Hermite-Hadamard inequality for Sugeno integral based on harmonically convex functions

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Abstract. For the classical Hermite-Hadamard inequality of harmonically convex functions, i.e.,

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

an upper bound is proved in the framework of the Sugeno integral.

Keywords: the Sugeno integral; the Hermite-Hadamard inequality; harmonically convex function.

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1. Introduction

One of the most important integral inequalities which is related to harmonically convex functions is classical Hermite-Hadamard integral inequality. Double inequality

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

is known as Hermite-Hadamard integral inequality for harmonically convex functions, where $f \in L([a,b])$ [7, 5]. When we are trying to obtain these inequalities in the spirit of monotone measures and non-additive integrals, we get different results than the classic form.

The concept of the fuzzy integral was introduced and initially examined by Sugeno [17]. Further theoretical investigations of the integral and its generalizations have been pursued by many researchers [14, 15, 12, 2, 8, 1]. The study of inequalities for the Sugeno integral was initiated by Román-Flores and Chalco-Cano [13]. In this article, at the first we prove some Hermite-Hadamard type inequalities for harmonically convex functions in the case of non-additive integrals. Consequently, upper bound for these functions are established. In fact, the main purpose of this article is to obtain an approximation for non-solvable integral of this type.

This paper is organized as follows. Some necessary preliminaries are presented in Section 2. We address the essential problems in Sections 3 and upper bound for the Sugeno integral based on a harmonically convex function is presented. Finally, a conclusion is drawn and a problem for further investigations is given in Section 4.

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2. Preliminaries

In this section, we are going to review some well known results from the theory of non-additive measures.

Definition 2.1. [8, 18] Let Σ be a Σ-algebra of subsets of X and let $\mu : \Sigma \to [0, \infty)$ be a non-negative, extended real-valued set function, we say that μ is a monotone measure (or fuzzy measure) iff:

(FM1): $\mu(\emptyset) = 0$;
(FM2): $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);
(FM3): $(E_n) \subseteq \Sigma, \quad E_1 \subseteq E_2 \subseteq \dots$ imply $\lim_{n\to+\infty} \mu(E_n) = \mu(\bigcup_{i=1}^{\infty} E_i)$ (continuity from below);
(FM4): $(E_n) \subseteq \Sigma, \quad E_1 \supseteq E_2 \supseteq \dots, \mu(E_1) < \infty$ imply $\lim_{n\to+\infty} \mu(E_n) = \mu(\bigcap_{i=1}^{\infty} E_i)$ (continuity from above).

Let (X, Σ, μ) be a monotone measure space and f is a non-negative real-valued function on X. We denote the set of all non-negative measurable functions f by $\mathcal{F}_+$ and F_α denote the set $\{x \in X \mid f(x) \geq \alpha\}$, the α-level of f, for $\alpha \geq 0$. $F_0 = \{x \in X \mid f(x) > 0\} = supp(f)$ is the support of f. We know that: $\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$.

Definition 2.2. [17, 8, 18] Let μ be a monotone measure (or fuzzy measure) on (X, Σ). If $f \in \mathcal{F}_+$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the monotone measure μ is defined by

$$\fint_A f d\mu := \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$, $\wedge$ denotes the operation sup and inf on $[0, \infty)$ respectively. In particular if $A = X$, then

$$\fint_X f d\mu := \fint f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).$$

The following properties of the Sugeno integral are well known and can be found in [18, 19].

Proposition 2.3. Let (X, Σ, μ) be a fuzzy measure space, with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. We have

1. $\fint_A f d\mu \leq \mu(A)$;
2. $\fint_A k d\mu \leq k \wedge \mu(A)$, for k non-negative constant;
3. if $f \leq g$ on A, then $\fint_A f d\mu \leq \fint_A g d\mu$;
4. if $A \subset B$, then $\fint_A f d\mu \leq \fint_B f d\mu$;
5. if $\mu(A) < \infty$, then $\fint_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$;
6. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow \fint_A f d\mu \leq \alpha$;
7. $\fint_A f d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$;
8. $\fint_A f d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.4. Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap F_\alpha)$. Then, due to (5) and (6) of Proposition 2.3, we have that

$$F(\alpha) = \alpha \Rightarrow \fint_A f d\mu = \alpha.$$

Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

The following proposition shows how to transform a Sugeno integral $⨍_A f \mathrm{d}\mu$, which is defined on a monotone measure space (X, Σ, μ), into another Sugeno integral $⨍ g \mathrm{d}m$ defined on the Lebesgue measure space $([0,\infty), \overline{B_+}, m)$, where $\overline{B_+}$ is the class of all Borel sets in $[0,\infty)$ and m is the Lebesgue measure.

Proposition 2.5. [18] For any $A \in \Sigma$

$$⨍_A f \mathrm{d}\mu = ⨍ \mu(A \cap F_\alpha) \mathrm{d}m,$$

where $F_\alpha = \{x \in X \mid f(x) \geq \alpha\}$ and m is the Lebesgue measure.

Definition 2.6. [16] A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(T_1): $T(x, 1) = T(1, x) = x$ for any $x \in [0, 1]$;
(T_2): For any $x_1, x_2, y_1, y_2 \in [0, 1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, $T(x_1, y_1) \leq T(x_2, y_2)$;
(T_3): $T(x, y) = T(y, x)$ for any $x, y \in [0, 1]$;
(T_4): $T(T(x, y), z) = T(x, T(y, z))$ for any $x, y, z \in [0, 1]$.

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm [9] if there is a t-norm T such that $S(x, y) = 1 - T(1 - x, 1 - y)$.

Example 2.7. The following functions are t-norms:

1: $T_M(x, y) = x \wedge y$.
2: $T_P(x, y) = x.y$.
3: $T_L(x, y) = (x + y - 1) \vee 0$.

Hereafter, we assume that (X, Σ, μ) is a monotone measure space. To simplify the calculation of the Sugeno integral, for a given $f \in \mathcal{F}_+(X)$ and $A \in \Sigma$, we write

$$\Gamma = \{\alpha : \alpha \geq 0, \;\; \mu(A \cap F_\alpha) > \mu(A \cap F_\beta) \;\; for \;\; any \;\; \beta > \alpha\}.$$

It is easy to see that

$$⨍_A f \mathrm{d}\mu = \bigvee_{\alpha \in \Gamma} (\alpha \wedge \mu(A \cap F_\alpha)).$$

Remark 2.8. A binary operator T on $[0, 1]$ is called a t-seminorm[16] if it satisfies the above condition (T_1) and (T_2). Notice that if T is a t-seminorm, for any $x, y \in [0, 1]$, we have $T(x, y) \leq T(x, 1) = x$ and $T(x, y) \leq T(1, y) = y$, and consequently, $T(x, y) \leq T_M(x, y)$.

By using the concept of t-seminorm, García and Álvarez [16] proposed the following family of fuzzy integral.

Definition 2.9. Let T be a t-seminorm. Then the seminormed Sugeno's fuzzy integral of a function $f \in \mathcal{F}_+$ over $A \in \Sigma$ with respect to T and the fuzzy measure μ is defined by

$$\int_{T,A} f \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} T(\alpha, \mu(A \cap F_\alpha)).$$

Notice that the Sugeno integral of $f \in \mathcal{F}_+$ over $A \in \Sigma$ is the seminormed Sugeno's fuzzy integral of f over $A \in \Sigma$ with respect to the t-seminorm T_M.

Proposition 2.10. (García and Álvarez [16])Let (X, Σ, μ) be a monotone measure space and T be a t-seminorm. Then,

1: For any $A \in \Sigma$ and $f, g \in \mathcal{F}_+$ with $f \leq g$, we have

$$\int_{T,A} f \mathrm{d}\mu \leq \int_{T,A} g \mathrm{d}\mu.$$

2: For $A, B \in \Sigma$ with $A \subset B$ and any $f \in \mathcal{F}_+$,

$$\int_{T,A} f \mathrm{d}\mu \leq \int_{T,B} f \mathrm{d}\mu.$$

Definition 2.11. [7] Let $I \subset \mathbb{R} - \{0\}$ is a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex on I if the inequality

$$f\left(\frac{ab}{ta + (1-t)b}\right) \leq tf(b) + (1-t)f(a) \tag{2.1}$$

holds, for all $a, b \in I$ and $t \in [0, 1]$. If the inequality (2.1) is reversed, then f is said to be harmonically concave. We note that for $t = \frac{1}{2}$, we have the definition of Jensen type of harmonic convex functions, that is

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{f(a) + f(b)}{2}, \ \forall a, b \in I.$$

Proposition 2.12. [7] Let $I \subset \mathbb{R} - \{0\}$ be a real interval and $f : I \to \mathbb{R}$ is function, then:

1: if $I \subset (0, +\infty)$ and f is convex and nondecreasing, then f is harmonically convex.
2: if $I \subset (0, +\infty)$ and f is harmonically convex and nonincreasing, then f is convex.
3: if $I \subset (-\infty, 0)$ and f is harmonically convex and nondecreasing, then f is convex.
4: if $I \subset (-\infty, 0)$ and f is convex and nonincreasing, then f is harmonically convex.

Proposition 2.13. [4] If $[a, b] \subset I \subseteq (0, \infty)$ and we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(t) = f(\frac{1}{t})$, then f is harmonically convex on $[a, b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Proposition 2.14. [6] A function $f : (0, \infty) \to \mathbb{R}$ is harmonically convex if and only if $xf(x)$ is convex.

Theorem 2.15. Let $f : [a, b] \subseteq (0, \infty) \to [0, +\infty)$ be a convex function with $f(a) \neq f(b)$.Then

$$⨍_a^b f \mathrm{d}\mu \leq \bigvee_{\alpha \in \Gamma} \left(\alpha \wedge \mu\left([a, b] \cap \left\{x \geq \frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}\right\}\right)\right)$$

where $\Gamma = [f(a), f(b))$ for $f(b) > f(a)$ and $\Gamma = [f(b), f(a))$ for $f(a) > f(b)$.

Proof. As f is convex function, for $x \in [a, b]$ we have,

$$f(x) = f\left((1 - \frac{x-a}{b-a})a + \frac{x-a}{b-a}b\right) \leq (1 - \frac{x-a}{b-a})f(a) + \frac{x-a}{b-a}f(b)$$

and so by (3) of Proposition 2.3

$$-\!\!\!\!\!\!\int_a^b f\mathrm{d}\mu \leq -\!\!\!\!\!\!\int_a^b \left((1-\frac{x-a}{b-a})f(a)+\frac{x-a}{b-a}f(b)\right)\mathrm{d}\mu = -\!\!\!\!\!\!\int_a^b g(x)\mathrm{d}\mu.$$

In order to calculate the integral in the right hard part of the last inequality, we consider the distribution function $F(\alpha)$ given by

$$F(\alpha)=\mu([a,b]\cap\{g\geq\alpha\})=\mu\left([a,b]\cap\left\{\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)\geq\alpha\right\}\right).$$

If $f(a)<f(b)$, then

$$F(\alpha)=\mu\left([a,b]\cap\left\{x\geq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)=\mu\left([\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)},b]\right).$$

Thus $\Gamma=[f(a),f(b))$ and we only consider $\alpha\in[f(a),f(b))$.

If $f(a)>f(b)$, then

$$F(\alpha)=\mu\left([a,b]\cap\left\{x\leq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)=\mu\left([a,\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\right).$$

Thus $\Gamma=[f(b),f(a))$ and only need $\alpha\in[f(b),f(a))$.
This completes the proof. □

Remark 2.16. In the case $f(a)=f(b)$ in Theorem 2.15, we have $g(x)=f(x)$ and so

$$-\!\!\!\!\!\!\int_a^b f\mathrm{d}\mu \leq -\!\!\!\!\!\!\int_a^b g\mathrm{d}\mu = -\!\!\!\!\!\!\int_a^b f(a)\mathrm{d}\mu = f(a)\wedge\mu([a,b]).$$

Corollary 2.17. Let $f:[a,b]\subseteq(0,\infty)\to(0,\infty)$ be a convex function and Σ be the Borel field and μ be the Lebesgue measure on $X=\mathbb{R}$, then

$$-\!\!\!\!\!\!\int_a^b f\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge(b-\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)})\right) & ,f(a)<f(b)\\ f(a)\wedge(b-a) & ,f(a)=f(b)\\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge(\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}-a)\right) & ,f(a)>f(b)\end{cases}$$

So

$$-\!\!\!\!\!\!\int_a^b f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}\wedge(b-a) & ,f(a)<f(b)\\ f(a)\wedge(b-a) & ,f(a)=f(b)\\ \frac{(b-a)f(a)}{f(a)-f(b)+(b-a)}\wedge(b-a) & ,f(a)>f(b).\end{cases}$$

Proof. In the case where $f(a)<f(b)$, we have

$$\bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge(b-\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)})\right)=\frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}.$$

In fact, $\alpha = \frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}$ is as the solution of the equation $F(\alpha) = \alpha$, where F is the distribution function. So taking into account (1) of Proposition 2.3 ($⨍_a^b f\mathrm{d}\mu \leq \mu([a,b]) = b-a$) and Remark 2.4 we have

$$⨍_a^b f\mathrm{d}\mu \leq \frac{(b-a)f(b)}{f(b)-f(a)+(b-a)} \wedge (b-a).$$

Proofs the other cases is analogous. □

Note that Corollary 2.17 is the same as the Sadarangani Theorem [3].

3. Main Results

Let $I \subset \mathbb{R} - \{0\}$ be a harmonically convex function and $a, b \in I$ with $a < b$ and $f \in L([a,b])$. The following inequalities

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{3.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for harmonically convex functions.

Unfortunately, as we will see in the following example, in general, the Hermite-Hadamard inequality is not valid in the fuzzy context.

Example 3.1. Let μ be the usual Lebesgue measure on $\mathbb{R}$ and the function $f(x) = \frac{3}{7}x^2$ on $X = [\frac{1}{2}, 1]$. Obviously, this function is convex and nondecreasing as a result f is harmonically convex function on $[\frac{1}{2}, 1]$. With the above inequality we have

$$⨍_{\frac{1}{2}}^1 \frac{f(x)}{x^2}dx = ⨍_{\frac{1}{2}}^1 \frac{3}{7}dx = \frac{3}{7} \wedge \mu([\frac{1}{2}, 1]) = \frac{3}{7} \simeq 0.42.$$

on the other hand, $\frac{f(\frac{1}{2})+f(1)}{2} = \frac{15}{56} \simeq 0.26$.

This proves that the right-hand side of inequality (3.1) is not satisfied for the Sugeno integrals.

The aim of this work is to show a the Hermite-Hadamard type inequality for the Sugeno integral in the case where f is a harmonically convex function.

Lemma 3.2. Let $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave, then

$$⨍_a^b f\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha \in [f(a), f(b))} \left(\alpha \wedge \mu[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}, b]\right) & , f(a) < f(b) \\ f(a) \wedge \mu([a,b]) & , f(a) = f(b) \\ \bigvee_{\alpha \in [f(b), f(a))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\right) & , f(a) > f(b). \end{cases}$$

Proof. Since $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ is harmonically convex function on the interval $[a,b]$, then by Proposition 2.13 the function $g : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$, $g(s) = f(\frac{1}{s})$ is convex on $[\frac{1}{b}, \frac{1}{a}]$. Obviously for any $x \in [a,b]$, $f(x) = g(\frac{1}{x})$, and therefor applying Theorem 2.15 to g, we have

$$\fint_a^b f(x)\mathrm{d}\mu = \fint_a^b g(\frac{1}{x})\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[g(\frac{1}{a}),g(\frac{1}{b}))} \Big(\alpha\wedge\mu[\frac{\alpha(b-a)+ag(\frac{1}{b})-bg(\frac{1}{a})}{g(\frac{1}{b})-g(\frac{1}{a})},b]\Big) & ,g(\frac{1}{a})<g(\frac{1}{b}) \\ g(\frac{1}{a})\wedge\mu([a,b]) & ,g(\frac{1}{a})=g(\frac{1}{b}) \\ \bigvee_{\alpha\in[g(\frac{1}{b}),g(\frac{1}{a}))} \Big(\alpha\wedge\mu[a,\frac{\alpha(b-a)+ag(\frac{1}{b})-bg(\frac{1}{a})}{g(\frac{1}{b})-g(\frac{1}{a})}]\Big) & ,g(\frac{1}{a})>g(\frac{1}{b}) \end{cases}$$

$$= \begin{cases} \bigvee_{\alpha\in[f(a),f(b))} \Big(\alpha\wedge\mu[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)},b]\Big) & ,f(a)<f(b) \\ f(a)\wedge\mu([a,b]) & ,f(a)=f(b) \\ \bigvee_{\alpha\in[f(b),f(a))} \Big(\alpha\wedge\mu[a,\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\Big) & ,f(a)>f(b). \end{cases}$$

□

Corollary 3.3. Let $f:[a,b]\subseteq(0,\infty)\to(0,\infty)$ be a harmonically convex function which is not concave, Σ be the Borel field and μ be the Lebesgue measure on $X=\mathbb{R}$, then

$$\fint_a^b f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(b)}{f(b)-f(a)+b-a}\wedge(b-a) & ,f(a)<f(b) \\ f(a)\wedge(b-a) & ,f(a)=f(b) \\ \frac{(b-a)f(a)}{f(a)-f(b)+b-a}\wedge(b-a) & ,f(a)>f(b). \end{cases}$$

Remark 3.4. If $[a,b]\subseteq(0,\infty)$ and f is harmonically convex and nonincreasing, then taking into account (2) of Proposition 2.12 the function f is convex and hance the upper bound for the Sugeno integral of f mentioned in article "Hermite-Hadamard inequality for fuzzy integral", were written by K. sadarangani is established.

Remark 3.5. If $[a,b]\subseteq(-\infty,0)$ and f is harmonically convex and nondecreasing, then taking into account (3) of Proposition 2.12 the function f is convex and hance the upper bound for the Sugeno integral of f is established.

Example 3.6. Let μ be a Lebesgue measure and consider function $f(x) = e^{-\frac{1}{x}}$ on $[\frac{1}{3}, \frac{3}{4}]$. Obviously, this function is non-negative and harmonically convex but neither convex, nor concave. we have,

$$\begin{aligned}
\fint_{\frac{1}{3}}^{\frac{3}{4}} f\mathrm{d}\mu &= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([\frac{1}{3}, \frac{3}{4}] \cap \left\{e^{-\frac{1}{x}} \geq \alpha\right\}\right)\right) \\
&= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([\frac{1}{3}, \frac{3}{4}] \cap \left\{-\frac{1}{x} \geq \ln\alpha\right\}\right)\right) \\
&= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([\frac{1}{3}, \frac{3}{4}] \cap \{-1 \geq xln\alpha\}\right)\right) \\
&= \bigvee_{\alpha\geq 0}\left(\alpha \wedge \mu\left([\frac{1}{3}, \frac{3}{4}] \cap \left\{x \geq \frac{-1}{ln\alpha}\right\}\right)\right).
\end{aligned}$$

As result with the solution of the equation

$$\frac{1}{ln\alpha} + \frac{3}{4} = \alpha$$

we conclude that $\alpha \simeq 0/175$. Then $\fint_{\frac{1}{3}}^{\frac{3}{4}} f\mathrm{d}\mu \simeq 0/175$.

On the other hand, since $f(\frac{3}{4}) = \frac{1}{e^{\frac{4}{3}}}$ and $f(\frac{1}{3}) = \frac{1}{e^3}$. By Corollary 3.3, we have

$$\fint_{\frac{1}{3}}^{\frac{3}{4}} f\mathrm{d}\mu \leq \frac{f(\frac{3}{4})(\frac{3}{4}-\frac{1}{3})}{f(\frac{3}{4}) - f(\frac{1}{3}) + (\frac{3}{4}-\frac{1}{3})} \wedge (\frac{3}{4} - \frac{1}{3})$$

$$\simeq 0/234 \wedge \frac{5}{12} = 0/234 \wedge 0/416 = 0/234$$

that is a logical inequality.

Example 3.7. The function $f(x) = x - \ln(x+1)$ is nondecreasing and harmonic convex function on $[\frac{1}{2}, 1]$. $f(1) = 1 - \ln 2$ and $f(\frac{1}{2}) = \frac{1}{2} - \ln(\frac{3}{2})$. As $f(1) > f(\frac{1}{2})$, Corollary 3.3 gives us,

$$\fint_{\frac{1}{2}}^{1} f\mathrm{d}\mu \leq \frac{(1-\frac{1}{2})f(1)}{f(1) - f(\frac{1}{2}) + \frac{1}{2}} \wedge (\frac{1}{2}) \simeq 0.718 \wedge \frac{1}{2} = \frac{1}{2}.$$

Thus, we find an upper bound for the Sugeno integral of this function on $[\frac{1}{2}, 1]$.

Example 3.8. The function $f(x) = e^{x^2+x}$ is nondecreasing and harmonic convex function on $[1, 2]$ and $f(1) = e^2$ and $f(2) = e^5$. As follows we find an upper bound for the Sugeno integral of this function,

$$\fint_{1}^{2} e^{x^2+x}\mathrm{d}\mu \leq \frac{e^5}{e^5 - e^2 + 1} \wedge (1) \simeq 1.0449 \wedge 1 = 1.$$

Remark 3.9. $f(x) = log(x)$ is a harmonically convex function but not convex, that is why in the Corollary 3.3, does not apply because it is concave. For concave function, we use the Sadarangani paper.

Corollary 3.10. Let $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be a harmonically convex function which is not concave and $g : \mathbb{R} \to \mathbb{R}$ is a linear function, then $f \circ g$ is harmonically convex[10] and so,

$$\fint_a^b (f \circ g)\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[f(g(a)),f(g(b)))} \left(\alpha \wedge \mu[\frac{\alpha(b-a)+af(g(b))-bf(g(a))}{f(g(b))-f(g(a))}, b]\right) & , f(g(a)) < f(g(b)) \\ f(g(a)) \wedge \mu([a, b]) & , f(g(a)) = f(g(b)) \\ \bigvee_{\alpha\in[f(g(b)),f(g(a)))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(g(b))-bf(g(a))}{af(g(b))-bf(g(a))}]\right) & , f(g(a)) > f(g(b)). \end{cases}$$

Corollary 3.11. Let $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ be a harmonically convex function which is not concave and $g : \mathbb{R} \to \mathbb{R}$ is a linear function, Σ be the Borel field and μ be the Lebesgue measure on $X = \mathbb{R}$, then $f \circ g$ is harmonic convex function[10] and so,

$$\fint_a^b (f \circ g)\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(g(b))}{f(g(b))-f(g(a))+b-a} \wedge (b-a) & , f(g(a)) < f(g(b)) \\ f(g(a)) \wedge (b-a) & , f(g(a)) = f(g(b)) \\ \frac{(b-a)f(g(a))}{f(g(a))-f(g(b))+b-a} \wedge (b-a) & , f(g(a)) > f(g(b)). \end{cases}$$

Remark 3.12. In the case g be harmonic convex function and f be relative convex function, we know that $f \circ g$ is harmonically convex function [11]. Thus similar results of Corollary 3.10 and Corollary 3.11 hold.

Corollary 3.13. Let $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ be a harmonically convex function which is not concave function, Σ be the Borel field and μ be the Lebesgue measure on $X = \mathbb{R}$, then

$$\int_{T_P,[a,b]} f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)^2 f(b)}{f(b)-f(a)+b-a} & , f(a) < f(b) \\ (b-a)f(a) & , f(a) = f(b) \\ \frac{(b-a)^2 f(a)}{f(a)-f(b)+b-a} & , f(a) > f(b). \end{cases}$$

Proof. For harmonically convex function $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ with $f(a) \neq f(b)$ according to Proposition 2.10 and Corollary 3.3 with t-norm T_p, we have

$$\int_{T_P,[a,b]} f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(b)}{f(b)-f(a)+b-a}.(b-a) & , f(a) < f(b) \\ f(a).(b-a) & , f(a) = f(b) \\ \frac{(b-a)f(a)}{f(a)-f(b)+b-a}.(b-a) & , f(a) > f(b) \end{cases}$$

$$= \begin{cases} \frac{(b-a)^2 f(b)}{f(b)-f(a)+b-a} & , f(a) < f(b) \\ (b-a)f(a) & , f(a) = f(b) \\ \frac{(b-a)^2 f(a)}{f(a)-f(b)+b-a} & , f(a) > f(b). \end{cases}$$

□

Example 3.14. Let μ be the Lebesgue measure on $\mathbb{R}$. Consider the function $f(x) = \frac{1}{x^2}$ on $X = [1,3]$. Obviously, this function is harmonically convex and positive on $X = [1,3]$. As $f(1) = 1$ and $f(3) = \frac{1}{9}$, using Corollary 3.13, we can get the following estimate:

$$\int_{T_P,[1,3]} \frac{1}{x^2} \mathrm{d}\mu \leq \frac{(3-1)^2 f(1)}{f(1) - f(3) + (3-1)} = \frac{18}{13}.$$

Now, let's introduce the most important theorem of this article. With the help of it, an upper bound in the framework of the Sugeno integral for Hermite-Hadamard inequality of harmonically convex functions can be established.

Theorem 3.15. Let $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave, then

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu \leq \fint_a^b f \mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha \in [f(a), f(b))} \left(\alpha \wedge \mu[\frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}, b] \right) & , f(a) < f(b) \\ f(a) \wedge \mu([a,b]) & , f(a) = f(b) \\ \bigvee_{\alpha \in [f(b), f(a))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}] \right) & , f(a) > f(b) \end{cases}$$

where $m_0 = min\{a^2, b^2\}$.

Proof. Let f be a harmonically convex function which is not concave and $m_0 = min\{a^2, b^2\}$. By Proposition 2.5 we have,

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu = \int_a^b \mu([a,b] \cap F_\alpha) \mathrm{d}m \tag{3.2}$$

where m is the Lebesgue measure and

$$F_\alpha = \{x \in X : m_0 \frac{f(x)}{x^2} \geq \alpha\}.$$

Obviously,

$$\left([a,b] \cap \left\{ f(x) \geq \frac{x^2}{m_0} \alpha \right\} \right) \subseteq ([a,b] \cap \{f(x) \geq \alpha\}).$$

By monotonicity μ, we deduce

$$\mu \left([a,b] \cap \left\{ f(x) \geq \frac{x^2}{m_0} \alpha \right\} \right) \leq \mu ([a,b] \cap \{f(x) \geq \alpha\}).$$

Now, by Proposition 2.3 and Proposition 2.5, we obtain

$$\int_a^b \mu \left([a,b] \cap \{f \geq \frac{x^2}{m_0} \alpha\} \right) \mathrm{d}m \leq \int_a^b \mu ([a,b] \cap \{f \geq \alpha\}) \mathrm{d}m = \fint_a^b f \mathrm{d}\mu. \tag{3.3}$$

Combining (3.2 , 3.3), we have

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu \leq \fint_a^b f \mathrm{d}\mu.$$

The last inequality follows from Lemma 3.2. □

Corollary 3.16. If $f:[a,b]\subseteq(0,\infty)\to(0,\infty)$ be a harmonically convex function which is not concave then,

$$\fint_a^b xf(x)\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[af(a),bf(b))}\left(\alpha\wedge\mu[\frac{\alpha(b-a)+abf(b)-baf(a)}{bf(b)-af(a)},b]\right) & ,af(a)<bf(b) \\ af(a)\wedge\mu([a,b]) & ,af(a)=bf(b) \\ \bigvee_{\alpha\in[bf(b),af(a))}\left(\alpha\wedge\mu[a,\frac{\alpha(b-a)+abf(b)-baf(a)}{bf(b)-af(a)}]\right) & ,af(a)>bf(b). \end{cases}$$

Proof. f is harmonically convex function.Therefore, according to the Proposition 2.14 $xf(x)$ is convex. Finally, the proof is complete by using Theorem 2.15. □

Corollary 3.17. If $f:[a,b]\subseteq(0,\infty)\to(0,\infty)$ be a harmonically convex function which is not concave, Σ be Borel field and μ be a Lebesgue measure on $X=\mathbb{R}$, then

$$\fint_a^b xf(x)\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)bf(b)}{bf(b)-af(a)+b-a}\wedge(b-a) & ,af(a)<bf(b) \\ af(a)\wedge(b-a) & ,af(a)=bf(b) \\ \frac{(b-a)af(a)}{af(a)-bf(b)+b-a}\wedge(b-a) & ,af(a)>bf(b). \end{cases}$$

Example 3.18. Let μ be the usual Lebesgue measure on X and the function $f(x)=\frac{3}{5}x^2$ on $X=[1,2]$. Obviously, this function is convex and nondecreasing. So by (1) of Proposition 2.12 f is harmonically convex on $[1,2]$. With use the Corollary 3.17 we have

$$\fint_1^2 xf(x)dx \leq \frac{(2-1)2f(2)}{2f(2)-f(1)+(2-1)}\wedge(2-1)\simeq 0.923.$$

On the other hand, $\fint_1^2 xf(x)dx\simeq 0.87$. This show that the Corollary 3.17 is valid.

4. Conclusion

In this paper, we have researched the Hermite-Hadamard inequality for the Sugeno integral based on harmonically convex functions. For further investigations we propose to consider the Hermite-Hadamard inequality for the Choquet integral, and also for some other non-additive integrals. In the future research, we will continue to explore other integral inequalities for non-additive measures and integrals based on harmonically convex function.

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