A Systematic Analysis of a Novel Hankel Transform Wavelet Framework and Its Complexity Implications

Nagma Irfan[1], Masood Alam[2], *Khursheed Alam[3]

[1] Uttarakhand Technical University, India (nagmairfanmath@gmail.com)

[2]Department of Mathematics and IT Center for Preparatory Studies Sultan Qaboos University Muscat, Oman (malam@squ.edu.om)

[3]Department of Mathematics and The A. H. Siddiqi Centre for Advanced Research in Applied Mathematics & Physics (CARAMP), Sharda University, India (khursheed.alam@sharda.ac.in)

***Corresponding Author**: Khursheed Alam (khursheed.alam@sharda.ac.in)

Abstract

In this paper, a comprehensive analysis attempted to study the application of classical wavelets in Hankel Transform's numerical computation by systematically reviewing the previous studies on this subject, which have been published in recent years, and identifying areas of research for future development. We discussed the mathematical foundations of the Hankel transform, the wavelet transforms, and their integration to form a Hankel transform wavelet framework. We provide numerical examples to validate the efficiency of the recommended framework in signal and image processing. Finally, we discuss the potential future implications of this framework in various fields, including medical imaging, acoustic and electromagnetic wave propagation, and more. The research employed the systematic review approach. The results for Hankel Transform digital calculation using various classical wavelets have been explored, several research gaps have been pointed out, and possible implications for this topic including chaotic time series prediction are recommended.

Mathematics Subject Classifications: 44A15, 65R10

Keywords: Bessel functions, Hankel transforms, Wavelets, CAS Wavelets, Sine-cosine wavelets, Chaotic time series.

Introduction

Wavelet transforms have been an active area of research in the last few decades, providing a powerful mathematical tool for analyzing signals and images in various domains. Wavelet transforms have been widely used in image processing, signal processing, pattern recognition, and chaos theory. On the other hand, the Hankel transform is a prevailing mathematical instrument for studying cylindrical and spherical objects. Hankel transform is broadly used in optics, acoustics, and electromagnetic wave transmission. Recently, researchers have integrated the Hankel transform and wavelet transform to form a new framework called Hankel transform wavelet framework. The proposed framework can be used to analyze signals and images with cylindrical or spherical symmetry. The framework has shown great potential in various fields such as medical imaging, material science, Chaotic time series(Feng,2019), and environmental science. This paper systematically studies the Hankel transform wavelet framework and its future implications.

A Bessel kernel is included in one dimension of the functional transform known as the Hankel transform. Additionally, it provides the radial solution to any dimension's angular Fourier symmetrical transformation, which is incredibly helpful in a variety of applications. In the domains of space science, geophysical science, fluid mechanics, electrical dynamics, thermodynamics, and audibility, the NASA Astronomical Data Service generated more than 700 papers that included the word "Hankel transform." In this essay, we look at the basic definitions and characteristics of a few terminologies. In contrast to the Fourier transformation, the calculation of similar problems with the Hankel transformation offers the conceptual advantage of decreasing the problem's dimensionality in unity unrelatedly to the initial dimension. This can be an effective instrument to overcome the transformation analytically in the study. This naively seeks to improve efficiency numerically.

In the last few decades, Scientists and researchers have expanded their use of wavelet technology and numerical evaluation to solve both linear and non-linear failures. In addition, Wavelets connect quick numerical algorithms including chaotic time series(Orthogonal polynomials have drawn a lot of interest in solving a variety of practical issues. Siddiqi (2018) researched wavelet applications for the numerical models, and partial differential equations, demonstrating challenges that occur practically, signals, and image handling, particularly for medicinal signals like EEG and ECG. The Wavelet approach has also been used for pattern recognition, weather research and forecasting, identifying airplanes and submarines, and more (Siddiqi, 2012). We are conducting a systematic analysis of the prior research on this topic in this paper.

The Objectives of the Systematic Review

This pilot study had the following objectives:

- To explore the status of research studies on numerical computation of Hankel Transforms.
- Investigate the contribution of those studies in improving the results.
- Identify research gaps in this research and where further study should be conducted.

Preliminaries

Wavelets

A singleton function known as the mother wavelet is used to build the class of functions known as wavelets. The following family of continuous wavelets can be created by continuously varying the dilation a and translation b parameters.

$$W\_(a,b)\ (t) = |a|\hat{}(-1\ /\ 2)\ ((t-b)/a), \qquad a,b \in R, a \neq 0.$$

Restricting the parameters a and b to discreet values as $a = 2^{-k}, b = n2^{-k}$,

$a = 2^{(-k)}, b = n2^{(-k)}$,

We got a family of discrete wavelets.

$$W_{kn}(t) = 2^{k/2} W(2^k t - n), \qquad k,n \in Z,$$

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where the function W(the mother wavelet), satisfies $\int RW(t)\,dt = 0$.

Haar constructed the orthonormal basis of compactly supported wavelets for (Daubechies,1992, Chui,1992)$L^2(R)$ for the first time in 1910.

CAS Wavelets

CAS Wavelets $\psi_{nm}(t) = \psi(k,n,m,t)$ involve four arguments $n,k,m$and$t$, (Nagma2015) :

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} CAS_m(2^k t - n), for\ \frac{n}{2^k} \le t < \frac{n+1}{2^k}, \\ 0, otherwise, \end{cases}$$

where $CAS_m(t) = cos(2m\pi t) + sin(2m\pi t)$.

(Nagma,2015) proposed an efficient process for the Fourier-Bessel transform and applied CAS wavelets which was not proposed earlier in any of research.

Sine-Cosine wavelets

Sine-cosine wavelets $\psi_{n,m}(t) = \psi(n,k,m,t)$ involves four arguments;$n = 0,1,2,\dots, 2^k - 1, k = 0,1,2,$ .....(Nagma 2016). Sine-Cosine wavelets are defined on the interval [0, 1) as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} f_m(2^k t - n), \frac{n}{2^k} \le t < \frac{n+1}{2^k}, \\ 0 otherwise \end{cases}$$

with $$f_m(t) = \begin{cases} \frac{1}{\sqrt{2}}, m = 0 \\ cos(2m\pi t), m = 1,2, \dots\dots\dots, L, \\ sin(2(m-L)\pi t), m = L+1, L+2, \dots\dots\dots, 2L, \end{cases}$$

It is evident that the set of Sine-cosine wavelets also has an orthonormal basis for $L^2([0,1])$ that is formed by the collection of Sine-Cosine wavelets. In a published paper, (Nagma, 2016) proposed an effective and quick method to calculate the Hankel transform based on Sine-cosine wavelets. They claim that the Sine-Cosine Wavelet Method is particularly user-friendly and appealing. Comparing the new methodology to the approaches already in use, it is more practical and accurate. Their assertion is supported by the numerical example and the contrasted outcomes. To prove the precision of numerical calculations, for few cases the difference between the precise and approximate solutions was graphically represented with error estimation.

Hankel Transform-a definition and review

Hermann Hankel is credited with developing the Hankel transform, which links to Bessel functions in spherical or cylindrical dimensions. The Fourier-Bessel transform is another name for it. Sneddon created the Finite Hankel transform in 1972. Hankel transforms are quite useful for problems with circular symmetry. The Hankel transform can be used to convert Laplace's partial differential equation in cylindrical coordinates into an ordinary differential equation. The Hankel transform is crucial in the processing of optical data since it is a two-dimensional Fourier transformation of a circularly symmetric function. Hankel transformations have also been shown to be very useful for resolving problems in a variety of other domains, including seismology, electro-scattering, acoustics, hydrodynamics, image processing, and geophysics Kisselev(2018). The Hankel transform (HT), which is applied to the mathematical analysis of radiation, diffraction, and field projection, is very useful for analyzing wave fields. Numerous engineering and academic disciplines have used the Hankel transform. The production of diffusion profiles and diffraction patterns, reconstructions in tomography and imaging, boundary value issues, beam and wave propagation, and other applications are only a few examples. Since the Hankel transform of zeroth order is a 2D Fourier transform of a rotationally symmetric function, the Hankel transform and the Fourier transform are just identical. Both the 2D Fourier transform and the spherical Hankel transform include the Hankel transform. The 3D Fourier transform's definition includes the spherical Hankel transform. The spherical Hankel transform is also mentioned in Natalie (2019)'s explanation of the 3D Fourier transform in spherical polar coordinates.

When analyzing wave fields, the Hankel transform (HT), which is used in the mathematical treatment of radiation, diffraction, and field projection, is quite helpful.

Many engineering and academic disciplines have used the Hankel transform. Applications include the creation of diffusion profiles and diffraction patterns, reconstructions in tomography and imaging, boundary value problems, beam and wave propagation, and more. Hankel transforms can be applied to networks that produce the Bessel differential equation or variants of this equation since they are symmetric, unlike the Laplace and Mellin transform, and the transformed variable is a real, not a complex variable. As the Hankel transform of order zero is a 2D Fourier transform of a rotationally symmetric function, So the Hankel transform and the Fourier transform are simply comparable. The 2D Fourier transform and the spherical Hankel transform both include the Hankel transform as a component. The 3D Fourier transform's definition also includes the spherical Hankel transform. This could be a helpful tool in the transformation's analysis. It makes false claims about increasing productivity in numbers.

Mathematical Background

Definition:

A function defined in the Cartesian coordinate system is related to a function described in the cylindrical coordinate system by the Hankel transform, a sort of integral transform. The definition of the generic Hankel transform pair with kernel K_ν is defined as

$$\boldsymbol{H_\nu(p) = \int_0^\infty r h(r) K_\nu(pr)\, dr},$$

Hankel transform is self-reciprocal. Inverse Hankel transform is

$$h(r) = \int_0^\infty p H_\nu(p) K_\nu(pr)\, dp,$$

K_ν is the νth-order Bessel function. In the case of the Finite Hankel transform only a direct transform has an integral form. Without loss of generality its expression is

$$H_\nu(p) = \int_0^1 r h(r) K_\nu(pr)\, dr$$

In past numerous investigation has been done for the numerical computation of the Hankel transform for both order zero[Zykov(2004),), Barakat et al. (1998), Yu et al. (1998), Markham et al. (2003)}and for high-order Hankel transform as well [Murphy et al. (2003), Agnesi et al. (1993), Suter et al. (2001), Postnikov (2003)].

Unfortunately, the efficiency of a technique for computing the Hankel transform strongly depends on the function that needs to be modified, making it difficult to choose the optimum algorithm for a particular function. By Barakat et al. in 1998, the zero-order Hankel transform was evaluated using the Filon quadrature methodology. They separated the integrand into a component product that changes slowly and a component product that changes quickly (in this case, the former is and the latter is).This method makes computing the inverse Hankel transform more difficult because the function is no longer smooth but instead rapidly oscillating. Additionally, between 0 and 1, the error is discernible. Yu et al. in 1998 offered a different method for computing the zero-order quasi-discrete Hankel transform. Perciante (2004)'s suggested remedy means solution of nth order. The zero-order Hankel transform algorithm developed by Yu et al. was later enhanced by Guizar-Sicairos et al. (2004) to offer a potent method for computing the integer order Hankel transform. For the first time, Postnikov (2002, 2003) published a revolutionary technique for computing the zero and first order Hankel transforms using Haar wavelets. In 2008, Singh et al. created a successful method for numerical evaluation of the Hankel transform of order using linear Legendre multi-wavelets and wavelets lead, and they asserted a stable algorithm for carrying out the Hankel transform in 2010.Pandey used Haar wavelets to solve the Hankel transform in 2010.

From previous research, it is clear that for each case, the convergence and stability analysis for a certain function approximation are proven, as well as the numerical accuracy stability of the hybrid algorithm.

Wavelets Application in Numerical Computation of Hankel Transform:

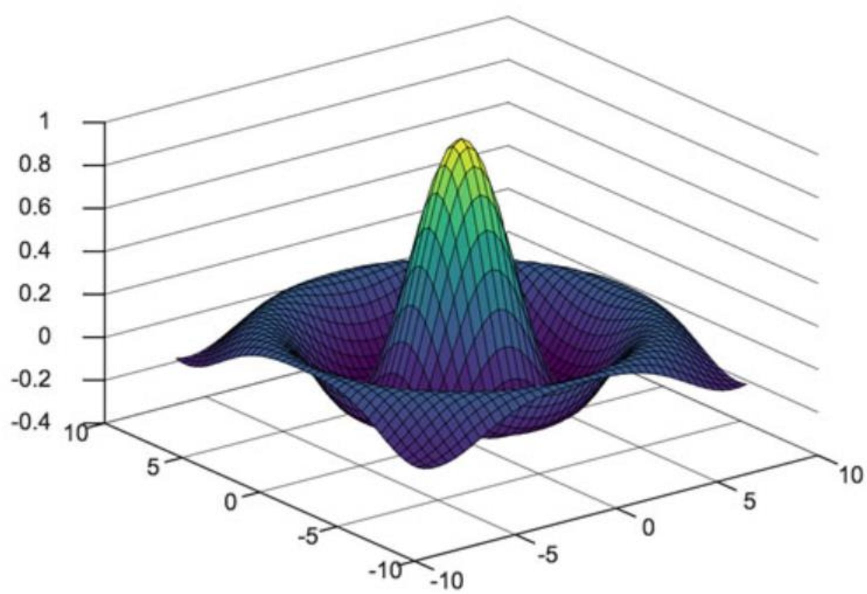
Hankel transforms are one of several mathematical problems that wavelets have been utilised to solve. Functions can be transformed from one domain (often the spatial domain) to another domain (typically the frequency domain) using the Hankel transform, a sort of integral transform. Hankel transform solutions that employ wavelets are frequently referred to as "wavelet-based numerical methods for solving Hankel transforms." These techniques discretize the integral equation using wavelets, then solve the resulting linear equation system.One advantage of using wavelets in the solution of Hankel transforms is that wavelets have good localization properties in both time and frequency domains, which makes them well-suited for representing functions that have localized features. This can result in more accurate and efficient solutions compared to other numerical methods.

Numerical examples

Exampe1: Consider a signal with cylindrical symmetry given by:

$f(r) = \exp(-r^2/2) \cos(2\pi r)$

We apply the Hankel transform wavelet framework to analyze this signal. We use the Mexican hat wavelet as the wavelet function. The Hankel transform wavelet of the signal is shown in Figure 1.The signal $f(r) = \exp(-r^2/2) \cos(2\pi r)$ has a cylindrical symmetry. The image generated from this signal would have circular shapes with oscillations in the radial direction. The circular shapes would have a Gaussian-like envelope with a maximum value at the center and gradually decreasing towards the edges. The oscillations in the radial direction would be more prominent towards the edges of the circular shapes and would have a frequency of 2π. The image would represent the features of the signal in the cylindrical coordinate system.



Sombrero Function

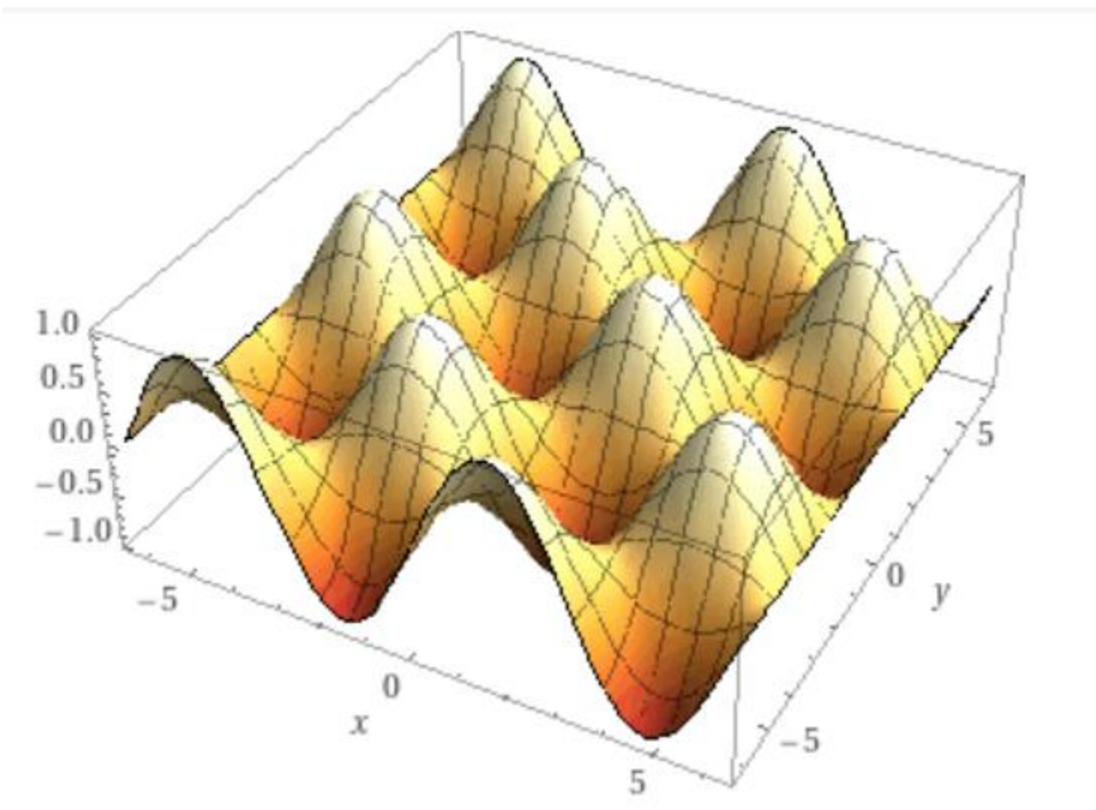
Fig1.

The Hankel transform wavelet of the signal clearly shows the location and frequency of the oscillations in the signal. The Hankel transform wavelet can be used to detect and analyze the features of the signal in the cylindrical coordinate system.

Example2: Edge Detection

Consider an image with spherical symmetry given by:

$$f(r,\theta,\varphi) = \sin(\theta)\cos(\varphi)$$



We apply the Hankel transform wavelet framework to detect the edges in the image. We use the Haar wavelet as the wavelet function. The Hankel transform wavelet of the image is shown in Figure 4. The Hankel transform wavelet of the given image with spherical symmetry can be obtained by first computing the Hankel transform of the image with respect to the radial variable r.

Future work

1. The Hankel transform wavelet framework has many potential future implications in various fields. In medical imaging, the framework can be used to analyze MRI and CT images with cylindrical and spherical symmetry. In acoustic and electromagnetic wave propagation, the framework can be used to analyze the properties of cylindrical and spherical objects, such as scattering and diffraction. In material science and environmental science, the framework can be used to analyze the properties of cylindrical and spherical particles and structures.
2. Because computational work completely supports the suggested algorithm's compatibility, it can be applied to other physical issues as well. The validity of this technique for such issues is expressly reflected by accurate percentage of results. Due to the localization aspect of the wavelet basis, the approximate solution incorporates both time and frequency information. With some modifications, this approach can be implemented on different real world problems with the aid of different wavelet basis which were neglected in earlier investigations.
3. By using wavelet or hybrid function Bernoulli polynomials to expand the integral, the projected technique can be converted into a broad class.
4. Additional Wavelets that are accessible can be used to solve the Hankel Transform, and their performance can be compared to determine which is superior.
5. **Numerical estimation of Hankel transform by using Haar Vilenkin wavelets:** it may be a good idea to develop an algorithm based on Haar Vilenkin wavelets to numerically evaluate Hankel transform.
6. **Numerical evaluation of HT with shearlets:** One can develop new numerical technique to solve Hankel transform using Shearlets.
7. In earlier cases Mathcad and Mathematica software is used to find out computational graphs and results other mathematical software like MatLab & Python could be used and compared for better graphs.
8. Error analysis part could be extended to show theoretical framework of results.

Results and Discussions:

In this investigation, we presented a systematic study of the Hankel transform wavelet framework and its future implications. We discussed the mathematical foundations of the Hankel transform, wavelet transform, and their integration to form the Hankel transform wavelet framework. We provided numerical examples to demonstrate the effectiveness of the framework in signal and image processing. Finally, we discussed the potential future implications of this framework in various fields. The Hankel transform wavelet framework is a powerful mathematical tool for analyzing signals and images with cylindrical or spherical symmetry and has many potential future implications in various fields.

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References

[1] A.H. Siddiqi, 2012, Wavelets in oil industry, AIP Conference Proceedings, vol. 1463, p. 52.
[2] A.V. Kisselev(2018), Ramanujan's Master Theorem and two formulas for zero-order Hankel transform, arXiv:1801.06390 [math.CA]
[3] Agnesi A, Reali GC, Patrini G, Tomaselli A (1993) Numerical evaluation of the Hankel transform: remarks. J Opt Soc Am A 10:1872–1874. doi:10.1364/JOSAA.10.001872
[4] AH Siddiqi(2018) , Wavelet Theory, Functional Analysis and Applications, Springer 399-463
[5] B.W. Suter(2001) , Fast n th order Hankel transform algorithm, IEEE Trans. Signal Process. 39 532–536 .
[6] Baddour, Natalie. (2019). The Discrete Hankel Transform. Fourier Transforms - Century of Digitalization and Increasing Expectations. 10.5772/intechopen.84399.
[7] C.K.Chui., Wavelets: A Tutorial in Theory and Applications, Academic Press, New York, 1992.
[8] E. B. Postnikov (2002), “About calculation of the Hankel transform using preliminary wavelet transform,” Journal of Applied Mathematics, vol. 6, pp. 319–325.
[9] E.B.Postnikov (2003), About calculation of the Hankel transform using preliminary wavelet transform, Journal of Applied Mathematics,6,319-325 http://dx.doi.org/10.1364/JOSA.73.001130
[10] I.Daubechies (1992), Ten Lectures on Wavelets, CBMS-NSF.
[11] I.N. Sneddon, The use of Integral Transforms. McGraw-Hill, 1972.
[12] M.Guizar-Sicairos and J.C.Gutierrez-vega(2004), Computation of quasi discrete Hankel transforms of integer order for propagating optical wave fields, J.Opt. Soc. Am. A, Col-21, 1, 53. doi:10.1364/JOSAA.21.000053
[13] Markham J, Conchello JA (2003) Numerical evaluation of Hankel transform for oscillatingfunction. J Opt Soc Am A 20(4):621–630. doi:10.1364/JOSAA.20.000621
[14] Mohamed ahmed, Boudref. (2019). Titchmarsh's theorem of Hankel transform. Daghestan Electronic Mathematical Reports. 10.31029/demr.11.2.
[15] Nagma Irfan & S.Kapoor(2011)Quick Glance on Different Wavelets And Their Operational Matrix Properties, International Journal of Research and Reviews in Applied Sciences (IJRRAS),ISSN-2076-7366, Vol-8, Issue1 , pp. 65-78.
[16] Nagma Irfan and A.H.Siddiqi (2015),Application of CAS wavelets in Numerical Evaluation of Hankel Transform occurring in Seismology, Mathematical Models, Methods and Applications, pp 285-298. DOI 10.1007/978-981-287-973-8_17

[17] Nagma Irfan and A.H.Siddiqi(2015), A Novel computational Hybrid approach In Solving Hankel Transform, Applied mathematics and computation,Volume281,pp.121-129.ISSN:0096-3003. DOI:10.1016/j.amc.2016.01.028.

[18] Nagma Irfan and A.H.Siddiqi,(2015),A wavelet algorithm for Fourier-Bessel Transform arising in optics", International Journal of Engineering Mathematics , Article ID 789675, 9pages.http://dx.doi.org/10.1155/2015/789675 .

[19] Nagma Irfan and A.H.Siddiqi,(2015),An Application of Wavelet Technique in Numerical Evaluation of Hankel Transform, International Journal of Nonlinear sciences and numerical simulation , 16(6), 293-299. DOI 10.1515/ijnsns-2015-0031.

[20] Nagma Irfan, A.H.Siddiqi (2016), Sine-Cosine Wavelets Approach in Numerical Evaluation of Hankel Transform for Seismology, Applied MathematicalModelling,Volume40,Issue(7-8),pp.4900-4907, ISSN: 0307-904X.DOI: 10.1016/j.apm.2015.12.019

[21] P. S. Zykov and E. B. Postnikov (2004), "Application of the wavelet transform with a piecewise linear basis to the evaluation of the Hankel transform," Computational Mathematics and Mathematical Physics, vol. 44, no. 3, pp. 396–400.

[22] P.K. Murphy and N.C. Gallagher (2003), Fast algorithm for computation of zero-order Hankel transform, J. Opt. Soc. Am, 73, 1130-1137.

[23] Perciante, Cesar & Ferrari, Jose. (2004). Fast Hankel transform of nth order with improved performance. Journal of the Optical Society of America. A, Optics, image science, and vision. 21. 1811-2. 10.1364/JOSAA.21.001811.

[24] R. Barakat, E. Parshall, and B. H. Sandler (1998), "Zero-order Hankel transformation algorithms based on Filon quadrature philosophy for diffraction optics and beam propagation," Journal of the Optical Society of America A, vol. 15, no. 3, pp. 652–659.

[25] R.K. Pandey, O.P. Singh, V.K. Singh (2010), A Stable algorithm for numerical evaluation of Hankel transforms using Haar Wavelets, Numerical Algorithms, 53.451-466. doi: 10.1007/s11075-009-9313-0

[26] Steven G. Murray, Francis J. Poulin, (2019). Hankel: A Python library for performing simple and accurate Hankel transformations, 4(37), 37.https://doi.org/10.21105/joss.01397

[27] T. Feng, S. Yang, and F. Han, (2019), "Chaotic time series prediction using wavelet transform and multi-model hybrid method," Journal of Vibroengineering, Vol. 21, No. 7, pp. 1983–1999. https://doi.org/10.21595/jve.2019.20579

[28] V.K. Singh , R.K. Pandey , S. Singh(2010) , A stable algorithm for Hankel transforms using hybrid of Block-pulse and Legendre polynomials, Comput. Phys. Commun. 181, 1–10 .

[29] V.K. Singh, O.P. Singh and R.K. Pandey (2008), Numerical evaluation of the Hankel transform by using linear Legendre multi-wavelets, Comput. Phys. Comm, 179, 424-429. doi:10.1016/j.cpc.2008.04.006

[30] V.K.Singh, O.P. Singh and R.K.Pandey(2008), Efficient algorithms to compute Hankel transforms using wavelets, Compu. Phys. Commun, 179, 812-818. doi:10.1016/j.cpc.2008.07.005

[31] Yu.L, M. Huang, M. Chen, W. Chen, W. Huang, and Z. Zhu(1998), "Quasi-discrete Hankel transform," Optics Letters, vol. 23, no. 6, pp. 409–411.