Connected Certified Domination Number of Degree Splitting Graphs of Certain Graphs

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Abstract

A dominating set S of a graph $G = (V, E)$ is called a certified dominating set of G. If every vertices in S has either zero or at least two neighbours in $V(G) - S$. A certified dominating set S of G is said to be connected certified dominating set if the subgraph induced by S is connected. The minimum cardinality taken over all the connected certified dominating set is called the connected certified domination number of G and is denoted by $\gamma_{cer}^{c}(G)$. in this paper, we investigate the connected certified domination number of degree splitting graphs of certain graphs.

Keywords: Dominating set, certified dominating set, certified domination number, connected certified domination, Degree splitting graphs.

AMS: 05C69

1. Introduction

Let $G = (V, E)$ be a finite, undirected graph without loops and multiple edges. The graph G has $n = |V|$ vertices and $m = |E|$ edges. A path P_n is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i v_{i+1}\}$, where $i = 1,2, \ldots, n-1$. A cycle is a path from a vertex back to itself.(So the first and last vertices are not distinct). A complete graph K_n is a graph in which any two distinct vertices are adjacent. A complete bipartite graph, denoted by $K_{m,n}$ is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y. A star is a complete bipartite graph $K_{1,n}$. The join $G + H$ of graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv; u \in V(G)\ and\ v \in V(H)\}$. The fan graph of order n is defined as $K_1 + P_n$ and is denoted by F_n or $F_{1,n}$. The wheel graph of order $n \geq 3$ is defined as $K_1 + C_n$ and is denoted by W_n or $W_{1,n}$.

Domination in graphs is one of the interesting areas in graph theory which has wide applications in Engineering and Science. There are more than 300 domination parameters available in the literature. Around 1960 Berge and ore started the mathematical exploration of domination theory in graphs. There is a plethora of material on domination theory; we recommend readers outstanding books [2,3] on domination-related parameters.

Suppose that we are given a group of X officials and a group of Y civilians. There $x \in X$ for each civil $y \in Y$ who can attend x, and every time any such y is attending x, there must be also another civil $z \in Y$ that observes y. That is z must act as a kind of witness, to sidestep any mismanagement from y. In the case of a certain social network, what is the minimum number of connected officials necessary to ensure such a service? This aforementioned issue motivates us to propose the concept of connected certified domination.

The theory of certified domination was introduced by Dettlaff, Lemanska, Topp, Ziemann and Zylnski [9] and further studded in [8]. It has many applications in real life situations. The concept of connected certified domination was introduced by A. Ilyass and V.S.Goswami[10]. This motivated we to study the connected certified number in central graphs of certain standard graphs such as complete, complete bipartite graph, path graph, cycle graph, wheel graph, fan graph and double star graph.

In [9], authors studied certified domination number in graphs which is defined as follows:

Definition 1.1

Let $G = (V, E)$ be any graph of order n. A subset $S \subseteq V(G)$ is said to be a certified dominating set of G if S is a dominating set of G and every vertex in S has either zero or at least two neighbours in $V - S$. The certified domination number denoted by $\gamma_{cer}(G)$ is the minimum cardinality of certified dominating sets in G.

Definition 1.2.

Let $G = (V, E)$ be any connected graph of order n. A certified dominating set $S \subseteq V(G)$ is called a connected certified dominating set of G if its induced subgraph $G[S]$ is connected. The connected certified domination number is the minimum cardinality of a connected dominating set of G and we denoted it by $\gamma_{cer}^{c}(G)$.

2. Preliminaries

Theorem 2.1 [9] For any graph G of order $n \geq 2$, every certified dominating set of G contains its support vertices.

Theorem 2.2 [9] For any graph G of order n, $1 \leq \gamma_{cer}^{c}(G) \leq n$.

Observation 2.3 [10]

1) Let $K_{m,n}$ be a complete bipartite graph, then $\gamma_{cer}^{c}(K_{m,n}) = 2$ for $3 \leq m \leq n$.
2) Let $K_{1,n-1}$ be a star graph, then $\gamma_{cer}^{c}(K_{1,n}) = 1$ for $n \geq 2$.
3) Let W_n be a wheel graph, then $\gamma_{cer}^{c}(W_n) = 1$.
4) Let $S_{1,n,n}$ be a double star graph, then $\gamma_{cer}^{c}(S_{1,n,n}) = 2$, where $n \geq 2$.

Observation 2.4 [10]

1) If K_n is a complete graph, then $\gamma_{cer}^{c}(K_n) = 1$ for $n \geq 3$.
2) If P_n is a path graph, then $\gamma_{cer}^{c}(P_n) = n$ for $n \geq 4$.
3) If C_n is a cycle graph, then $\gamma_{cer}^{c}(C_n) = n$ for $n \geq 4$.
4) If F_n is a fan graph, then $\gamma_{cer}^{c}(F_n) = 1$ for $n \geq 3$.

Observation 2.5 [10]

For any connected graph G, $\gamma_{cer}(G) \leq \gamma_{cer}^{c}(G)$.

3. Degree Splitting Graphs

Definition: 3.1 [5]

Let $G = (V, E)$ be any graph with $V(G) = S_1 \cup S_2 \cup, \dots, \cup S_l \cup L$, where S_i is the set having at least two vertices of G of the same degree and $L = V(G) - \cup S_i$, where $1 \leq i \leq l$, then the degree splitting $DS(G)$ of G is defined as a graph obtained from G by adding vertices $y_1, y_2, \dots, y_l$ and joining y_i to each vertices of S_i for $1 \leq i \leq l$.

If $V(G) = \bigcup_{i=1}^{l} S_i$ then L must be empty.

Example 3.2:

Consider the graph G and their corresponding degree splitting graph $DS(G)$ is given in Figure 3.1. Here $S_1 = \{x_1, x_7, x_8\}, S_2 = \{x_2, x_6\}, S_3 = \{x_4, x_5\}$ and $L = \{x_3\}$.

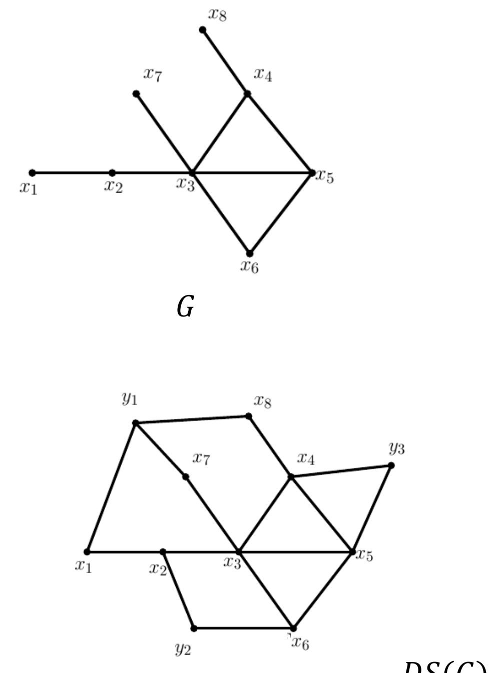
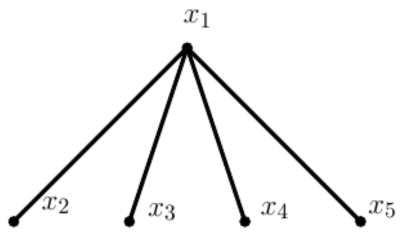


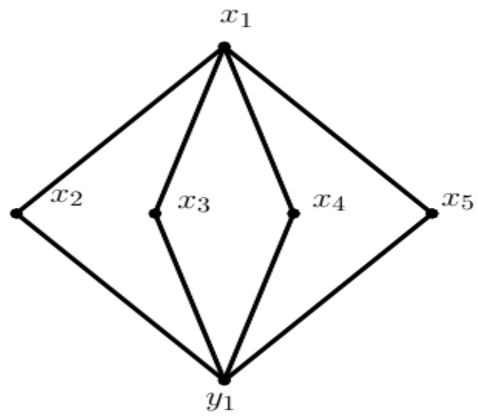
Figure. 3.1

Example 3.3

Consider the graph G and its degree splitting graph $DS(G)$ given in Figure 3.2



G



$DS(G)$

Figure. 3.2

Here $S = \{x_1, y_1\}$ is the unique minimum certified dominating set of $DS(G)$ and so, $\gamma_{cer}^{c}(DS(G)) = 2$. But $< S >$ is not connected so that S is not a connected certified dominating set of $DS(G)$.

Since $x_1 - y_1$ is a connected path of distance 2 through x_2, x_3, x_4 and x_5 respectively. Therefore, either x_2 or x_3 or x_4 must be in every connected certified dominating set of $DS(G)$. If $S' = S \cup \{x_2\}$, the every vertex in S has exactly three neighbours in $V(DS(G))\backslash S'$ and $\{x_2\}$ has zero neighbours in $V(DS(G))\backslash S'$. Also, $< S' >$ is connected. Therefore that S' is a minimum connected certified dominating set of $DS(G)$ and so $\gamma_{cer}^{c}(DS(G)) = |S'| = 3$.

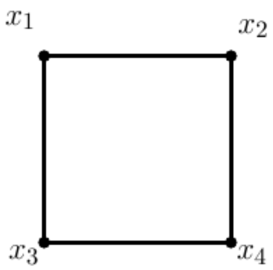
Furthermore, in Figure 3.2, $\gamma_{cer}^{c}(G) = 1$. Thus the connected certified dominating set of G and $DS(G)$ are different.

Observation 3.4

For any connected graph G, there is no obvious relation connecting $\gamma_{cer}^{c}(G)$ and $\gamma_{cer}^{c}(DS(G))$.

Example for $\gamma_{cer}^{c}(DS(G)) < \gamma_{cer}^{c}(G)$.

Consider the connected graph G given in Figure 3.3

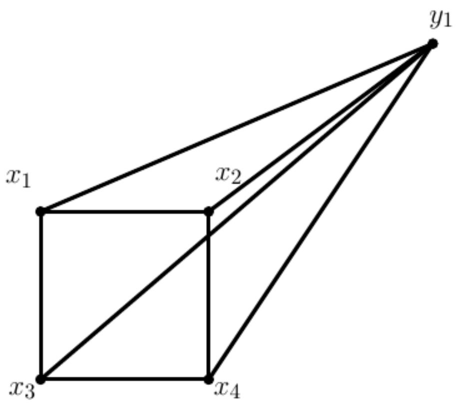


G

Figure. 3.3

Here $S = V(G)$ is the unique minimum connected certified dominating set of G and hence $\gamma_{cer}^{c}(G) = |S| = 4$.

Now we degree split every edge of G. A new connected graph $DS(G)$ obtained and is given in Figure 3.4



$DS(G)$

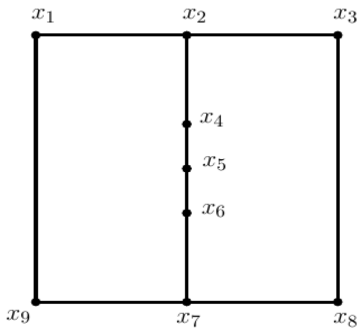
Figure. 3.4

Here $S' = \{y_1\}$ is the unique minimum connected certified dominating set of $DS(G)$ and hence, $\gamma_{cer}^{c}\big(DS(G)\big) = |S'| = 1$.

Thus, in this case $\gamma_{cer}^{c}\big(DS(G)\big) < \gamma_{cer}^{c}(G)$.

Example for, $\gamma_{cer}^{c}\big(DS(G)\big) = \gamma_{cer}^{c}(G)$

Consider the graph G given in Figure 3.5

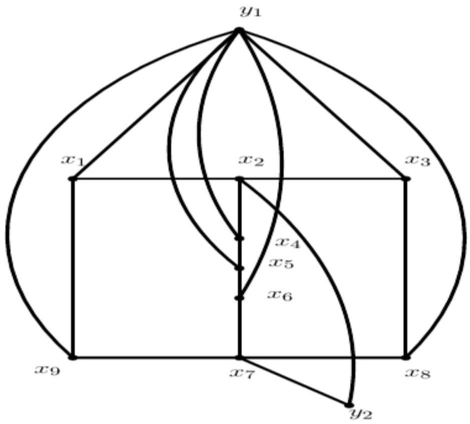


G

Figure. 3.5

Here $S = \{x_2, x_4, x_5, x_6, x_7\}$ is a minimum connected certified dominating set of G and hence, $\gamma_{cer}^{c}(G) = |S| = 5$.

Now we degree split every edge of G. A new connected graph $DS(G)$ is obtained and is given in Figure 3.6.



$DS(G)$

Figure. 3.6

Here $S' = \{x_2, x_3, x_7, x_8, y_1\}$ is a minimum connected certified dominating set of $DS(G)$ and hence $\gamma_{cer}^{c}(DS(G)) = |S'| = 5$.

Thus in this case $\gamma_{cer}^{c}(DS(G)) = \gamma_{cer}^{c}(G)$.

Example for $\gamma_{cer}^{c}(G) < \gamma_{cer}^{c}(DS(G))$, we consider the graph G given in Figure 3.2.

Here $\gamma_{cer}^{c}(G) = 1$ and $\gamma_{cer}^{c}(DS(G)) = 3$.

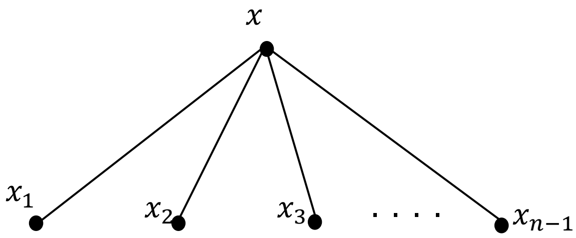
Thus $\gamma_{cer}^{c}(G) < \gamma_{cer}^{c}(DS(G))$.

Theorem 3.5

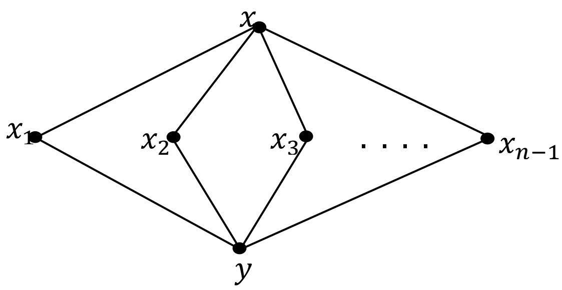
For any integer $n \geq 3, \gamma_{cer}^{c}\left(DS(K_{1,n-1})\right) = 3$.

Proof

Let $x_1, x_2, \dots, x_{n-1}$ be the end vertices and x be the full degree vertex of the star $K_{1,n-1}$ and y be the corresponding degree splited vertex which is added to obtain the graph $DS(K_{1,n-1})$. Then $V\left(DS(K_{1,n-1})\right) = \{x, x_1, x_2, x_3, \dots, x_{n-1}, y\}$ and so $\left|V\left(DS(K_{1,n-1})\right)\right| = n + 1$.



$K_{1,n-1}$



$DS(K_{1,n-1})$

Figure. 3.7

Since $DS(K_{1,n-1})$ is connected and $\Delta\left(DS(K_{1,n-1})\right) \neq V\left(DS(K_{1,n-1})\right) - 1$, then $\gamma_{cer}^{c}\left(DS(K_{1,n-1})\right) \geq 2$. Consider $S = \{x, y\}$. Clearly x_i for $1 \leq i \leq n-1$ is dominated by x and y so that S dominates $V\left(DS(K_{1,n-1})\right)$. Since $n \geq 3$ every vertices in S has at least two neighbours in $V\left(DS(K_{1,n-1})\right) \setminus S$. Therefore that S is a certified dominating set of $DS(K_{1,n-1})$. But $< S >$ is not connected. Also there does not exists a connected certified dominating set of cardinality two. Thus $\gamma_{cer}^{c}\left(DS(K_{1,n-1})\right) \geq 3$. Now consider $S' = S \cup \{x_i\}$ for $1 \leq i \leq n-1$. Clearly x_i incident with x and y only. So that $S'$is a connected certified dominating set of $DS(K_{1,n-1})$ and so $\gamma_{cer}^{c}\left(DS(K_{1,n-1})\right) \leq |S'| = 3$. Hence $\gamma_{cer}^{c}\left(DS(K_{1,n-1})\right) = 3$.

Example 3.6

Consider $DS(K_{1,6})$ given in Figure 3.8, by theorem 3.5, $S = \{x, y, x_1\}$ is a minimum connected certified dominating set of $DS(K_{1,6})$ and hence, $\gamma_{cer}^{c}(DS(K_{1,6}) = |S| = 3$.

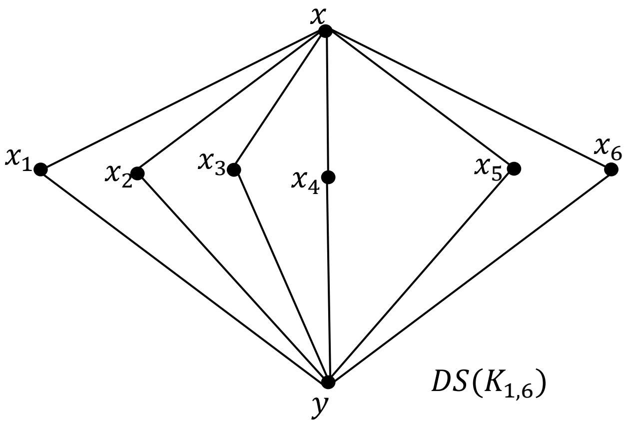


Figure. 3.8

Theorem 3.7

For the bistar graph $B_{m,n}$, $(m, n \geq 1)$,

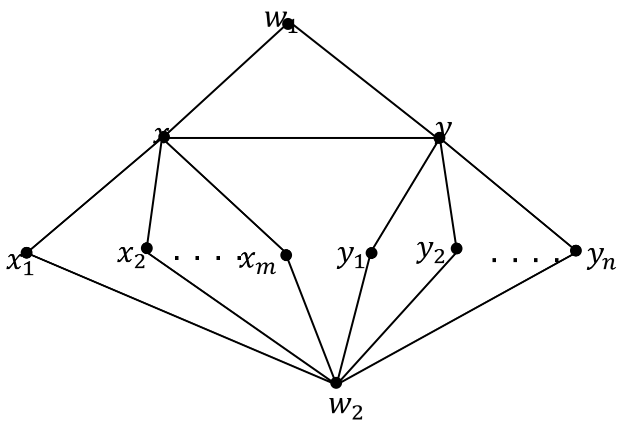
$$\gamma_{cer}^{c}\left(DS(B_{m,n})\right) = \begin{cases} 3 \; if \; n = 1 \; and \; m > 1 \; or \; n \geq 2 \\ 6 \; if \; m = n = 1 \end{cases}.$$

Proof

Consider the bistar graph $B_{m,n}$ with $V(B_{m,n}) = \{x, y, x_i, y_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. Here x_i and y_j are the vertices adjacent with x and y respectively. To obtain $DS(B_{m,n})$, three cases where arise:

Case(i) $n \geq 2$. Then $m \geq 2$. Two subcases arise.

Subcase(a) $m = n$. Let w_1 and w_2 be the corresponding vertices which are added to obtain $DS(B_{m,n})$. Then, $V\left(DS(B_{m,n})\right) = \{x, y, x_i, y_i, w_1, w_2; 1 \leq i \leq m\}$ and so $\left|V\left(DS(B_{m,n})\right)\right| = m + n + 4$. The graph is given in Figure 3.9

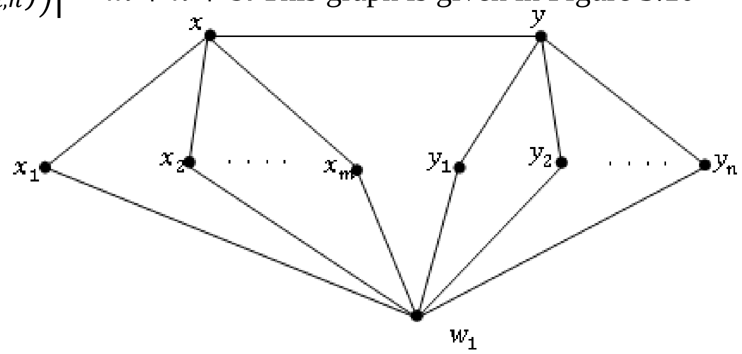


$DS(B_{m,n})$

Figure. 3.9

In this case we consider $S = \{x, w_2\}$. Clearly S is a certified dominating set of $DS(B_{m,n})$. But $< S >$ is not connected. Since x_1 is adjacent with only x and w_2. $S \cup \{x_1\}$ is a minimum connected certified dominating set of $DS(B_{m,n})$ and hence, $\gamma_{cer}^{c}\left(DS(B_{m,n})\right) = |S| = 3$.

Subcase (b) $m \neq n$. Then $m > n$. Let w_1 be the corresponding vertex which is added to obtain $DS(B_{m,n})$. Then $V\left(DS(B_{m,n})\right) = \{x, y, x_i, y_j, w_1; 1 \le i \le m, 1 \le j \le n\}$ and so $\left|V\left(DS(B_{m,n})\right)\right| = m + n + 3$. This graph is given in Figure 3.10



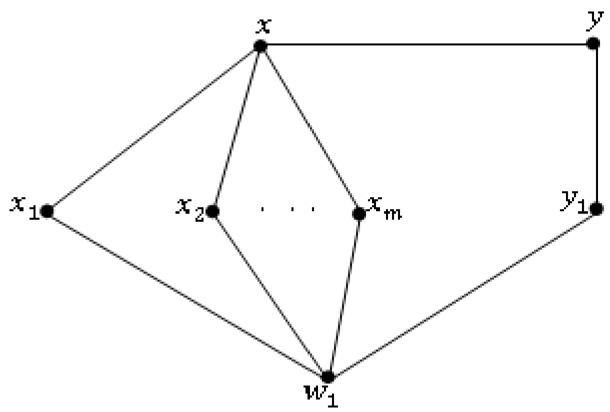
$DS(B_{m,n})$

Figure. 3.10

As similar in Subcase (a) here also $S = \{x, x_1, w_1\}$ is a minimum connected certified dominating set of $DS(B_{m,n})$ and hence $\gamma_{cer}^{c}\left(DS(B_{m,n})\right) = |S| = 3$.

Case (ii) $n = 1$ and $m > 1$

In this case in $B_{m,n}, y$ is adjacent with exactly one end verted y_1 and x is adjacent with more that two end vertices. Let w_1 be the corresponding degree splitted vertex which is added to obtain $DS(B_{m,n})$. Then $V\left(DS(B_{m,1})\right) = \{x, y, x_i, y_1, w_1; 1 \le i \le m\}$ and so $|V\left(DS(B_{m,1})\right)| = m + 4$. This graph is given in Figure 3.11.



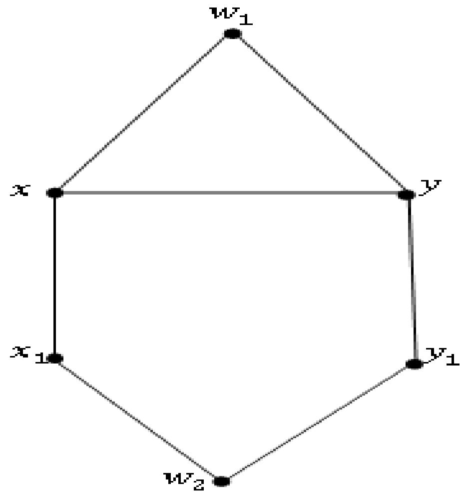
$DS(B_{m.1})$

Figure. 3.11

Since x dominates $y, x_1, x_2, \ldots, x_m$ and w_1 dominates $x_1, x_2, \ldots, x_m$ and $y_1, S = \{x, w_1\}$ is a dominating set of $DS(B_{m,1})$. Since $m > 1$, every vertices in S has at least two neighbours in $V\left(DS(B_{m,1})\right) \backslash S.$ So that S is a certified dominating set of$DS(K_{m,1})$. But $< S >$ is not connected. Since $x_1, x_2, \ldots, x_m$ dominated by both x and $w_1,$ that $S \cup \{x_i\}$ is a minimum connected certified dominating set of $DS(B_{m,1})$ and hence, $\gamma_{cer}^{c}(B_{m,1}) = 3$.

Case (iii) $m = n = 1$.

In this case in $B_{m,n}, x$ and y are adjacent to exactly one end vertex x_1 and y_1 respectively. So in $DS(B_{m,n})$, let w_1, w_2 be the corresponding vertex adjacent to both x_1, y_1 and x, y respectively and is given in Figure 3.12



$DS(B_{1,1})$

Figure. 3.12

Here $S = \{x, w_2\}$ be a certified dominating set of $DS(B_{1,1})$. But $< S >$ is not connected. Since x and w_2 are both adjacent to x_1 in the shortest $x - w_2$ path, that x_1 must be in every connected certified dominating set of $DS(B_{1,1})$. Let S' be a minimum connected certified dominating set of $DS(B_{1,1})$. Clearly $S \cup \{x_1\} \subseteq S'$. But in $S \cup \{x_1\}, w_2$ has exactly one neighbor in $V\left(DS(B_{1,1})\right) \backslash S \cup \{x_1\}$. So that $S \cup \{x_1\}$ itself is not a connected certified dominating set of $DS(B_{1,1})$. So y_1 must be in S'. As similar, that y also has exactly one neighbor w_1 in $V\left(DS(B_{1,1})\right) \backslash S \cup \{x_1, y_1, y\}$. Therefore w_2 must be in $S'.$ Thus every connected certified dominating set must contains at least 6 vertices. So, $\gamma_{cer}^{c}\left(DS(B_{1,1})\right) \geq 6$. Since $\left|V\left(DS(B_{1,1})\right)\right| = 6$, we conclude that $\gamma_{cer}^{c}\left(DS(B_{1,1})\right) = 6$.

Hence $\gamma_{cer}^{c}\left(DS(B_{m,n})\right) = \begin{cases} 3 \; if \; n = 1 \; and \; m > 1 \; or \; n \geq 2 \\ 6 \; if \; m = n = 1 \end{cases}$.

Example 3.8

Consider $DS(B_{4,4})$ given in Figure 3.13

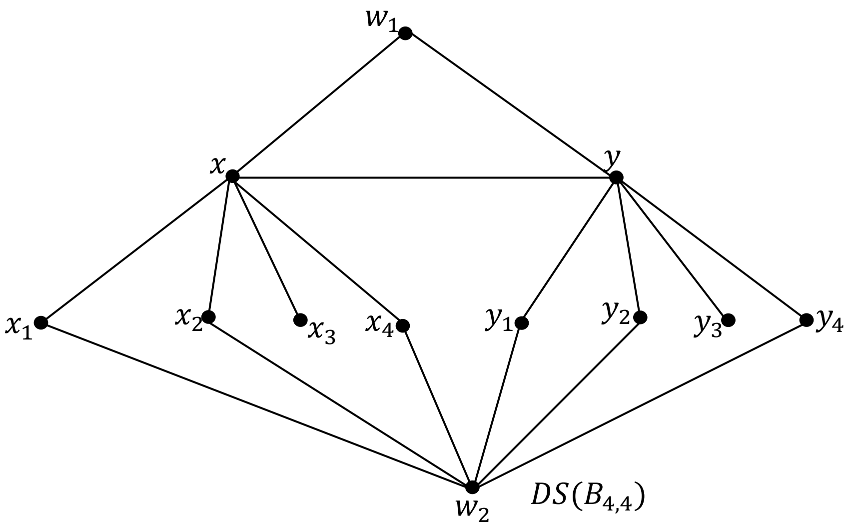


Figure. 3.13

By Theorem 3.7 $S = \{x, x_1, w_2\}$ is a minimum connected certified dominating set of $DS(B_{4,4})$ and hence $\gamma_{cer}^c\left(DS(B_{4,4})\right) = |S| = 3$.

Theorem 3.9

For integers $m, n \geq 2, \gamma_{cer}^c\left(DS(K_{m,n})\right) = \begin{cases} 1 \; if \; m = n \\ 2 \; if \; m \neq n \end{cases}$.

Proof

Consider $K_{m,n}$ with $V(K_{m,n}) = \{x_i, x_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ for $m, n \geq 2$ with partition $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. To obtain $\gamma_{cer}^c\left(DS(K_{m,n})\right)$, we consider two cases.

Case (i) $m = n$.

In this case every vertex is of some degree and let w be the added vertex which is adjacent to every x_i and $y_j, 1 \leq i \leq m$ and $1 \leq j \leq n$. Thus we obtained $DS(K_{m,n})$ and is given in Figure 3.14. Then $V\left(DS(K_{m,n})\right) = \left\{x_i, y_j, w; 1 \leq i \leq m, 1 \leq j \leq n\right\}$ and so $\left|V\left(DS(K_{m,n})\right)\right| = m + n + 1$.

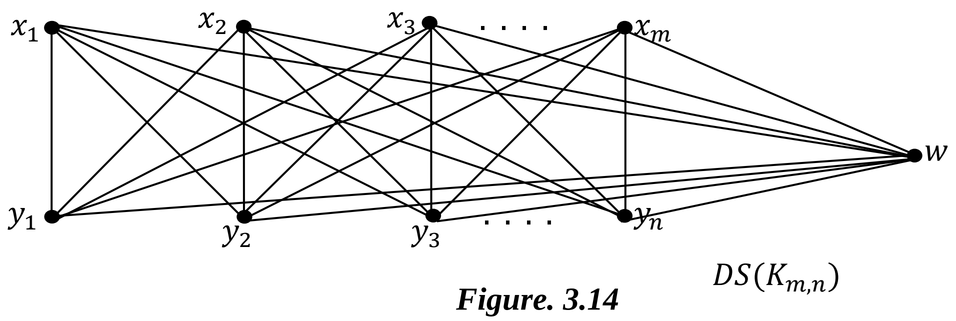


Figure. 3.14

Clearly w is adjacent with $x_i, 1 \leq i \leq m$ and $y_j, 1 \leq j \leq n$. Therefore w is a full degree vertex in $DS(K_{m,n})$. So, $\gamma_{cer}^{c}\left(DS(K_{m,n})\right) = 1$.

Case (ii) $m \neq n$

In this case, every vertex x_i is of same degree and every vertex y_j is of same degree, where $\deg(x_i) \neq \deg(y_j)$ for $1 \leq i \leq m, 1 \leq j \leq n$ in $K_{m,n}$. So let $w_1$and w_2 be the added vertices, where $w_1$is adjacent to every x_i and w_2 is adjacent to every y_j. Then $V\left(DS(K_{m,n})\right) = \{x_i, y_j, w_1, w_2; 1 \leq i \leq m, 1 \leq j \leq n\}$ and so $\left|V\left(DS(K_{m,n})\right)\right| = m + n + 2$. This graph is given in Figure 3.15

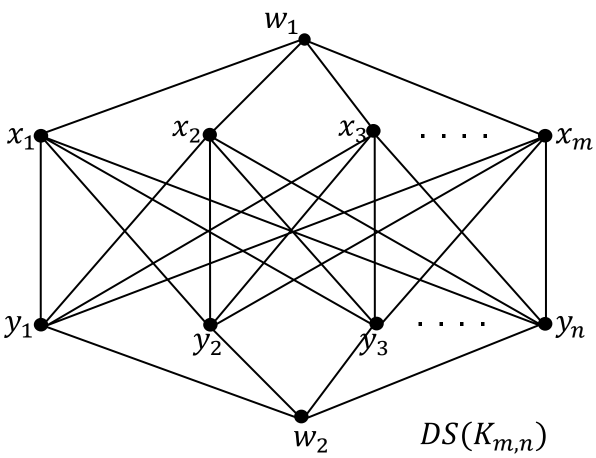


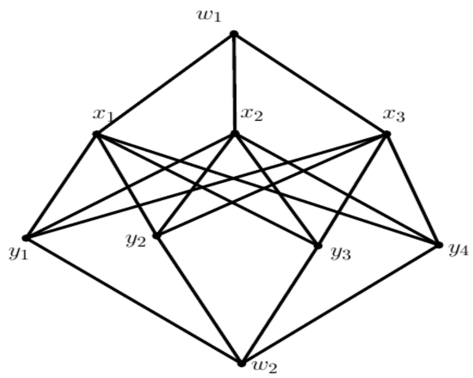
Figure. 3.15

Consider the set $S = \{x_i, y_j\}, 1 \leq i \leq m, 1 \leq j \leq n$. Here x_i is adjacent to every y_j and w_1. Also y_j is adjacent to every x_i and w_2. Thus S dominates $V\left(DS(K_{m,n})\right)$. Since $m, n \geq 2$, every vertex in S has at least two neighbours in $V\left(DS(K_{m,n})\right) \backslash S$. Also $< S >$ is connected certified dominating set of $DS(K_{m,n})$ and so $\gamma_{cer}^{c}\left(DS(K_{m,n})\right) \leq |S| = 2$. Since $\Delta\left(DS(K_{m.n})\right) \neq V\left(DS(K_{m,n})\right) - 1$, we conclude that $\gamma_{cer}^{c}\left(DS(K_{m,n})\right) = 2$.

Hence, $\gamma_{cer}^{c}\left(DS(K_{m,n})\right) = \begin{cases} 1 \; if \; m = n \\ 2 \; if \; m \neq n \end{cases}$.

Example 3.10

Consider $DS(K_{3,4})$ given in Figure 3.16 by Theorem 3.9, $S = \{x_1, y_1\}$ is a minimum connected certified dominating set of $DS(K_{3,4})$ and hence, $\gamma_{cer}^{c}\left(DS(K_{3,4})\right) = |S| = 2$.



$DS(K_{3,4})$

Figure. 3.16

Theorem 3.11

For integer $n \geq 2$, $\gamma_{cer}^{c}(DS(P_n)) = \begin{cases} n+1 \; if \; n = 3 \\ n+2 \; if \; n = 4 \\ n \; if \; n = 5 \\ n-1 \; otherwise \end{cases}$.

Proof

Let $n \geq 2$. If $n = 2$, then $DS(P_2) \cong K_3$ and so by Observation 2.4 $\gamma_{cer}^{c}(DS(P_2)) = 1 = n - 1$. Now assume $n \geq 3$.

Let $x_1, x_2, \dots, x_n$ be the path graph P_n with edge part partitions $S_1 = \{x_2, x_3, \dots, x_{n-1}\}$ and $S_2 = \{x_1, x_n\}$. To obtain $DS(P_3)$ from P_3 we add w, which corresponds to S_2. Therefore $DS(P_3) \cong C_4$ and so by Observation 2.4, $\gamma_{cer}^{c}(DS(P_3)) = 4 = n + 1$.

Now, assume $n \geq 4$. To obtain $DS(P_n)$ we add w_1 and w_2 which corresponds to S_1 and S_2 respectively. Then $V\left(DS(P_n)\right) = \{w_1, w_2, x_1, x_2, \dots, x_n\}$, where $\left|V\left(DS(P_n)\right)\right| = n + 2$. This graph is shown in Figure 3.17.

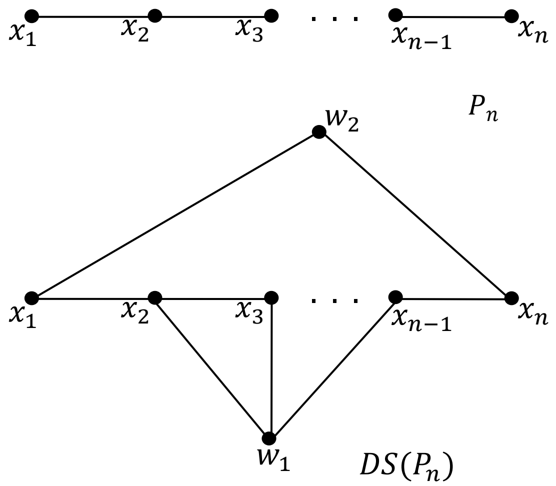


Figure. 3.17

For $n = 4, S = \{w_1, w_2\}$ or $S = \{w_2, x_2\}$ or $S = \{w_2, x_3\}$ are only minimum certified dominating set of $DS(P_4)$. But $< S >$ is not connected. Therefore, that S is not a connected certified dominating set of $DS(P_4)$. Now, we construct a connected certified dominating set S' for $DS(P_4)$. Since $w_2 - x_2$ and $w_2 - x_3$ are connected by unique path, we fix $S = \{w_1, w_2\}$. Assume $S \subseteq S'$clearly path from $w_1$to w_2, has exactly two shortest path of length 3 through x_1 and x_4, respectively. If we select x_1 or x_4 in S', then w_2 has exactly one neighbor in $V\big(DS(P_4)\big)\backslash S'$. So we must select x_2 or x_3 in S'. Then w_1 has exactly one neighbor in $V\big(DS(P_4)\big)\backslash S'$. Therefore x_1, x_2, x_3, x_4 must be in S'. Thus, $S' = \{w_1, w_2, x_1, x_2, x_3, x_4\}$ in a unique connected certified dominating set of $DS(P_4)$ and so $\gamma_{cer}^{c}\big(DS(P_4)\big) \geq |S'| = 6 = n + 2$. We conclude that $\gamma_{cer}^{c}\big(DS(P_4)\big) = 6 = n + 2$.

For $n = 5$, we construct a connected certified dominating set S' of $DS(P_5)$. If we consider $S = \{w_1, w_2\}$ then every vertices in$V\big(DS(P_5)\big)\backslash S$. In order to connectedness we must select all the vertices from $DS(P_4)$. So we select $S = \{w_2, x_2, x_4\}$. Clearly S is a certified dominating set of $DS(P_5)$. But $< S >$ is not connected. Therefore S itself is not a connected certified dominating set of $DS(P_5)$. Thus $S \subset S'$. If $x_1 \in S'$, then w_2 has exactly one neighbor x_5 in $V\big(DS(P_5)\big)\backslash S'$. So x_1, x_5 must be in S'. If $S \cup \{x_1, x_5\} = S'$, then S' itself be a minimum connected certified dominating set of $DS(P_5)$ and hence $\gamma_{cer}^{c}\big(DS(P_5)\big) = 5 = n$.

Consider $n \geq 6$. As similar in the case $n = 5$, we take $S = \{w_1, w_2\}$, then S is a minimum certified dominating set if $DS(P_n)$. But, since $< S >$ is not connected, that $\gamma_{cer}^{c}\big(DS(P_n)\big) \geq n + 2$. Therefore, we take S as without w_1. Let $S' = \left\{x_1, x_2, \dots, x_{\lceil\frac{n}{2}\rceil+1}, x_{\lceil\frac{n}{2}\rceil+2}, x_{\lceil\frac{n}{2}\rceil+3}, \dots x_{n-1}, x_n, w_2\right\}$. Clearly certified S' is a minimum connected certified dominating set of cardinality $\left|V\big(DS(P_n)\big)\right| - 3$. Therefore $\gamma_{cer}^{c}\big(DS(P_n)\big) = |S'| = \left|V\big(DS(P_n)\big)\right| - 3 = n + 2 - 3 = n - 1$.

Hence $\gamma_{cer}^{c}\big(DS(P_n)\big) = \begin{cases} n+1\ if\ n=3 \\ n+2\ if\ n=4 \\ n\ if\ n=5 \\ n-1,\ otherwise \end{cases}$.

Example 3.12

Consider $DS(P_{10})$ given in Figure 3.18. By Theorem 3.11, $S = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_{10}, w_2\}$ is a minimum connected certified dominating set of $DS(P_{10})$ and hence $\gamma_{cer}^{c}\big(DS(P_n)\big) = |S| = 9$.

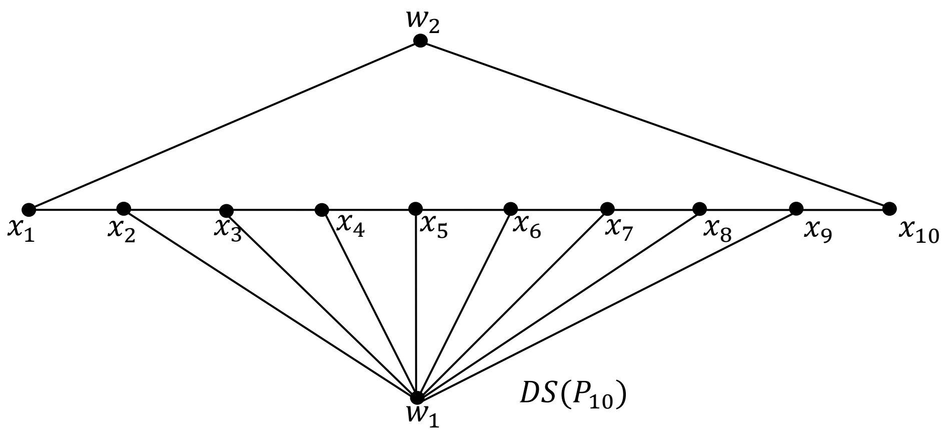


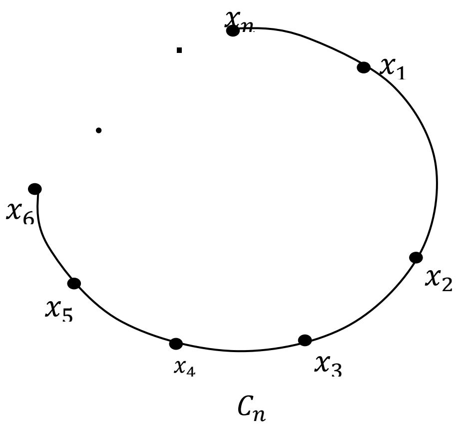
Figure. 3.18

Theorem 3.13

For integer $n \geq 3, \gamma_{cer}^{c}\big(DS(C_n)\big) = 1$.

Proof

Let $x_1, x_2, \dots, x_n$be the cycle graph C_n. To obtain $DS(C_n)$, we add a vertex w to which is adjacent to every vertex in C_n. Then $V\big(DS(C_n)\big) = \{w, x_1 x_2, \dots, x_n\}$ and so $\big|V\big(DS(C_n)\big)\big| = n+1$. This graph given in Figure 3.19.



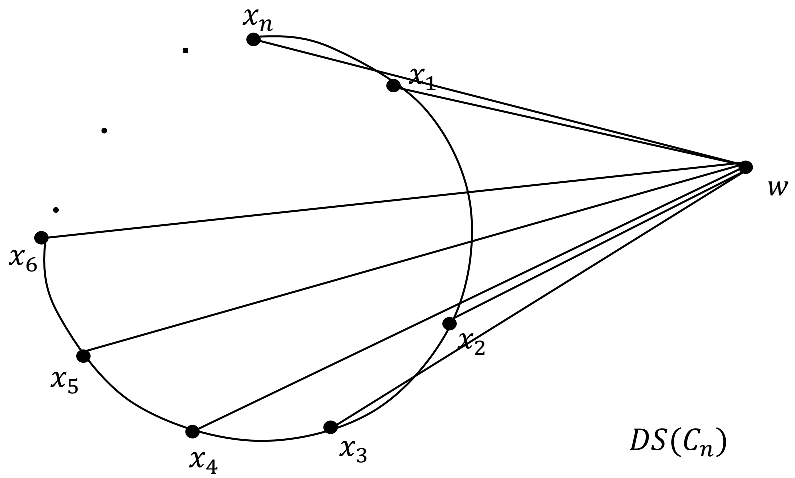


Figure. 3.19

Clearly $DS(C_n)$ is isomorphic to the wheel W_{n+1}. Therefore by observation 2.3, $\gamma_{cer}^{c}\big(DS(C_n)\big) = 1$.

Example 3.14

Consider $DS(C_{12})$ given in figure 3.20. By Theorem 3.13, $S = \{w\}$ is a minimum connected certified dominating set of $DS(C_{12})$ and hence $\gamma_{cer}^{c}\big(DS(C_{12})\big) = 1$.

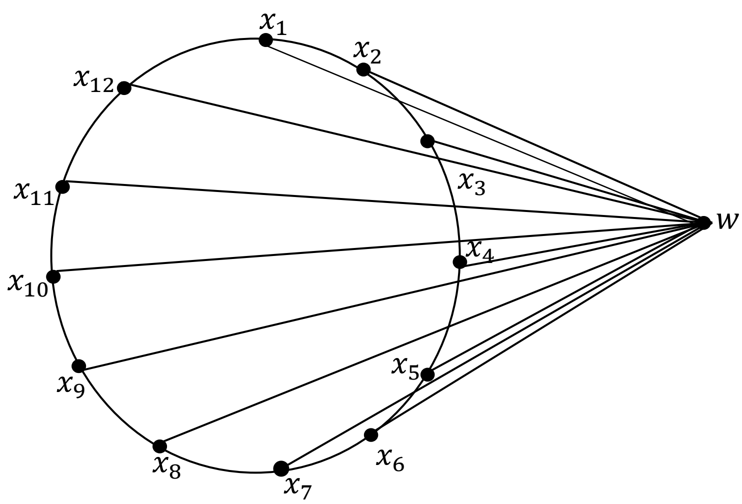


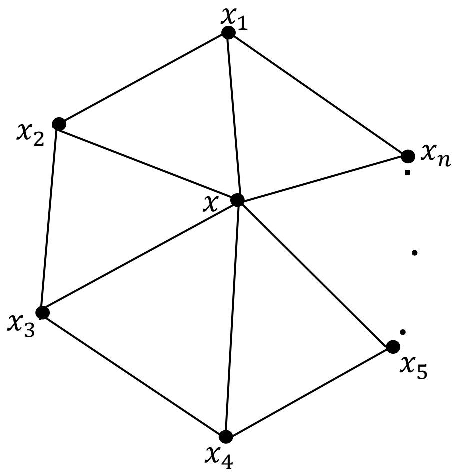
Figure. 3.20

Theorem 3.15

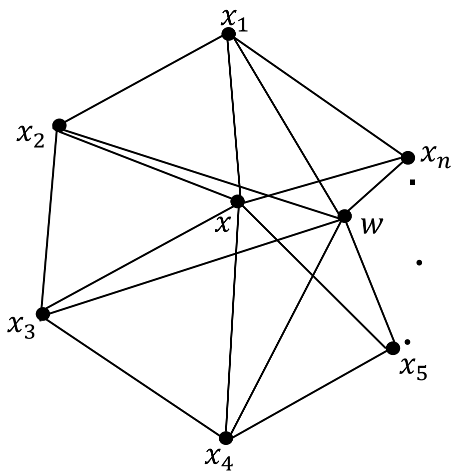
For integer $n \geq 5, \gamma_{cer}^{c}\big(DS(W_n)\big) = 2$.

Proof

Let$x_1, x_2, \dots, x_n$ be the rim vertices of W_n and x be the apex vertex of W_n. Let w be the corresponding degree splitted vertex which is added to every vertices $x_1, x_2, \dots, x_{n-1}$ to obtain the graph $DS(W_n)$. Then $V\big(DS(W_n)\big) = \{x_1, x_2, \dots, x_{n-1}, w\}$ and so $|V(DS(W_n)| = n + 1$.



W_n



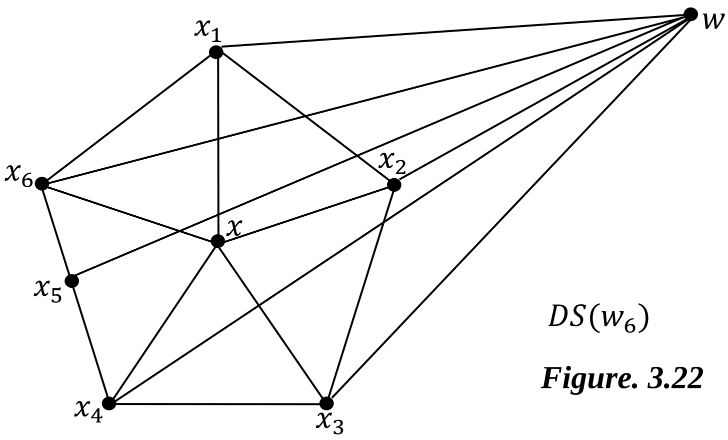
$DS(W_n)$

Figure. 3.21

Clearly $DS(W_n)$ has no full degree vertex. Also here w is adjacent with every vertices in $DS(W_n)$ other than x. To dominate x we select any x_i for $1 \le i \le n-1$. Consider $S = \{w, x_i\}$ for $1 \le i \le n-1$. Clearly, S is a connected certified dominating set of $DS(W_n)$ and so $\gamma_{cer}^c\big(DS(W_n)\big) \le |S| = 2$. Hence, $\gamma_{cer}^c\big(DS(W_n)\big) = |S| = 2$.

Example 3.16

Consider $DS(W_6)$given in Figure 3.22, by Theorem 3.15, $S = \{w, x_1\}$ is a minimum connected certified dominating set of $DS(W_6)$ and hence, $\gamma_{cer}^c\big(DS(W_6)\big) = 2$.



$DS(w_6)$

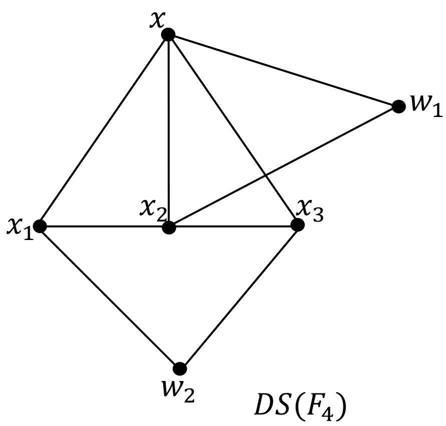
Figure. 3.22

Theorem 3.17

For integer $n \geq 4, \gamma_{cer}^{c}\big(DS(F_n)\big) = \begin{cases} 2 \; if \; n = 4 \\ 3 \; if \; n \geq 5 \end{cases}$.

Proof

Let $x_1, x_2, \ldots, x_{n-1}$ be the n-vertices of F_n, where x is the apex vertex of F_n. For $n = 4$, to obtain $DS(F_4)$ form F_4 we add w_1 and w_2 corresponds to $S_1 = \{x, x_2\}$ and $S_2 = \{x_1, x_3\}$ respectively. Then $V\big(DS(F_4)\big) = \{x, x_1, x_2, x_3, w_1, w_2\}$ and so $\big|V\big(DS(F_4)\big)\big| = 6.$ This graph is given in Figure 3.23.

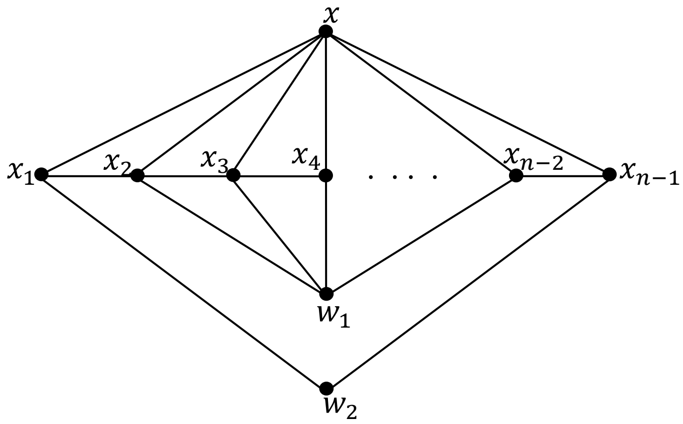


$DS(F_4)$

Figure. 3.23

If we select a vertex from S_1 and a vertex from S_2 which forms a certified dominating set of $DS(F_4)$. Let $S = \{x, x_3\}$ clearly, $< S >$ is connected Therefore S itself form a connected certified dominating set of $DS(F_4)$ and so $\gamma_{cer}^{c}\big(DS(F_4)\big) \leq |S| = 2.$ Since $\Delta(V\big(DS(F_4)\big) \neq V(DS(F_4)) - 1$, we conclude that $\gamma_{cer}^{c}\big(DS(F_4)\big) = 2.$

Now assume $n \geq 5$. Let $S = \{x_2, x_3 \ldots, x_{n-2}\}$ and $S_2 = \{x_1, x_{n-1}\}$ be the degree partition of F_n. To obtain $DS(F_n)$ we add w_1 and w_2 corresponds to S_1 and S_2 respectively. Then $V\big(DS(F_n)\big) = \{x, x_1, x_2, \ldots, x_{n-1}, w_1, w_2\}$ and $\big|V\big(DS(F_n)\big)\big| = n + 2.$ This graph is given in Figure 3.24.



$DS(F_n)$

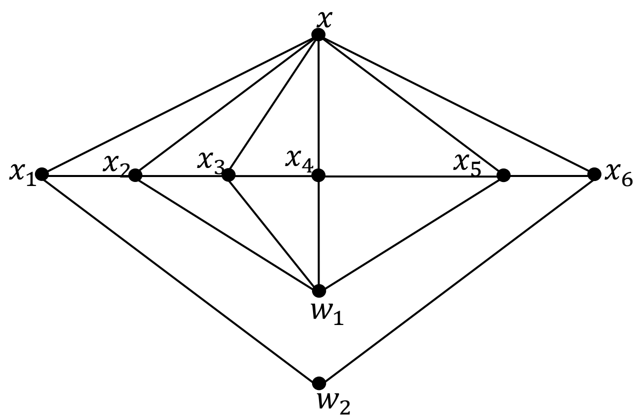
Figure. 3.24

Clearly x dominates $x_1, x_2, \dots, x_{n-1}$ and w_2 dominated by x_1 and x_{n-1} and w_1 dominated by $x_2, x_3, \dots x_{n-2}$. Consider $S = \{x, x_1, x_3\}$. Since $n \geq 5$, every vertices in S has more that two neighbor in $V\big(DS(F_n)\big)\backslash S$. Therefore, that S is a certified dominating set of $DS(F_n)$. Also x is adjacent with x_1 and x_3, that the subgraph induced by S is connected. Also if we remove any vertex from S, then S is not a connected certified dominating set of $DS(F_n)$. Thus S is a minimal connected certified dominating set of $DS(F_n)$. Moreover, there does not exists a connected certified dominating set of cardinality less than 3. Therefore S is a minimum connected certified dominating set of $DS(F_n)$ and hence, $\gamma_{cer}^{c}\big(DS(F_n)\big) = |S| = 3$.

Thus, $\gamma_{cer}^{c}\big(DS(F_n)\big) = \begin{cases} 2 \ if \ n = 4 \\ 3 \ if \ n \geq 5 \end{cases}$.

Example 3.18

Consider $DS(F_7)$ given in Figure 3.25. By Theorem 3.17, $S = \{x, x_1, x_3\}$ is a minimum connected certified dominating set of $DS(F_7)$ and hence, $\gamma_{cer}^{c}\big(DS(F_n)\big) = |S| = 3$.



$DS(F_7)$

Figure. 3.25

Theorem 3.19

For integer $n \geq 5, \gamma_{cer}^{c}\big(DS(H_n)\big) = 4$.

Proof

Let H_n be the helm graph obtained from the wheel $W_{1,n-1}$ by attaching pendent edge for each rim vertices $x_1, x_2, \ldots, x_{n-1}$. Let the pendent vertices of H_n be $y_1, y_2, \ldots, y_{n-1}$ and x be the apex vertex of $W_{1,n-1}$. To obtain $DS(H_n)$ we add w_1 and w_2 corresponding to the set of vertices $S_1 = \{x_1, x_2, \ldots, x_{n-1}\}$ and $S_2 = \{y_1, y_2, \ldots, y_{n-1}\}$ respectively. Then $V\big(DS(H_n)\big) = \{x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_{n-1}, x, w_1, w_2\}$ and so $|V(DS(H_n)| = 2n + 1$.

This graph is given in Figure 3.26

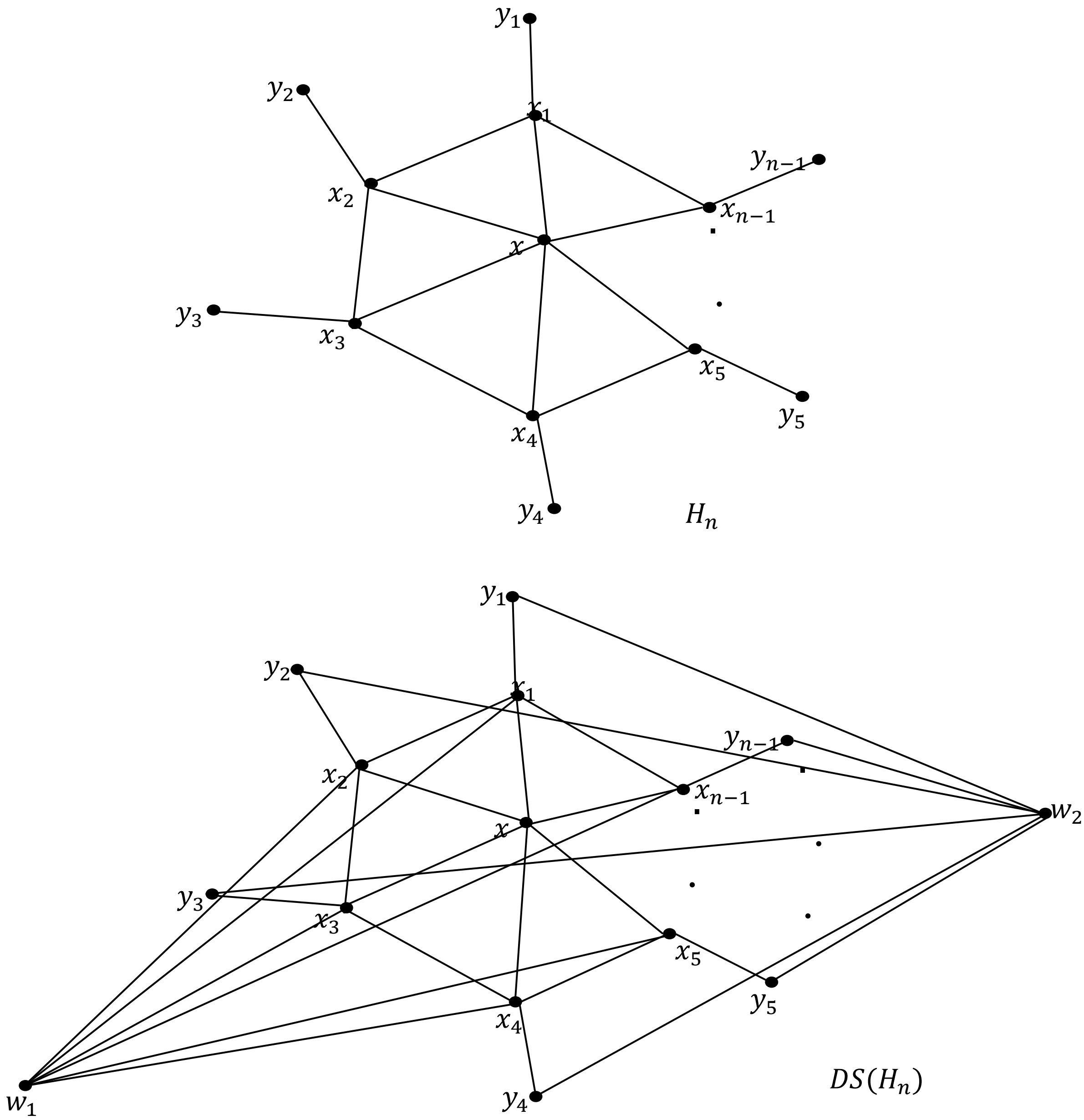
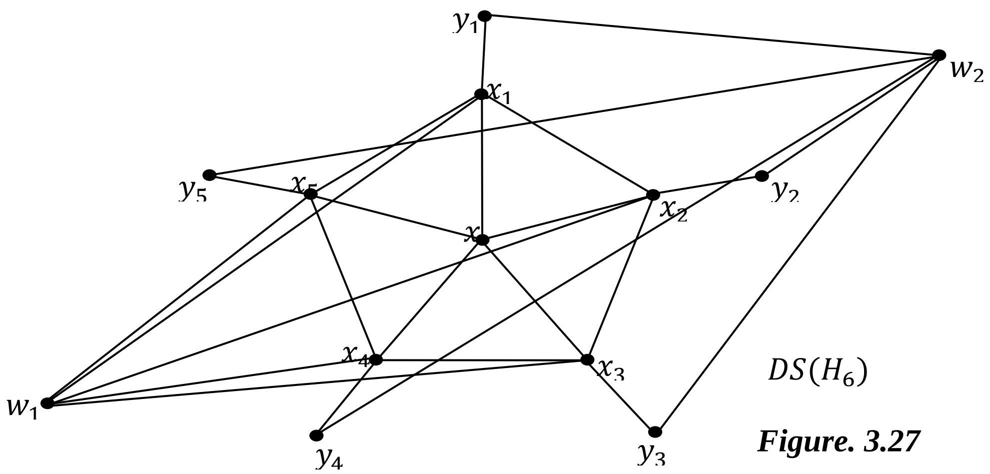


Figure. 3.26

Consider $S = \{w_1, x_i, w_2\}, 1 \leq i \leq n-1$. Clearly $w_1$dominates $x_1, x_2, \dots, x_{n-1}$ and w_2 dominates $y_1, y_2, \dots, y_{n-1}$. Also x_i dominates x. Since $n \geq 5$ every vertices in S has more that two neighbours in $V\big(DS(H_n)\big)\backslash S$. Therefore, that S is a certified dominating set of $DS(H_n)$. But w_2 is not adjacent with w_1 and x_i for $1 \leq i \leq n-1$. Therefore the subgraph induced by S is not connected. So that S is not a connected certified dominating set of $DS(H_n)$. Now consider $S' = S \cup \{y_i\}, 1 \leq i \leq n-1$. Clearly y_i is adjacent with x_i and w_2. Also x_i and w_2 are adjacent in $DS(H_n)$. Moreover y_i has zero neighbours in $V\big(DS(H_n)\big)\backslash S \cup \{y_i\}$. Therefore that $S \cup \{y_i\}$ forms a minimum connected certified dominating set of $DS(H_n)$ and hence $\gamma_{cer}^{c}\big(DS(H_n)\big) = |S \cup \{y_i\}| = 4$.

Example 3.20

Consider $DS(H_6)$ given in Figure 3.27. By Theorem 3.19, $S = \{w_1, w_2, x_1, y_1\}$ is a minimum connected certified dominating set of $DS(H_6)$ and hence $\gamma_{cer}^{c}\big(DS(H_6)\big) = |S| = 4$.



$DS(H_6)$

Figure. 3.27

Theorem 3.21

If G be a regular graph of order $n \geq 2$, then $\gamma_{cer}^{c}\big(DS(G)\big) = 1$.

Proof

Let G be any regular graph of order $n \geq 2$. Then $DS(G) \cong G + K_1$ and $\Delta\big(DS(G)\big) = V\big(DS(G)\big) - 1$. Therefore, $\gamma_{cer}^{c}\big(DS(G)\big) = 1$.

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