Perspective Graves Cycle of Triangles in Finite Projective Planes: A Matrix Representation Study

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Abstract

The study of the Graves Triad, a cyclic triad of triangles each circumscribing the next, was done in the Euclidean plane and projective plane. This current study identifies cyclic sequences of triangles called Graves cycles in finite projective planes, where each triangle in the sequence circumscribes the next one, and any two triangles from the sequence form a four-fold perspective pair with one center of perspectivity common to all. The length of such cycles depends on the order of the field. The study uses the matrix representation of a triangle by homogeneous coordinates with respect to a reference frame.

Keywords: Projective Plane, Homogeneous coordinates, Perspective Triangles, Graves Triad of Triangles
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1. Introduction and Preliminaries

Two mutually inscribed and circumscribed simplexes are referred to as a Mobius pair of simplexes[1][2]. Such pairs have been identified in odd-dimensional projective spaces, as shown by [3]. However, in even-dimensional projective spaces, particularly in the projective plane, a cyclic triad of trian- gles exists, where each triangle circumscribes the next—this configuration is known as the Graves triad.

Three triangles $(A) = A_1A_2A_3$, $(B) = B_1B_2B_3$ and $(C) = C_1C_2C_3$ are said to form a Graves triad if vertices of (B) are on the sides of (A), vertices of (C) are on the sides of (B) and the vertices of (A) are on the sides of (C) [4] [5].

[6] has studied Graves triad in the geometry of triangles. [5] examined Graves triads, Mobius pairs concerning perspectivities using matrices. [7] has given a synthetic proof of the existence of perspective Graves triads which holds in all harmonic planes.[5] and [7] studied the perspective Graves triad in a projective plane using homogeneous coordinates and matrices represent- ing triangles. The present study revisits the Graves triad in finite projective planes and studies cycles of perspective Graves triads not confined to three triangles. Homogeneous coordinates are used and a 3×3 matrix represents a triangle.

A vector space V isomorphic to F^3 can define a two-dimensional projective space, a projective plane, $PG(2, F)$, over the field F. Here, every one-dimensional subspace of V represents a point of the plane and every two-dimensional subspace represents a line [8]. The triangle $(A) = A_1A_2A_3$ whose vertices are represented by the one-dimensional subspaces generated by the unit vectors e_i, $i = 1, 2, 3$ is considered as the reference triangle with homogeneous coordinates of the respective points expressed as $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The unit point defined by $U = \sum_{i=1}^{3} e_i$ together with the reference triangle (A) completes the reference frame required for providing homogeneous coordinates for the points of $PG(2, F)$. The homogeneous coordinates of U is always $(1, 1, 1)$. The reference frame, denoted by $[(A), U]$, is responsible for the coordinate system in the plane. Any change in the reference triangle or the unit point will call for a coordinate change. Sometimes, the unit point is altered to simplify coordinate expressions.

Consider a triangle $(B) = B_1B_2B_3$. The homogeneous coordinates of the vertices of (B) with respect to the reference frame $[(A), U]$ can be expressed by the rows of a 3×3 matrix and we write $(B)_A = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ or $(B)_A = ((b_{ij}))$, $i, j = 1, 2, 3$ to represent a triangle (B) with respect to the reference frame $[(A), U]$.

If $(B)_A = M$, then $(A)_B = M^{-1}$. By employing the properties of homogeneous coordinates, one can effectively utilize the adjugate of M to express (A) with (B) as the triangle of reference. Similarly, if $(B)_A = M_1$ and $(C)_B = M_2$ then $(C)_A = (C)_B(B)_A = M_2M_1$.

Using the matrix for analyzing the property of a triangle one can multiply the matrix by any non-singular diagonal matrix without changing the triangle. However, while using the matrix for changing the reference frame, we may multiply the matrix by a non-zero scalar matrix i.e. the matrix as a whole is multiplied by a scalar thereby making the expression of the matrix simpler.

Two triangles (A) and (B) are perspective if the lines joining their corresponding vertices are concurrent at a point P. Here, P is called the center of perspectivity. Furthermore, the points of intersections of the corresponding sides of the triangles are collinear. This line is referred to as the line or axis of perspectivity. If (A) and (B) are perspective from the centre (p_1, p_2, p_3), then we have

$$(B)_A = \begin{bmatrix} a_1 & p_2 & p_3 \\ p_1 & a_2 & p_3 \\ p_1 & p_2 & a_3 \end{bmatrix} \tag{1}$$

where, a_1, a_2 and a_3 are some arbitrary values depending on the triangles and nature of perspectivity. The matrix for (B) can also take the following form

$$(B)_A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \tag{2}$$

where, the centre of perspectivity is $(h, \frac{fh}{g}, f)$. Additionally, if $af^2 = bg^2 = ch^2$, then (A) and (B) are four-fold perspective [9].

2. Matrix representation of Graves Triad

When the vertices of the triangle (B) are on the sides of (A), say the vertex B_1 is on the side A_2A_3, the vertex B_2 is on the side A_1A_3 and the vertex B_3 is on the side A_1A_2, then the homogeneous coordinates of the vertices B_1, B_2, and B_3 will have a zero at the first, second, and third coordinates respectively. This is because when the vertex is on the side of a reference triangle, it has no contribution from the opposite vertex for its homogeneous coordinates. So, the matrix $(B)_A$ becomes a zero-diagonal matrix. A zero diagonal matrix is called a proper zero diagonal if the off-diagonal elements of the matrix are non-zero. For a non-degenerate triangle (B) where the vertices of (B) are on the sides of (A) then the matrix $(B)_A$ becomes a proper zero diagonal [5].

Consider three triangles (A), (B), and (C), where (A) is the reference triangle. Let $(A)_C = M_1$ and $(C)_B = M_2$ be proper 3×3 zero diagonal matrices. So, the vertices of (A) are on the sides of (C) and the vertices of (C) are on the sides of (B).

$$\text{Let } (B)_A = \begin{bmatrix} \lambda & c_{12} & c_{13} \\ c_{21} & \mu & c_{23} \\ c_{31} & c_{32} & \gamma \end{bmatrix} = M_3$$

Therefore, $M_1M_2M_3 \equiv I$ as it is known that $(A)_C(C)_B(B)_A = (A)_A = I$.

In such a situation if any two of λ, μ and γ are zero, then all three are zero, i.e., if two of the vertices of (B) are on the sides of (A), then so will be the third [5]. Since $M_1M_2M_3 \equiv I$, we obtain $M_3 = M_2^{-1}M_1^{-1} = (M_1M_2)^{-1}$. However, we will use adjugate instead of the inverse.

3. Perspective Graves Cycle

Let $[(A), U]$ be the reference frame of a projective plane over the field F and P be a point not on the sides of (A) with coordinates (p_1, p_2, p_3). Let the lines A_iP meet the opposite sides A_jA_k of the triangle (A) at the point B_i where $i, j, k = 1, 2, 3$. Then we have the triangle $(B) = B_1B_2B_3$ whose vertices are on the sides of (A) and this pair of triangles is a perspective pair from the center P.

Here, the triangle (A) circumscribes the triangle (B) and they are per- spective from a center which is not on any sides of (A) or (B). This is the first requirement for a perspective Graves sequence of triangles. The triangles $\{(A), (B)\}$ form a fourfold perspective pair meaning that (A) is perspective with (B) in four different arrangements of (B). Four-fold perspective triangles are studied by [9] in their work on multi-perspective triangles. However, if this pair is observed minutely, we get the following.

$A_1A_2A_3$ is perspective with $B_1B_2B_3$ from the centre P and the axis of perspectivity may be called as p. $A_1A_2A_3$ is perspective with $B_1B_3B_2$ from the centre A_1 and the axis is the line B_2B_3. $A_1A_2A_3$ is perspective with

$B_3B_2B_1$ from the centre A_2 and the axis is the line B_1B_3. $A_1A_2A_3$ is perspective with $B_2B_1B_3$ from the centre A_3 and the axis is the line B_1B_2. Here, out of the four centers, three of the centers are the vertices of the triangle (A), and out of the four axes; three of the axes are the sides of the triangle (B). So, we arrive at the following theorem.

Theorem 1. *When a triangle* (A) *circumscribes a triangle* (B) *and the pair is perspective from a center not on any side of these two triangles then this pair is in four-fold perspective with the additional centers of perspectivity being the vertices of* (A) *and the additional axes of perspectivity being the sides of* (B).

The matrix representation of (B) with respect the reference frame $[(A), U]$ is expressed as $(B)_A = \begin{bmatrix} 0 & p_2 & p_3 \\ p_1 & 0 & p_3 \\ p_1 & p_2 & 0 \end{bmatrix} = M.$

If we choose $P = U$, the unit point, without loss of generality, then the matrix expression takes the form

$$(B)_A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = M \tag{3}$$

If we find out the coordinates of P, which is U as per our present choice, with triangle (B) as the reference then we get it as

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times Adjugate M = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

The reference frame $[(B), U]$ retains the same unit point as the reference frame $[(A), U]$. Now from (B), we get another triangle (C), the way we obtained (B) from (A). So, we get $(C)_B = M$. We may continue this process to get a sequence of triangles $\{(A), (B), (C), (D), ...\}$ where each triangle in the sequence circumscribes the next.

Let us consider $PG(2, 3)$

Let $(B)_A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = M$

and $(C)_B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Now, $(C)_A = (C)_B(B)_A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Let $(D)_C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Now, $(D)_A = (D)_C(C)_A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (A)$

Consequently, the fourth triangle *D* doesn't exist. Here (*A*), (*B*), and (C) form a Graves cycle. The vertices of (*B*) are on the sides of (*A*). The vertices of (*C*) are on the sides of (*B*) and the vertices of (*A*) are on the sides of (*C*).

Similarily in PG(2,5) Let $(B)_A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = M$

and $(C)_B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Now, $(C)_A = (C)_B(B)_A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

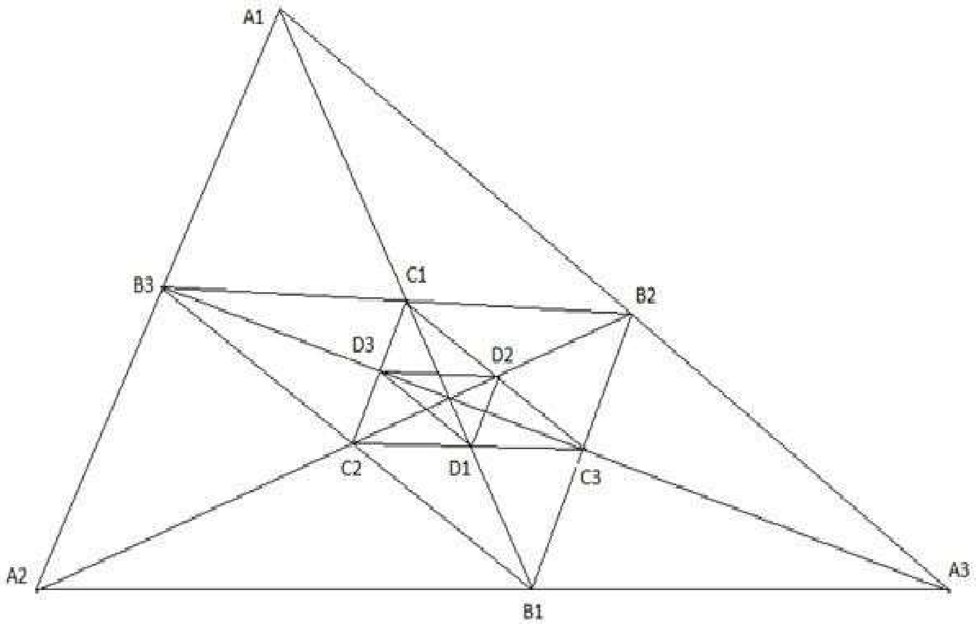
Let $(D)_C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Now, $(D)_A = (D)_C(C)_A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 3 & 2 \end{bmatrix}$

$$\text{Let } (E)_D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\text{Now, } (E)_A = (E)_D(D)_A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here, the fifth triangle triangle E doesn't exist. Hence, (A), (B), (C) and (D) form a Graves cycle. The vertices of (B) are on the sides of (A). The vertices of (C) are on the sides of (B) and the vertices of (D) are on the sides of (C) and the vertices of (A) are on the sides of (D).



Additionally, every consecutive pair in the sequence is a perspective pair from the same common center P, forming a four-fold perspective according to Theorem 1.

The above sequence, in general, is an infinite sequence of triangles where each triangle in the sequence circumscribes the next. We call it a Graves cy- cle (synonymous with Graves Triad) only if it is cyclic, that is, the sequence terminates, and if the last triangle of the sequence circumscribes the first.

The sequence will terminate at some point if the field F is finite. Before confirming a Graves cycle of triangles, it is necessary to investigate the connection between the initial and final triangles.

In this sequence, the matrix of a triangle is M as mentioned in equation (3), when represented with the previous triangle as the reference triangle. So, when the $(k+1)^{th}$ triangle of the sequence is expressed with respect to the reference triangle (A), then the matrix of the $(k+1)^{th}$ triangle with respect to the reference triangle (A) becomes M^k. Let's modify our notation to depict this sequence of triangles. Let the reference triangle (A) be denoted by ($A^{(0)}$), the triangle (B) be denoted by ($A^{(1)}$), the triangle (C) by ($A^{(2)}$) and so on. So, the earlier expression $(B)_A$ will be $(A^{(1)})_{(A(0))}$. With this notation the triangle ($A^{(0)}$) is the initial or 0^{th} triangle of the sequence and triangle ($A^{(k)}$) will be the k^{th} triangle of the sequence. In that case we have $(A^{(k)})_{(A(0))} = M^k$.

We get some initial powers as

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, M^3 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}, M^4 = \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix}$$

By observing the powers of M, we write $M^k = \begin{bmatrix} d_k & m_k & m_k \\ m_k & d_k & m_k \\ m_k & m_k & d_k \end{bmatrix}$. It can be observed that $d_k = m_k + (-1)^k$.

When we multiply M with M^{k-1} to get M^k, it is observed that $m_k = 2m_{k-1} - (-1)^k$. So we arrive at a lemma.

Lemma 2. *For the matrix* $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ *its* k^{th} *power can be expressed as* $M^k = \begin{bmatrix} d_k & m_k & m_k \\ m_k & d_k & m_k \\ m_k & m_k & d_k \end{bmatrix}$*,where* $m_k = 2m_{k-1} - (-1)^k$ *and* $d_k = m_k + (-1)^k$ *with initial values* $m_0 = 0$*,* $m_1 = 1$*,* $k \geq 1$*.*

Proof. The Principle of Mathematical Induction can prove this lemma.

Basis Step: For k = 1, we have $M^1 = \begin{bmatrix} d_1 & m_1 & m_1 \\ m_1 & d_1 & m_1 \\ m_1 & m_1 & d_1 \end{bmatrix}$.

With the initial condition $m_1 = 1$ we have $d_1 = m_1 + (-1)^1 = 1 + (-1) = 0$.

Therefore, $M^1 = M = \begin{bmatrix} d_1 & m_1 & m_1 \\ m_1 & d_1 & m_1 \\ m_1 & m_1 & d_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which is true.

Induction hypothesis: Let us assume that $M^k = \begin{bmatrix} d_k & m_k & m_k \\ m_k & d_k & m_k \\ m_k & m_k & d_k \end{bmatrix}$ be true

for some k = n i.e. let $M^n = \begin{bmatrix} d_n & m_n & m_n \\ m_n & d_n & m_n \\ m_n & m_n & d_n \end{bmatrix}$ with $m_n = 2m_{n-1} - (-1)^n$

and $d_n = m_n + (-1)^n$ be True.

So, $M^n = \begin{bmatrix} m_n + (-1)^n & m_n & m_n \\ m_n & m_n + (-1)^n & m_n \\ m_n & m_n & m_n + (-1)^n \end{bmatrix}$

Inductive Step: To show that $M^{n+1} = \begin{bmatrix} d_{n+1} & m_{n+1} & m_{n+1} \\ m_{n+1} & d_{n+1} & m_{n+1} \\ m_{n+1} & m_{n+1} & d_{n+1} \end{bmatrix}$.

Consider, $M^{n+1} = M.M^n = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_n + (-1)^n & m_n & m_n \\ m_n & m_n + (-1)^n & m_n \\ m_n & m_n & m_n + (-1)^n \end{bmatrix}$

$$= \begin{bmatrix} 2m_n & 2m_n + (-1)^n & 2m_n + (-1)^n \\ 2m_n + (-1)^n & 2m_n & 2m_n + (-1)^n \\ 2m_n + (-1)^n & 2m_n + (-1)^n & 2m_n \end{bmatrix}$$

$$= \begin{bmatrix} 2m_n + (-1)^n - (-1)^n & 2m_n + (-1)^n & 2m_n + (-1)^n \\ 2m_n + (-1)^n & 2m_n + (-1)^n - (-1)^n & 2m_n + (-1)^n \\ 2m_n + (-1)^n & 2m_n + (-1)^n & 2m_n + (-1)^n - (-1)^n \end{bmatrix}$$

$$= \begin{bmatrix} 2m_n - (-1)^{n+1} + (-1)^{n+1} & 2m_n - (-1)^{n+1} & 2m_n - (-1)^{n+1} \\ 2m_n - (-1)^{n+1} & 2m_n - (-1)^{n+1} + (-1)^{n+1} & 2m_n - (-1)^{n+1} \\ 2m_n - (-1)^{n+1} & 2m_n - (-1)^{n+1} & 2m_n - (-1)^{n+1} + (-1)^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} m_{n+1}-(-1)^n & m_{n+1} & m_{n+1} \\ m_{n+1} & m_{n+1}-(-1)^n & m_{n+1} \\ m_{n+1} & m_{n+1} & m_{n+1}-(-1)^n \end{bmatrix}$$
$$= \begin{bmatrix} d_{n+1} & m_{n+1} & m_{n+1} \\ m_{n+1} & d_{n+1} & m_{n+1} \\ m_{n+1} & m_{n+1} & d_{n+1} \end{bmatrix}$$

Hence, it is true for any k using Mathematical induction. □

The recurrence relation obtained in Lemma 2 can be solved using the generating function to get $m_k = \frac{2^k-(-1)^k}{3}$.

To solve this, let us take $f(x) = m_0 + m_1x + m_2x^2 + ... + m_{k-1}x^{k-1} + m_kx^k + ...$

$-2xf(x) = -2x[m_0 + m_1x + m_2x^2 + ... + m_{k-1}x^{k-1} + m_kx^k + ...]$

$\frac{1}{1+x} = 1 - x + x^2 + ... + (-1)^{k-1}x^{k-1} + (-1)^kx^k + ...$

By adding the above and using, $m_k - 2m_{k-1} + (-1)^k = 0$, we get $(1 - 2x)f(x) + \frac{1}{1+x} = m_0 + 1 = 1$

or $f(x) = \frac{1}{1-2x}(1 - \frac{1}{1+x})$

So, $f(x) = \frac{x}{(1+x)(1-2x)} = \frac{1}{3}[\frac{1}{1-2x} - \frac{1}{1+x}]$

Hence, $m_k = \frac{2^k-(-1)^k}{3}$.

This number frequently becomes a prime for different values of k. For k = 3, it is 3; for k = 4 it is 5; for k = 5 it is 11, for k = 7 it is 43; for k = 11 it is 683 and so on. Let m_k be a prime p for some k and let the order of our field be p^m for some positive integer m. In this case, the off-diagonal elements of M^k become zero and the diagonal elements equal 1 or –1. Hence, we can write $M^k \equiv I$ as our matrices can be multiplied by any scalar. So we arrive at the following lemma.

Lemma 3. *Let* $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ *and* $\frac{2^k-(-1)^k}{3} = p$ *for some* k *where* p *is a prime. Then* $M^k \equiv I$ *over the field* $GF(q)$, $q = p^m$, $m \in \mathbb{Z}^+$.

With the lemma 3 in hand let us consider the sequence of triangles $\{(A^{(i)})\}, i = 0, 1, 2, 3, \ldots$ where $(A^{(0)})$ is the reference triangle, $(A^{(1)})_{(A(0))} = M$ and $(A^{(k)})_{(A(0))} = M^k$. Let for some k, the value of $\frac{2^k-(-1)^k}{3} = p$ is a prime. Then, we have $(A^{(k)})_{(A(0))} = M^k \equiv I$ if we consider our field to be $GF(q)$, $q = p^m$. So, the triangle $((A)^{(k)})$ is same as the triangle $((A)^{(0)})$. In that case, the sequence $\{(A^{(i)})\}, i = 0, 1, 2, 3, \ldots$ terminates and we get a finite sequence $\{(A^{(i)})\}, i = 0, 1, 2, 3, \ldots(k - 1)$ having k triangles in it. This will be a cyclic sequence of k triangles, referred to as a Graves cycle of length k. Here each triangle circumscribes the next and any two triangles from the cycle are four-fold perspectives. Here, P is the common center of perspectivity. In the case of two consecutive triangles the other three centers of perspectivity are the vertices of the former triangle. When the triangles in the cycle are not consecutive, we obtain three new points as the centers of perspectivity, forming a new triangle which is four-fold perspective with the pair. This is due to the fact that when the triangle $((A)^{(j)})$ is expressed with respect to the triangle $((A)^{(i)})$, that is considering the triangle $((A)^{(i)})$ as reference, then we get $(A^{(j)})_{(A(i))} = M^{j-i(mod k)}$. However, any power of M takes the form given in (2) with the condition $af^2 = bg^2 = ch^2$ satisfied, indicating that the triangles are in a four-fold perspective. So we arrive at the following theorem.

Theorem 4. *Let* $\frac{2^k-(-1)^k}{3} = p$ *for some* k *where* p *is a prime. Then there exists a perspective Graves cycle of length* k *in the projective plane over the finite field* $GF(q)$, $q = p^m$ *where every pair of triangles form a four-fold perspective pair.*

In $PG(2, 3)$, $M^3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Thus, the length of the Graves cycle is 3 in PG(2,3).

In $PG(2, 5)$, $M^4 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ Therefore, the length of the Graves cycle is 4 in PG(2,5).

This theorem talks about the length of the perspective Graves cycle only when $\frac{2^k-(-1)^k}{3} = p$, a prime. However, not all primes take this form. If the order of our field is a power of those primes, which are not in the form of $\frac{2^k-(-1)^k}{3}$, even then the sequence will terminate resulting in a Graves cycle due to the finite order of the field. The study is continuing to find the length of such Graves cycles.

Declarations

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Graphical Abstract

Perspective Graves Cycle of Triangles in Finite Projective Planes: A Matrix Representation Study

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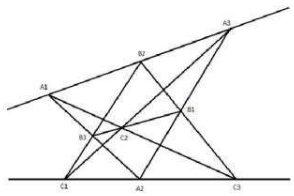
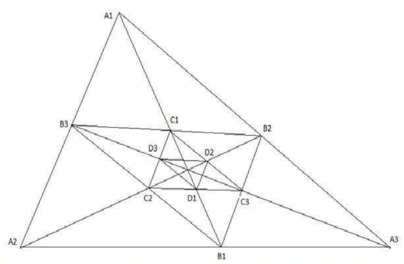


In the above Pappus configuration the triangles $(A) = A_1A_2A_3$, $(B) = B_1B_2B_3$ & $(C) = C_1C_2C_3$ forms a Graves Triad of triangles where the vertices of (B) are on the sides of (A), vertices of (C) are on the sides of (B) and the vertices of (A) are on the sides of (C). This triad also forms a perspective Graves triad as $A_1A_2A_3 \overset{A_3}{\underset{\wedge}{=}} B_2B_1B_3$, $A_1A_2A_3 \overset{C_2}{\underset{\wedge}{=}} C_3C_2C_1$ & $B_1B_2B_3 \overset{B_1}{\underset{\wedge}{=}} C_1C_3C_2$. Perspective Graves triad was studied from 1975 to 1977.



Considering the above figure on a projective plane the triangles $(A), (B), (C)$ & (D) are parts of a sequence of triangles with the vertices of each triangle on the sides of the former triangle. Any two triangles of the sequence are perspectives from a common center and any two adjacent triangles in the sequence are four-fold perspectives. This sequence is infinite in case the field over which the plane is defined is infinite. However, for finite fields, this sequence terminates, and in the case of fields of order $q = p^m$, where p is a prime of the form $\frac{2^k - (-1)^k}{3}$ the length of this sequence becomes k. Moreover, the vertices of the triangle (A) will be on the sides of the last triangle making it a cyclic sequence, a Graves cycle of triangles on a finite projective plane.

Highlights

Perspective Graves Cycle of Triangles in Finite Projective Planes: A Matrix Representation Study

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- Two mutually inscribed and circumscribed simplexes are called a Mobius pair of simplexes. Such pairs in odd-dimensional projective spaces have been established. However, in even-dimensional projective spaces, particularly on a projective plane, a cyclic triad of triangles exists, each circumscribing the next, called the Graves triad. The current research studies the existence of longer cycles of triangles in finite projective planes where each triangle circumscribes the next and is in a four-fold perspective. The length of such cycles is derived to be k if the odd characteristic of the field, which is a prime, over which the projective plane is defined, takes the form $2^k-(-1)^k/3$.
- The existence of zero diagonal matrices of order k over a finite field $GF(q)$ where $q = p^m$, p is an odd prime, is established. It exists if p takes the form $\frac{2^k-(-1)^k}{3}$.