On modified degenerate poly-tangent numbers and polynomials

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Abstract : In this paper we introduce the modified degenerate degenerate poly-tangent polynomials and numbers. We also give some properties, explicit formulas, several identities, a connection with modified degenerate poly-tangent numbers and polynomials, and some integral formulas. Finally, we investigate the zeros of the modified degenerate poly-tangent polynomials by using computer.

Key words : Tangent numbers and polynomials, degenerate poly-tangent numbers and polynomials, Cauchy numbers, Stirling numbers, modified degenerate poly-tangent polynomials.

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1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials, poly-Bernoulli numbers and polynomials, poly-Euler numbers and polynomials(see [1-11]). In this paper, we define modified degenerate poly-tangent polynomials and numbers and study some properties of the modified degenerate poly-tangent polynomials and numbers. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Carlitz [1] has defined the degenerate Stirling numbers of the first kind and second kind, $S_1(n,k,\lambda)$ and $S_2(n,k,\lambda)$ by means of

$$\left(\frac{1-(1-t)^\lambda}{\lambda}\right)^k = k!\sum_{n=k}^{\infty} S_1(n,k,\lambda)\frac{t^n}{n!}, \tag{1.1}$$

$$\left((1+\lambda t)^{1/\lambda}-1\right)^k = k!\sum_{n=k}^{\infty} S_2(n,k,\lambda)\frac{t^n}{n!}. \tag{1.2}$$

Howard [12] has defined the degenerate weighted Stirling numbers of the first kind and second kind, $S_1(n,k,x,\lambda)$ and $S_2(n,k,x,\lambda)$ by means of

$$(1-t)^{\lambda-x}\left(\frac{1-(1-t)^\lambda}{\lambda}\right)^k = k!\sum_{n=k}^{\infty} S_1(n,k,x,\lambda)\frac{t^n}{n!}, \tag{1.3}$$

$$(1+\lambda t)^{x/\lambda}\left((1+\lambda t)^{1/\lambda}-1\right)^k = k!\sum_{n=k}^{\infty} S_2(n,k,x,\lambda)\frac{t^n}{n!}. \tag{1.4}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k).$$

The generalized raising factorial $< x|\lambda >_n$ with increment λ is defined by

$$< x|\lambda >_n = \prod_{k=0}^{n-1}(x+\lambda k).$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}.$$

The degenerate poly-Bernoulli numbers $\mathcal{B}_n^{(k)}(\lambda)$ were introduced by Kaneko [5] by using the following generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-(1+\lambda t)^{-1/\lambda}} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(\lambda)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}), \tag{1.5}$$

where

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k} \tag{1.6}$$

is the kth polylogarithm function.

The degenerate poly-Euler polynomials $\mathcal{E}_n^{(k)}(x,\lambda)$ are defined by generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x,\lambda)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \tag{1.7}$$

The familiar degenerate tangent polynomials $\mathbf{T}_n(x,\lambda)$ are defined by the generating function([7]):

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathbf{T}_n(x,\lambda)\frac{t^n}{n!}, \quad (|2t| < \pi). \tag{1.8}$$

When $x = 0$, $\mathbf{T}_n(0,\lambda) = \mathbf{T}_n(\lambda)$ are called the degenerate tangent numbers. The degenerate tangent polynomials $\mathbf{T}_n^{(r)}(x,\lambda)$ of order r are defined by

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^r (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathbf{T}_n^{(r)}(x,\lambda)\frac{t^n}{n!}, \quad (|2t| < \pi). \tag{1.9}$$

It is clear that $r = 1$ we recover the degenerate tangent polynomials $\mathbf{T}_n(x,\lambda)$.

The degenerate Bernoulli polynomials $\mathbf{B}_n^{(r)}(x,\lambda)$ of order r are defined by the following generating function

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^r (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathbf{B}_n^{(r)}(x,\lambda)\frac{t^n}{n!}, \quad (|t| < 2\pi). \tag{1.10}$$

The degenerate Frobenius-Euler polynomials of order r, denoted by $\mathbf{H}_n^{(r)}(u,x,\lambda)$, are defined as

$$\left(\frac{1-u}{(1+\lambda t)^{1/\lambda}-u}\right)^r (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}. \tag{1.11}$$

The values at $x = 0$ are called degenerate Frobenius-Euler numbers of order r; when $r = 1$, the polynomials or numbers are called ordinary degenerate Frobenius-Euler polynomials or numbers.

The degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are defined by the generating function:

$$\frac{2\mathrm{Li}_k\left(1-e^{-t}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \tag{1.12}$$

When $x = 0$, $\mathcal{T}_n^{(k)}(0,\lambda) = \mathcal{T}_n^{(k)}(x,\lambda)$ are called the degenerate poly-tangent numbers. Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [1-12]). In the following section, we introduce the modified degenerate poly-tangent polynomials and numbers. After that we will investigate some their properties. We also give some relationships both between these polynomials and modified degenerate poly-tangent polynomials and between these polynomials and cauchy numbers. Finally, we investigate the zeros of the modified degenerate poly-tangent polynomials by using computer.

2. Modified degenerate poly-tangent polynomials

In this section, we define modified degenerate poly-tangent numbers and polynomials and provide some of their relevant properties.

The modified degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are defined by the generating function:

$$\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!},\quad (k\in\mathbb{Z}). \tag{2.1}$$

When $x=0$, $\mathcal{T}_n^{(k)}(0,\lambda)=\mathcal{T}_n^{(k)}(x,\lambda)$ are called the degenerate poly-tangent numbers. Upon setting $k=1$ in (2.1), we have

$$\mathcal{T}_n^{(1)}(x,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\lambda^{n-1}S_1(l,1)\mathbf{T}_{n-l}(x,\lambda)\text{ for }n\geq 1.$$

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda}\\&=\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

By comparing the coefficients on both sides of (2.2), we have the following theorem.

Theorem 2.1. For $n\in\mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(\lambda)(x|\lambda)_{n-l}.$$

The following elementary properties of the degenerate poly-tangent numbers $\mathcal{T}_n^{(k)}(\lambda)$ and polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are readily derived form (2.1). We, therefore, choose to omit details involved.

Theorem 2.2. For $k\in\mathbb{Z}$, we have

$$(1)\quad \mathcal{T}_n^{(k)}(x+y,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)(y|\lambda)_{n-l}.$$

$$(2)\quad \mathcal{T}_n^{(k)}(2-x,\lambda)=\sum_{l=0}^{n}(-1)^l\binom{n}{l}\mathcal{T}_{n-l}^{(k)}(2,\lambda)<x|\lambda>_l\ .$$

Theorem 2.3 For any positive integer n, we have

$$\begin{aligned}&(1)\quad \mathcal{T}_n^{(k)}(mx,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)((m-1)x|\lambda)_{n-l}.\\&(2)\quad \mathcal{T}_n^{(k)}(x+1,\lambda)-\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{n-1}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)(1|\lambda)_{n-l}.\end{aligned} \tag{2.3}$$

From (1.6), (1.8), and (2.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}=\left(2\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda}\\
&=\sum_{l=0}^{\infty}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{l+1}}{(l+1)^k}\frac{2(1+\lambda t)^{x/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\\
&=\sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i\frac{2(1+\lambda t)^{x/\lambda}(1+\lambda t)^{-i/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\\
&=\sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathbf{T}_j(x,\lambda)(-1)^{n-j}<i|\lambda>_{(n-j)}\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n}\frac{1}{(l+1)^k}\binom{l+1}{i}(-i)^{n+i-j}\binom{n}{j}\mathbf{T}_j(x,\lambda)<i|\lambda>_{(n-j)}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.4}$$

By comparing the coefficients on both sides of (2.4), we have the following theorem.

Theorem 2.4 For $n\in\mathbb{Z}_+$, we have

$$\begin{aligned}
\mathcal{T}_n^{(k)}(x,\lambda)&=\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n}\frac{(-i)^{n+i-j}}{(l+1)^k}\binom{l+1}{i}\binom{n}{j}\mathbf{T}_j(x,\lambda)<i|\lambda>_{(n-j)}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\frac{(-i)^i}{(l+1)^k}\binom{l+1}{i}\mathbf{T}_n(x-i,\lambda).
\end{aligned}$$

By (2.1), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}=2\sum_{l=0}^{\infty}(-1)^l(1+\lambda t)^{2l/\lambda}\sum_{l=0}^{\infty}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{l+1}}{(l+1)^k}(1+\lambda t)^{x/\lambda}\\
&=2\sum_{l=0}^{\infty}\sum_{i=0}^{l}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{i+1}}{(i+1)^k}(-1)^{l-i}(1+\lambda t)^{(2l-2i)/\lambda}(1+\lambda t)^{x/\lambda}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{2(-1)^{l+j-i}\binom{i+1}{j}}{(i+1)^k}(1+\lambda t)^{(2l-2i+x)/\lambda}(1+\lambda t)^{-j/\lambda}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\sum_{m=0}^{n}\frac{2(-1)^{l+j-i}\binom{i+1}{j}\binom{n}{m}(2l-2i+x|\lambda)_m<j|\lambda>_{(n-m)}}{(i+1)^k}\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we have the following theorem.

Theorem 2.5 For $n\in\mathbb{Z}_+$, we have

$$\begin{aligned}
\mathcal{T}_n^{(k)}(x,\lambda)&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\sum_{m=0}^{n}\frac{2(-1)^{l+j-i}\binom{i+1}{j}\binom{n}{m}(2l-2i+x|\lambda)_m<j|\lambda>_{(n-m)}}{(i+1)^k}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{2(-1)^{l+j-i}\binom{i+1}{j}(2l-2i-j+x|\lambda)_m}{(i+1)^k}.
\end{aligned}$$

3. Some identities involving degenerate poly-tangent numbers and polynomials

In this section, we give several combinatorics identities involving degenerate poly-tangent numbers and polynomials in terms of degenerate Stirling numbers, generalized falling factorial functions, generalized raising factorial functions, Beta functions, degenerate Bernoulli polynomials of higher order, and degenerate Frobenius-Euler functions of higher order.

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(1-(1-(1+\lambda t)^{-1/\lambda})\right)^{-x}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-(1+\lambda t)^{-1/\lambda})^l\\
&=\sum_{l=0}^{\infty}<x>_l\frac{((1+\lambda t)^{1/\lambda}-1)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{-l/\lambda}\right)\\
&=\sum_{l=0}^{\infty}<x>_l\sum_{n=0}^{\infty}S_2(n,l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(-l,\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}(-l,\lambda)<x>_l\right)\frac{t^n}{n!},
\end{aligned}\tag{3.1}$$

where $<x>_l=x(x+1)\cdots(x+l-1)(l\geq 1)$ with $<x>_0=1$.

By comparing the coefficients on both sides of (3.1), we have the following theorem.

Theorem 3.1 For $n\in\mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}(-l,\lambda)<x>_l.$$

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(1-(1-(1+\lambda t)^{-1/\lambda})\right)^{-x}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-(1+\lambda t)^{-1/\lambda})^l\\
&=\sum_{l=0}^{\infty}<x>_l\frac{(1+\lambda t)^{-l/\lambda}((1+\lambda t)^{1/\lambda}-1)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\sum_{l=0}^{\infty}<x>_l\sum_{n=0}^{\infty}S_2(n,l,-l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,-l,\lambda)\mathcal{T}_{n-i}^{(k)}(\lambda)<x>_l\right)\frac{t^n}{n!},
\end{aligned}\tag{3.2}$$

where $<x>_l=x(x+1)\cdots(x+l-1)(l\geq 1)$ with $<x>_0=1$.

By comparing the coefficients on both sides of (3.2), we have the following theorem.

Theorem 3.2 For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,-l,\lambda)\mathcal{T}_{n-i}^{(k)}(\lambda) < x >_l .$$

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!} &= \left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(((1+\lambda t)^{1/\lambda}-1)+1\right)^x \\
&= \frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x}{l}\left((1+\lambda t)^{1/\lambda}-1\right)^l \\
&= \sum_{l=0}^{\infty}(x)_l\frac{\left((1+\lambda t)^{1/\lambda}-1\right)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right) \qquad (3.3)\\
&= \sum_{l=0}^{\infty}(x)_l\sum_{n=0}^{\infty}S_2(n,l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_lS_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides of (3.3), we have the following theorem.

Theorem 3.3 For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_lS_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}.$$

By (1.2), (1.10), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!} \\
&= \left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} \\
&= \frac{((1+\lambda t)^{1/\lambda}-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!} \\
&= \frac{((1+\lambda t)^{1/\lambda}-1)^r}{r!}\left(\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\right)\frac{r!}{t^r} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r,\lambda)\sum_{i=0}^{n-l}\binom{n-l}{i}\mathbf{B}_i^{(r)}(x,\lambda)\mathcal{T}_{n-l-i}^{(k)}(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.4 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{n}\sum_{i=0}^{n-l}\frac{\binom{n}{l}\binom{n-l}{i}}{\binom{l+r}{r}}S_2(l+r,r)T_{n-l-i}^{(k)}\mathbf{B}_i^{(r)}(x,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&=\frac{((1+\lambda t)^{1/\lambda}-u)^r}{(1-u)^r}\left(\frac{1-u}{(1+\lambda t)^{1/\lambda}-u}\right)^r(1+\lambda t)^{x/\lambda}\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\\
&=\sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}\sum_{i=0}^{r}\binom{r}{i}(1+\lambda t)^{i/\lambda}(-u)^{r-i}\frac{1}{(1-u)^r}\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\\
&=\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(i,\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{l=0}^{n}\binom{n}{l}\mathbf{H}_l^{(r)}(u,x,\lambda)\mathcal{T}_{n-l}^{(k)}(i,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.5 For $n\in\mathbb{Z}_+$ and $r\in\mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathbf{H}_l^{(r)}(u,x,\lambda)\mathcal{T}_{n-l}^{(k)}(i,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\frac{(1+\lambda t)^{1/\lambda}+1}{(1+\lambda t)^{1/\lambda}+1}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}\left(\frac{(1+\lambda t)^{1/\lambda}}{(1+\lambda t)^{2/\lambda}+1}+\frac{1}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\left(\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{1}{2}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\mathcal{E}_{n-l}^{(k)}(x,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.6 For $n\in\mathbb{Z}_+$ and $r\in\mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\mathcal{E}_{n-l}^{(k)}(x,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\frac{1-(1+\lambda t)^{-1/\lambda}}{1-(1+\lambda t)^{-1/\lambda}}\\
&=\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{1-(1+\lambda t)^{-1/\lambda}}\left(\frac{2(1+\lambda t)^{x/\lambda}}{(1+\lambda t)^{2/\lambda}+1}-\frac{2(1+\lambda t)^{(x-1)/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\left(\sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\mathbf{T}_n(x,\lambda)-\mathbf{T}_n(x-1,\lambda)\right)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(x,\lambda)-\mathbf{T}_n(x-1,\lambda)\right)\mathcal{B}_{n-l}^{(k)}(x,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.7 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(x,\lambda) - \mathbf{T}_n(x-1,\lambda)\right) \mathcal{B}_{n-l}^{(k)}(\lambda).$$

By Theorem 3.6 and Theorem 3.7, we have the following corollary.

Corollary 3.8 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}&\sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(1,\lambda) + \mathbf{T}_n(\lambda)\right) \mathcal{E}_{n-l}^{(k)}(x,\lambda)\\ &= 2\sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(x,\lambda) - \mathbf{T}_n(x-1,\lambda)\right) \mathcal{B}_{n-l}^{(k)}(\lambda).\end{aligned}$$

3. Distribution of zeros of the degenerate poly-tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$. The degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ can be determined explicitly. A few of them are

$$\begin{aligned}
&\mathcal{T}_0^{(k)}(x,\lambda) = 0,\\
&\mathcal{T}_1^{(k)}(x,\lambda) = 1,\\
&\mathcal{T}_2^{(k)}(x,\lambda) = -3 + 2^{1-k} - \lambda + 2x\\
&\mathcal{T}_3^{(k)}(x,\lambda) = 4 - 3\cdot 2^{2-k} + 2\cdot 3^{1-k} + 9\lambda - 3\cdot 2^{1-k}\lambda + 2\lambda^2 - 9x\\
&\qquad + 3\cdot 2^{1-k}x - 6\lambda x + 3x^2,\\
&\mathcal{T}_4^{(k)}(x,\lambda) = 3 + 3^{3-2k} + 7\cdot 2^{1-k} + 3\cdot 2^{3-k} - 8\cdot 3^{1-k} - 4\cdot 3^{2-k} - 24\lambda\\
&\qquad + 3\cdot 2^{3-k}\lambda + 3\cdot 2^{4-k}\lambda - 4\cdot 3^{2-k}\lambda - 33\lambda^2 + 11\cdot 2^{1-k}\lambda^2 - 6\lambda^3\\
&\qquad + 16x - 3\cdot 2^{4-k}x + 8\cdot 3^{1-k}x + 54\lambda x - 3\cdot 2^{2-k}\lambda x - 3\cdot 2^{3-k}\lambda x\\
&\qquad + 22\lambda^2 x - 18x^2 + 3\cdot 2^{2-k}x^2 - 18\lambda x^2 + 4x^3.
\end{aligned}$$

We investigate the beautiful zeros of thedegenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ by using a computer. We plot the zeros of the poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ for $n = 30, k = -5, -1, 1, 5, \lambda = 1/2$, and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 30$ and $k = -5$. In Figure 1(top-right), we choose $n = 30$ and $k = -1$. In Figure 1(bottom-left), we choose $n = 30$ and $k = 1$. In Figure 1(bottom-right), we choose $n = 30$ and $k = 5$. Stacks of zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$ from a 3-D structure are presented(Figure 2). In Figure 2(left), we choose $k = -5$. In Figure 2(middle), we choose $k = 1$. In Figure 2(right), we choose $k = 5$. Our numerical results for approximate solutions of real zeros of $\mathcal{T}_n^{(k)}(x,\lambda), \lambda = 1/2$ are displayed(Tables 1, 2).

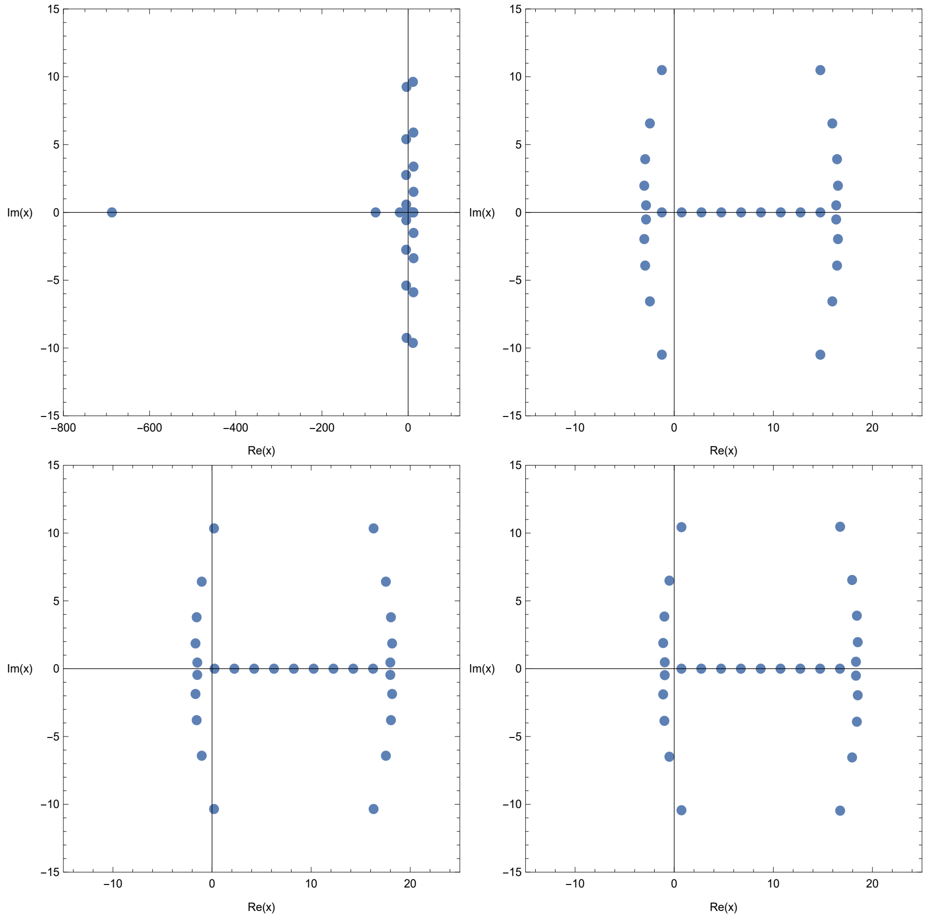

Figure 1: Zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$

Table 1. Numbers of real and complex zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$

	$k=-10$		$k=1$		$k=10$	
degree n	real	complex zeros	real	complex zeros	real	complex zeros
2	1	0	1	0	1	0
3	2	0	2	0	2	0
4	3	0	3	0	3	0
5	4	0	4	0	4	0
6	5	0	5	0	5	0
7	6	0	2	4	2	4
8	5	2	3	4	3	4
9	6	2	4	4	4	4
10	5	4	5	4	5	4
11	6	4	6	4	6	4
12	7	4	7	4	5	6

The plot of real zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \leq n \leq 30$ structure are presented(Figure 3). In Figure 3(left), we choose $k=-5$ and $\lambda=1/2$. In Figure 3(middle),we choose $k=1$ and $\lambda=1/2$. In Figure 3(right), we choose $k=5$ and $\lambda=1/2$.

We observe a remarkable regular structure of the complex roots of the degenerate poly-tangent

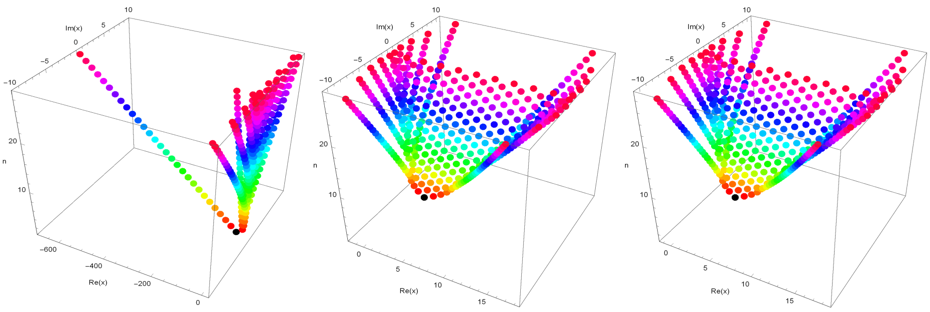

Figure 2: Stacks of zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$

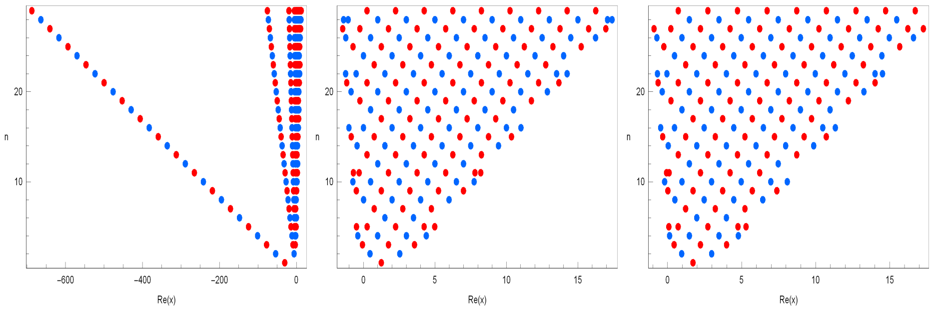

Figure 3: Real zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$

polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$. We also hope to verify a remarkable regular structure of the complex roots of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$(Table 1).

Next, we calculated an approximate solution satisfying poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda) = 0$ for $x \in \mathbb{R}$. The results are given in Table 2 and Table 3.

Table 2. Approximate solutions of $\mathcal{T}_n^{(k)}(x,\lambda) = 0, \lambda = 1/2, k = -5$

degree n	x
2	30.250
3	-53.896, -6.1044
4	-77.421, -8.8591, -2.9699
5	-100.91, -11.489, -3.9628, -1.6365
6	-124.39, -14.080, -4.7720, -2.3421, -0.66874
7	-147.85, -16.655, -5.4611, -3.0181, -1.0879, 0.076439

Table 3. Approximate solutions of $\mathcal{T}_n^{(k)}(x,\lambda)=0, \lambda=1/2, k=5$

degree n	x
2	1.7188
3	0.95682, 2.9807
4	0.44597, 2.2234, 3.9869
5	0.13979, 1.4750, 3.4758, 4.7844
6	0.090663, 0.71964, 2.7246, 4.7571, 5.3017
7	1.9752, 3.9751

By numerical computations, we will make a series of the following conjectures:

Conjecture 4.1. Prove that $\mathcal{T}_n^{(k)}(x,\lambda), x\in\mathbb{C}$, has $Im(x,\lambda)=0$ reflection symmetry analytic complex functions. However, $T_n^{(k)}(x,\lambda), k\neq 1$, has not $Re(x,\lambda)=a$ reflection symmetry for $a\in\mathbb{R}$.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n(see Figures 1, 2, 3). We are able to decide if $\mathcal{T}_n^{(k)}(x,\lambda))=0$ has $n-1$ distinct solutions(see Tables 1, 2, 3).

Conjecture 4.2. Prove that $\mathcal{T}_n^{(k)}(x,\lambda))=0$ has $n-1$ distinct solutions.

Since $n-1$ is the degree of the polynomial $\mathcal{T}_n^{(k)}(x,\lambda)$, the number of real zeros $R_{\mathcal{T}_n^{(k)}(x,\lambda)}$ lying on the real plane $Im(x,\lambda)=0$ is then $R_{\mathcal{T}_n^{(k)}(x,\lambda)}=n-1-C_{\mathcal{T}_n^{(k)}(x,\lambda)}$, where $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{\mathcal{T}_n^{(k)}(x,\lambda)}$ and $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ which appear in mathematics and physics.

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