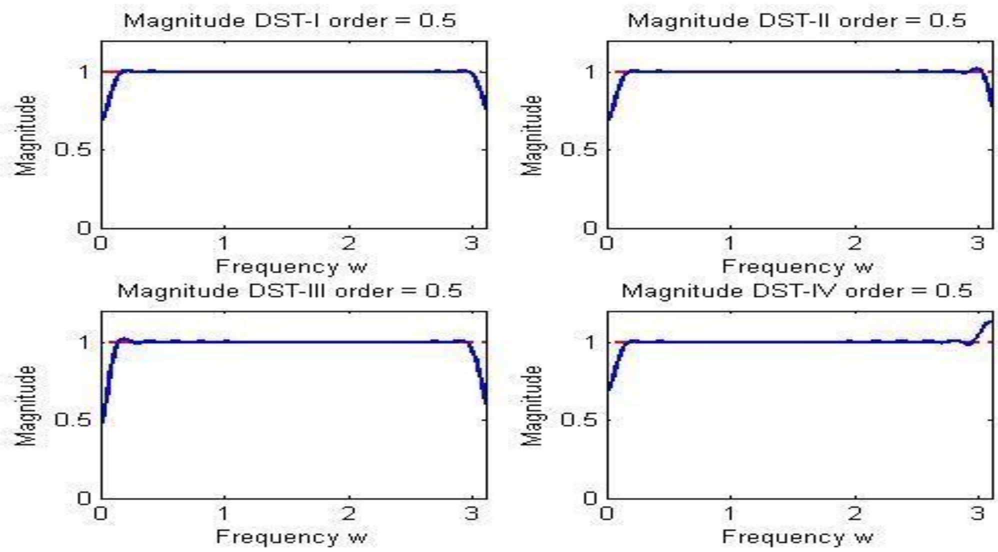

Fig.5: Magnitude response for $\alpha = 0.5$

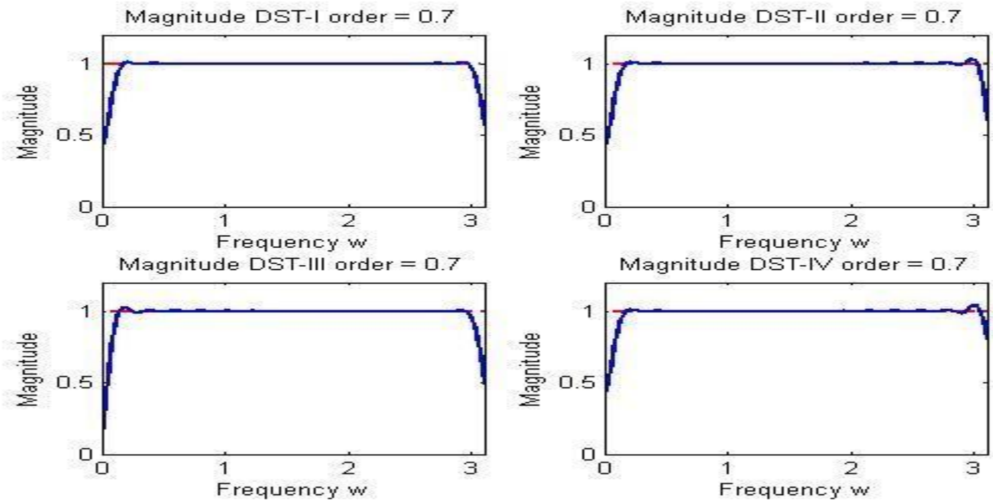

Fig.6: Magnitude response for $\alpha = 0.7$

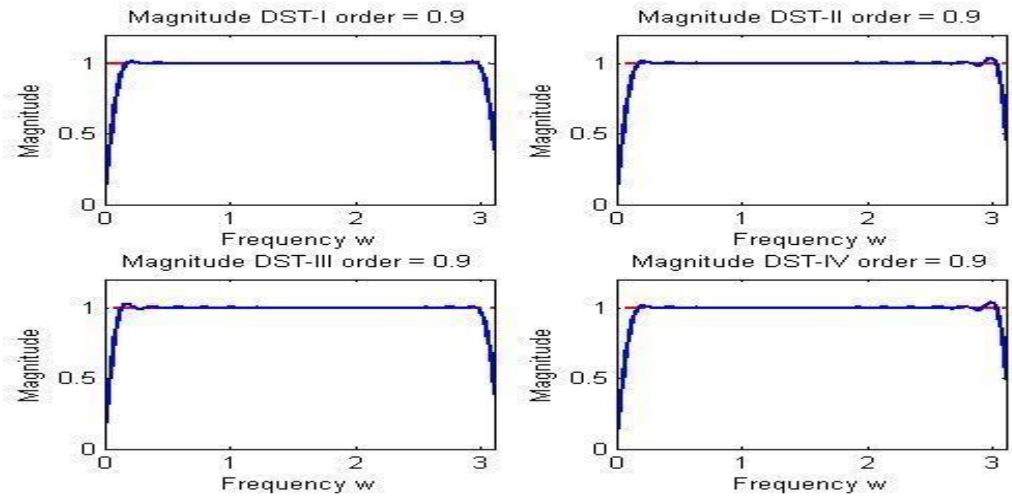

Fig.7: Magnitude response for $\alpha = 0.9$

5. Conclusion

In this paper arbitrary order Hilbert transformer designing approach of window technique for DST-II methods and frequency response for various DST methods for different arbitrary order are presented. Window design methods are well suited for the arbitrary order Hilbert differentiator design. Hilbert transformer method (proposed method) using DST performs better for window design method. We can also extend this result to all discrete sine and discrete cosine transforms with respect to different fractional derivatives like, Riemann-Liouville, Caputo, Reisz, Grunwald-Letinkov, Weyl's etc. This work can also be extended to 2D discrete sine and discrete cosine transforms.

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