Studies on the Higher Order Difference Equation $x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\dfrac{ax_{n-t}}{bx_{n-t}+c}$

M. M. El-Dessoky[1,2]
[1]King Abdulaziz University, Faculty of Science, Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Mansoura University, Faculty of Science, Department of Mathematics,
Mansoura 35516, Egypt.
E-mail: dessokym@mans.edu.eg

ABSTRACT

The main objective of this paper is to study the local and the global stability of the solutions, the periodic character and the boundedness of the difference equation

$$x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\frac{ax_{n-t}}{bx_{n-t}+c},\qquad n=0,\ 1,\ ...,$$

where the parameters $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1},...,\ x_{-1},$ x_0 are positive real numbers where $s=max\{l,\ k,\ t\}$.

Keywords: Difference equations, Stability, Global stability, Boundedness, Periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

The higher-order difference equations are of paramount importance in applications. Such equations also seem naturally as discrete analogues and as numerical solutions of differential which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics and so on [1-9]. The theory of difference equations gets a central position in applicable analysis. That is, the theory of difference equations will continue to play an important role in mathematics as a whole. Hence, it is very interesting to study the behavior of solutions of a difference equations and to discuss the local and global asymptotic stability of their equilibrium points [10-15]. In recent years, the behavior of solutions of various difference equations has been one of the main topics in the theory of difference equations [16-34].

Abo-Zeid [35] obtained the global asymptotic stability of all solutions of the difference equation

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_nx_{n-1}x_{n-2}},\qquad n=0,1,...,$$

where $A,\ B,\ C$ are positive real numbers and the initial conditions $x_{-2},\ x_{-1},\ x_0$ are real numbers.

Abu-Saris et al. [36] studied the globally asymptotically stability of the equilibrium solution of the rational difference equation

$$x_{n+1}=\frac{a+x_nx_{n-k}}{x_n+x_{n-k}},\qquad n=0,1,...,$$

where k is a nonnegative integer, $a\geq 0$, and $x_{-k},\ ...,\ x_0>0$.

You-Hui et al. [37] investigated the global attractivity of the nonlinear difference equation

$$y_{n+1}=\frac{p+qy_n}{1+y_n+ry_{n-k}},\qquad n=0,1,...,$$

where $p,\ q,\ r\in[0,\ \infty)$, $k\geq 1$ is a positive integer and the initial conditions $y_{-k},\ \ldots,\ y_{-1}$ are nonnegative real numbers and y_0 is a positive real number.

Zayed et al. [38] investigated the boundedness character, the periodicity character, the convergence and the global stability of positive solutions of the difference equation,

$$x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}}, \qquad n = 0, 1, ...,$$

where the coefficients $\alpha_i,\ \beta_i \in (0,\ \infty)$ for $i = 0,\ 1,\ 2$, and $l,\ k$ are positive integers such that $l < k$. The initial conditions $x_{-k},\ ,\ ...,\ x_{-l},\ ...,\ x_{-2},\ x_{-1}, x_0$ are arbitrary positive real numbers.

El-Dessoky [39] investigated some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{dx_{n-s} - e}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $a,\ b,\ c,\ d$ and e are positive real numbers and the initial conditions $x_{-t},\ x_{-t+1}, ...,\ x_{-1}$, x_0 are positive real numbers where $t = max\{l,\ k,\ s\}$.

Our goal is to obtain some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1} = \beta x_{n-l} + \alpha x_{n-k} + \frac{ax_{n-t}}{bx_{n-t} + c}, \qquad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1}, ...,\ x_{-1}$, x_0 are positive real numbers where $s = max\{l,\ k,\ t\}$.

2. LOCAL STABILITY

In this section, we study the local stability of the equilibrium point of equation (1).

The equilibrium points of Eq. (1) are given by

$$\overline{x} = \beta\overline{x} + \alpha\overline{x} + \frac{a\overline{x}}{b\overline{x} + c},$$

$$b(1 - \alpha - \beta)\overline{x}^2 + c(1 - \alpha - \beta)\overline{x} = a\overline{x}.$$

So, $\overline{x}_0 = 0$ is forever an equilibrium point of the difference equation (1). If $\alpha + \beta < 1$, then the positive equilibrium point of the Eq. (1) is given by

$$\overline{x}_1 = \frac{a}{b(1 - \alpha - \beta)} - \frac{c}{b}.$$

Let $f : (0,\ \infty)^3 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$f(u,\ v,\ w) = \beta u + \alpha v + \frac{aw}{bw + c}.$$

Therefore, it follows that

$$\frac{\partial f(u,\ v,\ w)}{\partial u} = \beta,\ \frac{\partial f(u,\ v,\ w)}{\partial v} = \alpha,\ \frac{\partial f(u,\ v,\ w)}{\partial w} = \frac{ac}{(bw + c)^2}. \tag{2}$$

THEOREM 2.1. *The zero equilibrium $\overline{x}_0$ of the difference equation (1) is locally asymptotically stable if*

$$c\,(\alpha + \beta) + a < c. \tag{3}$$

Proof: So, we can write Eq. (2) at zero equilibrium point $\overline{x}_0 = 0$ in the form

$$\frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial u} = \beta = p_1,\ \frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial v} = \alpha = p_2 \text{ and } \frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial w} = \frac{a}{c} = p_3.$$

Then the linearized equation of Eq. (1) about $\overline{x}_0$ is

$$y_{n+1} - p_1 y_{n-k} - p_2 y_{n-l} - p_3 y_{n-s} = 0,$$

According to Theorem 1.6 page 7 in [1], then Eq. (1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$|\beta| + |\alpha| + \left|\frac{a}{c}\right| < 1,$$

and so

$$c\,(\alpha + \beta) + a < c.$$

The proof is complete.

Example 1. Consider $l = 2$, $k = 1$, $t = 3$, $\beta = 0.3$, $\alpha = 0.2$, $a = 0.5$, $b = 0.7$ and $c = 6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the zero solution of the difference equation (1) is local stability (see Fig. 1).

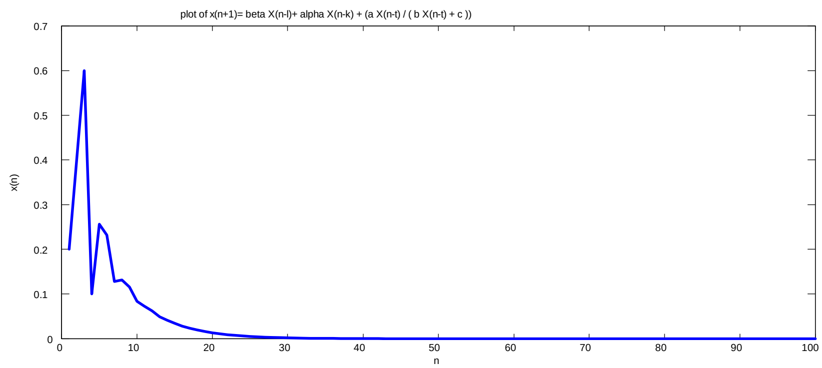

Figure 1. Plot the behavior of zero solution of Eq. (1) is local stable.

THEOREM 2.2. *The positive equilibrium $\overline{x}_1$ of the difference equation (1) is locally asymptotically stable if*

$$c\,(1 - \alpha - \beta) < a. \tag{4}$$

Proof: So, we can write Eq. (2) at the positive equilibrium point $\overline{x}_1 = \frac{a}{b(1-\alpha-\beta)} - \frac{c}{b}$

$$\frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial u} = \beta = p_1,\ \ \frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial v} = \alpha = p_2 \text{ and } \frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial w} = \frac{c(1-\alpha-\beta)^2}{a} = p_3.$$

Then the linearized equation of Eq. (1) about $\overline{x}_1$ is

$$y_{n+1} - p_1 y_{n-k} - p_2 y_{n-l} - p_3 y_{n-s} = 0,$$

According to Theorem 1.6 page 7 in [1], then Eq. (1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$|\beta| + |\alpha| + \left|\frac{c(1-\alpha-\beta)^2}{a}\right| < 1,$$

and so

$$\frac{c(1-\alpha-\beta)^2}{a} < 1 - a - b,$$

if $\alpha + \beta < 1$, then

$$c\,(1 - \alpha - \beta) < a.$$

The proof is complete.

Example 2. Figure (2) shows the solution of the difference equation (1) is local stability if $l = 2$, $k = 1$, $t = 3$, $\beta = 0.3$, $\alpha = 0.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$.

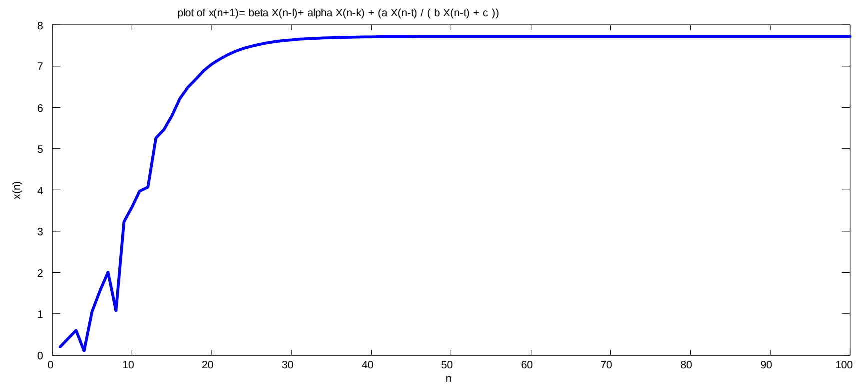

Figure 2. Draw the behavior of the positve solution of Eq. (1) is local stable.

Example 3. The solution of the difference equation (1) is unstable if $l = 2$, $k = 1$, $t = 3$, $\beta = 0.9$, $\alpha = 0.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 3).

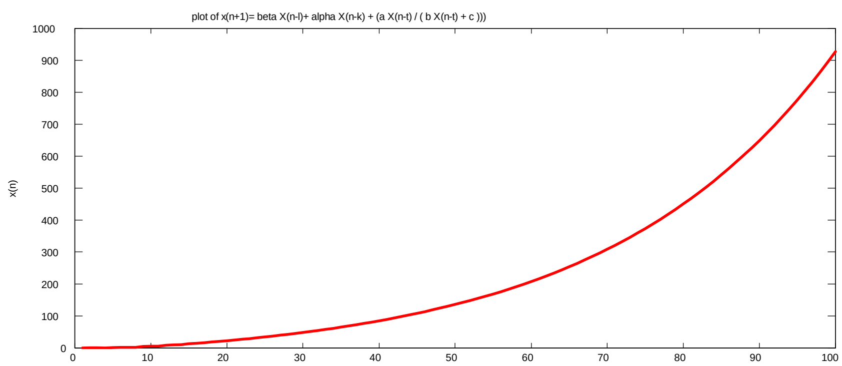

Figure 3. Sketch the behavior of the solution of Eq. (1) is unstable.

3. GLOBAL STABILITY

In this section, the global asymptotic stability of equation (1) is studied.

THEOREM 3.1. *The equilibrium point $\overline{x}_0$ is a global attractor of difference equation (1) if*

$$\alpha + \beta + \frac{a}{c} < 1. \tag{5}$$

Proof: Suppose that ζ and η are real numbers and assume that $F : [\zeta, \eta]^3 \longrightarrow [\zeta, \eta]$ is a function defined by

$$F(x,\ y,\ z) = \beta x + \alpha y + \tfrac{az}{bz+c}.$$

Then
$$\tfrac{\partial F(x,\ y,\ z)}{\partial x} = \beta,\ \tfrac{\partial F(x,\ y,\ z)}{\partial y} = \alpha \text{ and } \tfrac{\partial F(x,\ y,\ z)}{\partial z} = \tfrac{ac}{(bz+c)^2}.$$

Now, we can see that the function $F(x,\ y,\ z)$ increasing in $x,\ y$ and z. Then

$$\begin{aligned} & \left[\beta x + \alpha x + \frac{ax}{bx+c} - x\right](x - \overline{x}_0) \\ \leq \quad & \left[-(1-\alpha-\beta)x + \frac{ax}{bx} + \frac{ax}{c}\right](x-0) \leq -\left(1-\alpha-\beta-\frac{a}{c}\right)x^2 < 0 \end{aligned}$$

If $\alpha+\beta+\frac{a}{c}<1$, then $F(x,\ y,\ z)$ satisfies the negative feedback property

$$[F(x,\ x,\ x) - x](x - \overline{x}_0) < 0, \quad \text{for } \overline{x}_0 = 0.$$

According to Theorem 1.10 page 15 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

Example 4. Consider $l = 2,\ k = 1,\ t = 3,\ \beta = 0.03,\ \alpha = 0.02,\ a = 0.5,\ b = 0.7$ and $c = 4$ and the initial conditions $x_{-3} = 0.5,\ x_{-2} = 0.7,\ x_{-1} = 0.6$ and $x_0 = 1.1$, the zero solution of the difference equation (1) is global stability (see Fig. 4).

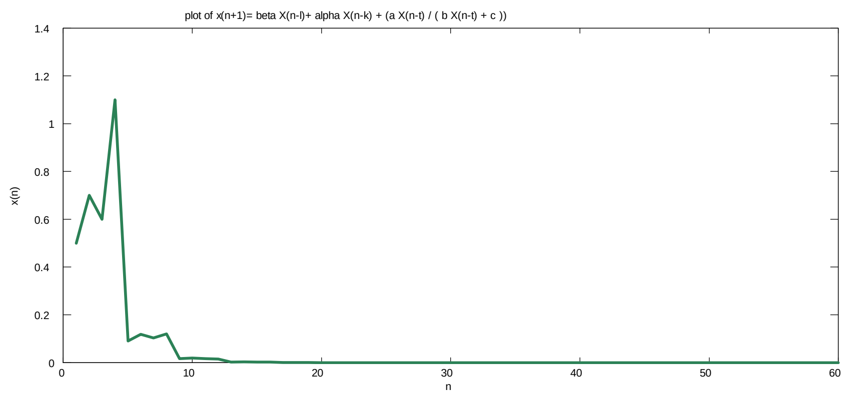

Figure 4. Plot the behavior of the zero solution of Eq. (1) is global stability.

THEOREM 3.2. *The equilibrium point $\overline{x}_1$ is a global attractor of difference equation (1) if*

$$\beta + \alpha < 1. \tag{6}$$

Proof: Suppose that ζ and η are real numbers and assume that $g : [\zeta, \eta]^3 \longrightarrow [\zeta, \eta]$ is a function defined by

$$g(u,\ v,\ w) = \beta u + \alpha v + \tfrac{aw}{bw+c}.$$

Then

$$\tfrac{\partial g(u,\ v,\ w)}{\partial u} = \beta,\ \tfrac{\partial g(u,\ v,\ w)}{\partial v} = \alpha \text{ and } \tfrac{\partial g(u,\ v,\ w)}{\partial w} = \tfrac{ac}{(bw+c)^2}.$$

Now, we can see that the function $g(u,\ v,\ w)$ increasing in $u,\ v$ and w.

Let (m, M) be a solution of the system $M = g(M,\ M,\ M)$ and $m = g(m,\ m,\ m)$. Then from Eq. (1), we see that

$$M = \beta M + \alpha M + \frac{aM}{bM+c},\ m = \beta m + \alpha m + \frac{am}{bm+c},$$

thus

$$\begin{aligned} b(1\alpha-\beta)M^2 - c(1-\alpha-\beta)M &= aM, \\ b(1\alpha-\beta)m^2 - c(1-\alpha-\beta)m &= am. \end{aligned}$$

Subtracting we obtain

$$\begin{aligned} b(1-\alpha-\beta)(M^2-m^2)-(a+c(1-\alpha-\beta))(M-m) &= 0, \\ (M-m)\{b(1-\alpha-\beta)(M+m)-a-c(1-\alpha-\beta)\} &= 0 \end{aligned}$$

under the condition $0 \neq b(1-\alpha-\beta)$, we see that

$$M = m.$$

According to Theorem 1.15 page 18 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

Example 5. The solution of the difference equation (1) is global stability when $l = 2$, $k = 1$, $t = 3$, $\beta = 0.1$, $\alpha = 0.2$, $a = 2$, $b = 1$ and $c = 0.01$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 5).

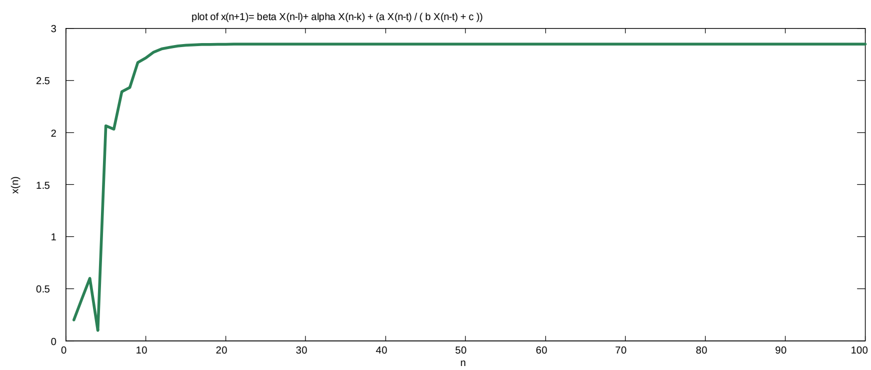

Figure 5. Plot the behavior of the solution of Eq. (1) is global stability.

4. PERIODIC SOLUTIONS

THEOREM 4.1. *Let l, k and t are both odd positive integers then for all β, α, a, b and c are positive real numbers, then Eq. (1) has a prime period two solution if*

$$\alpha + \beta < 1 \text{ and } c(1-\alpha-\beta) < a. \tag{7}$$

Proof: First, suppose that there exists distinct nonnegative solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when l, k and t are both odd, then $x_{n+1} = x_{n-l} = x_{n-k} = x_{n-t} = P$. It follows Eq. (1) that

$$P = \beta P + \alpha P + \tfrac{aP}{bP+c} \text{ and } Q = \beta Q + \alpha Q + \tfrac{aQ}{bQ+c}.$$

Therefore,

$$b(1-\alpha-\beta)P^2 + (c(1-\alpha-\beta)-a)P = 0, \tag{8}$$

$$b(1-\alpha-\beta)Q^2 + (c(1-\alpha-\beta)-a)Q = 0, \tag{9}$$

Subtracting (9) from (8) gives

$$P + Q = \tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} \tag{10}$$

Again, adding (8) and (9) yields

$$PQ = 0. \tag{11}$$

where $a > c(1-\alpha-\beta)$ and $1 > \alpha+\beta$. Let P and Q are the two distinct nonnegative real roots of the quadratic

$$b(1-\alpha-\beta)t^2 - (a - c(1-\alpha-\beta))t = 0, \tag{12}$$

and so

$$a - c(1-\alpha-\beta) > 0 \text{ and } 1-\alpha-\beta > 0, \tag{13}$$

from Inequality (13), we obtain Inequality (7).

Second suppose that Inequality (7) is true. We will show that Eq. (1) has a prime period two solution.

Therefore P and Q are distinct nonnegative real numbers.

Set

$$x_{-l} = P,\ x_{-k} = P,\ x_{-t} = P,\ ,...,\ x_{-3} = P,\ x_{-2} = Q,\ x_{-1} = P,\ x_0 = Q.$$

We would like to show that

$$x_1 = x_{-1} = P = \tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} \quad \text{and} \quad x_2 = x_0 = Q = 0.$$

It follows from Eq. (1) that

$$\begin{aligned} x_1 &= \beta P + \alpha P + \tfrac{aP}{bP+c} = (\beta+\alpha)P + \frac{a\left(\frac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right)}{b\left(\frac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right)+c}, \\ &= (\beta+\alpha)\left(\tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right) + \tfrac{a(a-c(1-\alpha-\beta))}{b(a-c(1-\alpha-\beta))+cb(1-\alpha-\beta)} = (\beta+\alpha)\left(\tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right) + \tfrac{a(a-c(1-\alpha-\beta))}{ab}, \\ &= \tfrac{(\beta+\alpha)(a-c(1-\alpha-\beta))+(1-\alpha-\beta)(a-c(1-\alpha-\beta))}{b(1-\alpha-\beta)} = \tfrac{(a-c(1-\alpha-\beta))(\beta+\alpha+1-\alpha-\beta)}{b(1-\alpha-\beta)} = \tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} = P. \end{aligned}$$

and

$$x_2 = \beta Q + \alpha Q + \tfrac{aQ}{bQ+c} = 0 = Q,$$

Then by induction we get

$$x_{2n} = Q \quad \text{and} \quad x_{2n+1} = P \quad \text{for all} \quad n \geq -2.$$

Thus Eq. (1) has the prime period two solution

$$...,\ P,\ Q,\ P,\ Q,\ ...,$$

where P and Q are the distinct nonnegative roots of the quadratic Eq. (12) and the proof is complete.

Theorem 4.2. *Let $l,\ k$ and t are both even positive integers then for all $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Let that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $l,\ k$ and t are both even, then $x_{n+1} = P$ and $x_{n-l} = x_{n-k} = x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha Q + \tfrac{aQ}{bQ+c} \text{ and } Q = \beta P + \alpha P + \tfrac{aP}{bP+c}.$$

Therefore,

$$bPQ + cP = b(\beta+\alpha)Q^2 + (a + c(\beta+\alpha))Q, \tag{14}$$

$$bPQ + cQ = b(\beta+\alpha)P^2 + (a + c(\beta+\alpha))P, \tag{15}$$

Subtracting (15) from (14) gives

$$P + Q = -\frac{a+c(1+\beta+\alpha)}{b(\beta+\alpha)} \tag{16}$$

Again, adding (14) and (15) yields

$$PQ = \frac{c(a+c(1+\beta+\alpha))}{b^2(\beta+\alpha)(1+\beta+\alpha)}. \tag{17}$$

From (16) and (17), we have

$$(P+Q)\,PQ = -\frac{c(a+c(1+\beta+\alpha)+c)^2}{b^3(\beta+\alpha)^2(1+\beta+\alpha)} < 0$$

This contradicts the hypothesis that both P and Q are positive. Thus, the proof is now completed.

Theorem 4.3. *Let $l,\ k$ are even and t is odd positive integers then for all $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Let that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q,...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $l,\ k$ are even and t is odd, then $x_{n+1} = x_{n-t} = P$ and $x_{n-l} = x_{n-k} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha Q + \frac{aP}{bP+c} \text{ and } Q = \beta P + \alpha P + \frac{aQ}{bQ+c}.$$

Therefore,

$$bP^2 + cP = b(\alpha+\beta)PQ + c(\alpha+\beta)Q + aP, \tag{18}$$

$$bQ^2 + cQ = b(\alpha+\beta)PQ + c(\alpha+\beta)P + aQ, \tag{19}$$

By subtracting (18) from (19), we deduce

$$P + Q = \frac{a - c(1+\alpha+\beta)}{b} \tag{20}$$

Again, by adding (18) and (19), we get

$$PQ = -\left(\frac{c(\alpha+\beta)(a-c(1+\alpha+\beta))}{b^2(\alpha+\beta+1)}\right). \tag{21}$$

If $a > c(1+\alpha+\beta)$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Therefor, the proof is completed.

Theorem 4.4. *If $l,\ t$ are even and k is odd positive integers then Eq. (1) has no positive prime period two solution.*

Proof: Let there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q,...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $l,\ t$ are even and k is odd, then $x_{n+1} = x_{n-k} = P$ and $x_{n-l} = x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha P + \frac{aQ}{bQ+c} \text{ and } Q = \beta P + \alpha Q + \frac{aP}{bP+c}.$$

Therefore,

$$b(1-\alpha)PQ + c(1-\alpha)P = b\beta Q^2 + (c\beta + a)Q, \tag{22}$$

$$b(1-\alpha)PQ + c(1-\alpha)Q = b\beta P^2 + (c\beta + a)P, \tag{23}$$

By subtracting (23) from (22), we get

$$P + Q = -\left(\frac{a + c(1-\alpha+\beta)}{b\beta}\right) \tag{24}$$

While, by adding (22) and (23), we deduce

$$PQ = \frac{c(1-\alpha)(a + c(1-\alpha+\beta))}{b^2\beta(1-\alpha+\beta)}. \tag{25}$$

If $\alpha < 1$ and $\alpha < 1 + \beta$ then from (24) and (25), we have

$$PQ(P+Q) = -\frac{c(1-\alpha)(a + c(1-\alpha+\beta))^2}{b^3\beta^2(1-\alpha+\beta)} < 0$$

This contradicts the hypothesis that both P, Q are positive. Thus, the proof is now completed.

THEOREM 4.5. *Suppose that $k,\ t$ are even and l is odd positive integers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $k,\ t$ are even and l is odd, then $x_{n+1} = x_{n-l} = P$ and $x_{n-k} = x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta P + \alpha Q + \frac{aQ}{bQ+c} \text{ and } Q = \beta Q + \alpha P + \frac{aP}{bP+c}.$$

Therefore,

$$b(1-\beta)PQ + c(1-\beta)P = b\alpha Q^2 + (c\alpha + a)Q, \tag{26}$$

$$b(1-\beta)PQ + c(1-\beta)Q = b\alpha P^2 + (c\alpha + a)P, \tag{27}$$

By subtracting (27) from (26), we have

$$P + Q = -\left(\frac{a + c(1+\alpha-\beta)}{b\alpha}\right) \tag{28}$$

Again, by adding (26) and (27),we deduce

$$PQ = \frac{c(1-\beta)(a + c(1+\alpha-\beta))}{b^2\alpha(1+\alpha-\beta)}. \tag{29}$$

If $\beta < 1$ and $\beta < 1 + \alpha$ then from (28) and (29), we have

$$PQ\left(P+Q\right) = -\frac{c(1-\beta)(a+c(1+\alpha-\beta))^2}{b^2\alpha^2(1+\alpha-\beta)} < 0$$

This contradicts the hypothesis that both P, Q are positive. Thus, the proof is now completed.

THEOREM 4.6. *Let k is even and $l,\ t$ are odd positive integers then for all $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when k is even and $l,\ t$ are odd, then $x_{n+1} = x_{n-l} = x_{n-t} = P$ and $x_{n-k} = Q$. It follows Eq. (1) that

$$P = \beta P + \alpha Q + \frac{aP}{bP+c} \text{ and } Q = \beta Q + \alpha P + \frac{aQ}{bQ+c}.$$

Therefore,

$$b\left(1-\beta\right)P^2 + c\left(1-\beta\right)P = b\alpha PQ + c\alpha Q + aP, \tag{30}$$

$$b\left(1-\beta\right)Q^2 + c\left(1-\beta\right)Q = b\alpha PQ + c\alpha P + aQ, \tag{31}$$

By subtracting (31) from (30), we get

$$P + Q = \frac{a-c(1+\alpha-\beta)}{b(1-\beta)} \tag{32}$$

Again, by adding (30) and (31), we have

$$PQ = -\frac{c\alpha(a-c(1+\alpha-\beta))}{b^2(1-\beta)(1+\alpha-\beta)}. \tag{33}$$

where $\beta < 1$, $\beta < 1+\alpha$ and $c\left(1+\alpha-\beta\right) < a$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Therefor, the proof is completed.

THEOREM 4.7. *If l is even and $k,\ t$ are odd positive integers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when l is even and $k,\ t$ are odd, then $x_{n+1} = x_{n-k} = x_{n-t} = P$ and $x_{n-l} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha P + \frac{aP}{bP+c} \text{ and } Q = \beta P + \alpha Q + \frac{aQ}{bQ+c}.$$

Therefore,

$$b\left(1-\alpha\right)P^2 + c\left(1-\alpha\right)P = b\beta PQ + c\beta Q + aP, \tag{31}$$

$$b\left(1-\alpha\right)Q^2 + c\left(1-\alpha\right)Q = b\beta PQ + c\beta P + aQ, \tag{32}$$

Subtracting (32) from (31) gives

$$P + Q = \frac{a-c(1+\beta-\alpha)}{b(1-\alpha)} \tag{33}$$

Again, adding (31) and (32) yields

$$PQ = -\frac{c\beta(a-c(1+\beta-\alpha))}{b^2(1-\alpha)(1+\beta-\alpha)}. \quad (34)$$

If $\alpha < 1$, $\alpha < 1+\beta$ and $c(1+\beta-\alpha) < a$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Thus, the proof is completed.

THEOREM 4.8. *Let t is even and l, k are odd positive integers. If*

$$c(1-\alpha-\beta)+a \neq 0,$$

then Eq. (1) has no prime period two solution.

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when t is even and l, k are odd, then $x_{n+1} = x_{n-k} = x_{n-l} = P$ and $x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta P + \alpha P + \frac{aQ}{bQ+c} \text{ and } Q = \beta Q + \alpha Q + \frac{aP}{bP+c}.$$

Therefore,

$$b(1-\alpha-\beta)PQ + c(1-\alpha-\beta)P = aQ, \quad (35)$$

$$b(1-\alpha-\beta)PQ + c(1-\alpha-\beta)Q = aP, \quad (36)$$

Subtracting (47) from (46) gives

$$(c(1-\alpha-\beta)+a)(P-Q) = 0$$

Since $c(1-\alpha-\beta)+a \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is completed.

Example 6. Figure (6) shows the Eq. (1) has a period two solution when $l = 1$, $k = 3$, $t = 5$, $\beta = 0.1$, $\alpha = 0.2$, $a = 0.5$, $b = 0.07$ and $c = 0.05$ and the initial conditions $x_{-5} = P$, $x_{-4} = Q$, $x_{-3} = P$, $x_{-2} = Q$, $x_{-1} = P$ and $x_0 = Q$ where $P = \frac{a-c(1+\beta-\alpha)}{b(1-\alpha)}$ and $Q = 0$.

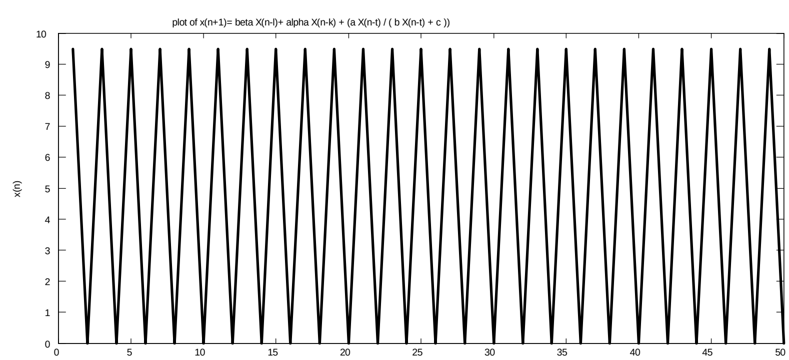

Figure 6. Sketch the solution of Eq. (1) has a period two solution.

Example 7. Consider $l = 5$, $k = 2$, $t = 4$, $\beta = 0.6$, $\alpha = 0.2$, $a = 0.4$, $b = 0.7$ and $c = 0.5$ and the initial conditions $x_{-5} = 1.2$, $x_{-4} = 1.4$, $x_{-3} = 0.6$, $x_{-2} = 1.1$, $x_{-1} = 0.3$ and $x_0 = 0.8$ the solution of Eq. (1) has no period two solution (See Fig. 7).

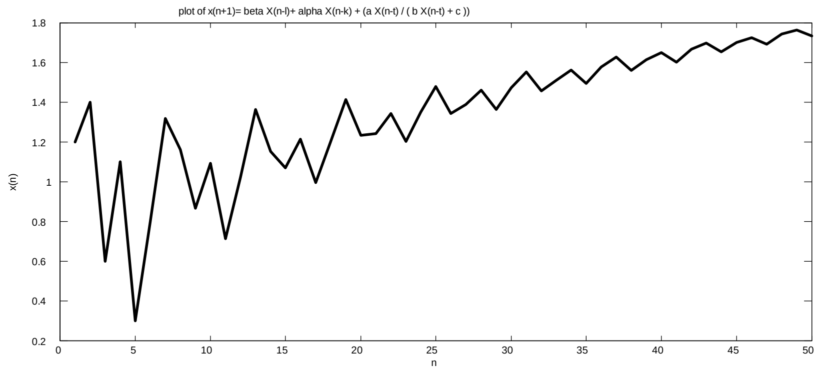

Figure 7. Draw the solution of Eq. (1) has no periodic.

5. BOUNDEDNESS OF THE SOLUTIONS

In this section, we investigate the boundedness nature of the positive solutions of equation (1).

Theorem 5.1. *Every solution of difference equation (1) is bounded if* $\beta+\alpha<1$.

Proof: Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned} x_{n+1} &= \beta x_{n-l}+\alpha x_{n-k}+\tfrac{ax_{n-t}}{bx_{n-t}+c}, \\ &\leqslant \beta x_{n-l}+\alpha x_{n-k}+\tfrac{ax_{n-t}}{bx_{n-t}}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{a}{b} \quad \text{for all } n\geq 1. \end{aligned}$$

By using a comparison, we can right hand side as follows

$$t_{n+1}=\beta t_{n-l}+\alpha t_{n-k}+\tfrac{a}{b}.$$

and this equation is locally asymptotically stable if $\beta+\alpha<1$, and converges to the equilibrium point $\overline{t}=\frac{a}{b(1-\beta-\alpha)}$.Therefore

$$\lim_{n\to\infty}\sup x_n \leqslant \tfrac{a}{b(1-\beta-\alpha)}.$$

Thus the solution is bounded.

Theorem 5.2. *Every solution of difference equation (1) is unbounded if* $\beta>1or\ \alpha>1$.

Proof: Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1).Then from Equation (1) we see that

$$x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{ax_{n-t}}{bx_{n-t}+c}>\beta x_{n-l} \quad for \text{ all } \quad n\geq 1.$$

We see that the right hand side can be written as follows

$$t_{n+1}=\beta t_{n-l}.$$

then

$$t_{ln+i}=\beta^n t_{l+i}+const., \qquad i=0,1,...,l,$$

and this equation is unstable because $\beta>1$, and $\lim\limits_{n\to\infty} t_n=\infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded from above.

Similarly from Equation (1) we see that

$$x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{ax_{n-t}}{bx_{n-t}+c}>\alpha x_{n-k}\ for\ all\ n\geq 1.$$

We see that the right hand side can be written as follows

$$t_{n+1}=\alpha t_{n-k}.$$

then

$$t_{kn+i} = \alpha^n t_{k+i} + const., \qquad i = 0, 1, ..., k,$$

and this equation is unstable because $\alpha > 1$, and $\lim_{n\to\infty} t_n = \infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded from above. Thus, the proof is now completed.

Example 8. We assume $l = 2$, $k = 1$, $t = 3$, $\beta = 0.4$, $\alpha = 1.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the solution of the difference equation (1) is unbounded (see Fig. 8).

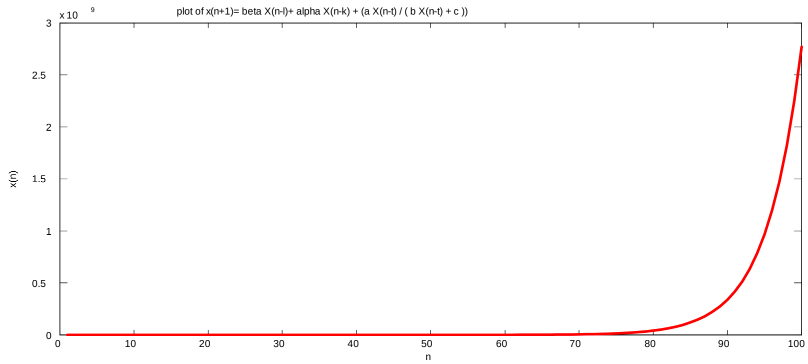

Figure 8. Plot the behavior of the solution of Eq. (1) is unbounded.

REFERENCES

1. E. A. Grove and G. Ladas, Periodicities in nonlinear difference equations, Vol., 4, Chapman & Hall / CRC Press, 2005.
2. V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.
3. M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.
4. E. C. Pielou, Population and Community Ecology, Gordon and Breach, New York, 1974.
5. Chang-you Wang, Shu Wang, Zhi-wei Wang, Fei Gong, and Rui-fang Wang, Asymptotic Stability for a Class of Nonlinear Difference Equations, Discrete Dyn. Nat. Soc., Vol., 2010, (2010), Article ID 791610, 10 pages.
6. R. P. Agarwal, Difference Equations and Inequalities, Vol. 228 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.
7. L. Erbe, J. Baoguo, and A. Peterson, Nonoscillation for second order sublinear dynamic equations on time scales, J. Comput. Appl. Math., Vol., 232(2), (2009), 594–599.
8. S. Stević, Boundedness and global stability of a higher-order difference equation, J. Difference Equ. Appl., Vol., 14(10-11), (2008), 1035-1044.
9. Lin-Xia Hu, Hong-Ming Xia, Global asymptotic stability of a second order rational difference equation, Appl. Math. Comput., 233, (2014), 377–382.
10. Tuo Li and Xiu-Mei Jia, Global Behavior of a Higher-Order Difference Equation, Discrete Dyn. Nat. Soc., Vol., 2010, (2010), Article ID 834020, 8 pages.
11. M. M. El-Dessoky, Dynamics and Behavior of the Higher Order Rational Difference equation, J. Comput. Anal. Appl., Vol., 21(4), (2016), 743-760.
12. Maoxin Liao , Xianhua Tang, Changjin Xu: On the rational difference equation $x_{n+1} = 1 + \frac{(1-x_{n-k})(1-x_{n-l})(1-x_{n-m})}{x_{n-k}+x_{n-l}+x_{n-m}}$, J. Appl. Math. Comput., **35**, 63-71, (2011).
13. A. Cabada & J. B. Ferreiro, Existence of positive solutions for nth-order periodic difference equations, J. Difference Equ. Appl., Vol., 17(6), (2011), 935-954.
14. R. Abo-Zeid, Global behavior of a higher order difference equation, Mathematica Slovaca, 64(4), (2014), 931-940.

15. M. A. El-Moneam, S. O. Alamoudy, On Study of the Asymptotic Behavior of Some Rational Difference Equations, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 21, (2014), 89-109.
16. M. M. El-Dessoky, Qualitative behavior of rational difference equation of big Order, Discrete Dyn. Nat. Soc., Vol., 2013, (2013), Article ID 495838, 6 pages.
17. H. Sedaghata, Global attractivity in a class of non-autonomous, nonlinear, higher order difference equations, J. Difference Equ. Appl., Vol., 19(7), (2013), 1049-1064.
18. Chuanxi Qiana, Global attractivity in a higher order difference equation with variable coefficients, J. Difference Equ. Appl., Vol., 18(7), (2012), 1121-1132.
19. E. M. Elsayed and M. M. El-Dessoky, Dynamics and global behavior for a fourth-order rational difference equation, Hacettepe J. Math. and Stat., 42(5) (2013), 479–494.
20. M. A. Obaid, E. M. Elsayed and M. M. El-Dessoky, Global Attractivity and Periodic Character of Difference Equation of Order Four, Discrete Dyn. Nat. Soc., Vol., 2012, (2012), Article ID 746738, 20 pages.
21. Mehmet Gűműş, The Periodicity of Positive Solutions of the Nonlinear Difference Equation $x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}$, Discrete Dyn. Nat. Soc., Vol., 2013, (2013), Article ID 742912, 3 pages.
22. E. M. Elsayed, Dynamics and behavior of a higher order rational difference equation, J. Nonlinear Sci. Appl., 9 (4), (2016), 1463–1474.
23. E. M. Elsayed, M. M. El-Dessoky and Ebraheem O. Alzahrani, The Form of The Solution and Dynamics of a Rational Recursive Sequence, J. Comput. Anal. Appl., Vol. 17(1), (2014), 172-186.
24. E. M. E. Zayed, On the dynamics of the nonlinear rational di erence equation, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 22, (2015), 61-71.
25. I. Yalcinkaya, A. E. Hamza and C. Cinar, Global behavior of a recursive sequence, Selcuk J. Appl. Math., 14, (2013), 3-10.
26. Abdul Khaliq, E. M. Elsayed, The dynamics and solution of some difference equations, J. Nonlinear Sci. Appl., 9 (3), (2016), 1052–1063.
27. E. M. Elsayed, M. M. El-Dessoky and Asim Asiri, Dynamics and Behavior of a Second Order Rational Difference equation, J. Comput. Anal. Appl., Vol. 16(4), (2014), 794-807.
28. E. M. E. Zayed, Dynamics of a higher order nonlinear rational difference equation, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 23, (2016), 143-152.
29. M. M. El-Dessoky, On the dynamics of higher Order difference equations $x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}$, J. Comput. Anal. Appl., **22**, (2017), 1309-1322.
30. M. M. El-Dessoky and Aatef Hobiny, Dynamics of a Higher Order Difference Equations $x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l} + bx_{n-k}}{cx_{n-l} + dx_{n-k}}$, J. Comput. Anal. Appl., **24**, (2018), 1353-1365.
31. M. M. El-Dessoky and M. A. El-Moneam, On the Difference equation $x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + \frac{\gamma x_{n-k}}{Dx_{n-s} + Ex_{n-t}}$, J. Comput. Anal. Appl., **25**, (2018), 342-354.
32. M. M. El-Dessoky, On the dynamics of a higher Order rational difference equations, Journal of the Egyptian Mathematical Society, Vol. 25(1), (2017), 28–36.
33. Asim Asiri, M. M. El-Dessoky and E. M. Elsayed, Solution of a third order fractional system of difference equations , J. Comput. Anal. Appl., Vol., 24(3), (2018), 444-453.
34. M. M. El-Dessoky, E. M. Elabbasy and Asim Asiri, Dynamics and solutions of a fifth-order nonlinear difference equation, Discrete Dyn. Nat. Soc. , Vol. 2018, (2018), Article ID 9129354, 21 pages.
35. R. Abo-Zeid, On the oscillation of a third order rational difference equation, J. Egyptian Math. Soc., 23, (2015), 62–66.
36. R. Abu-Saris, C. Cinar, I. Yalçınkaya, On the asymptotic stability of $x_{n+1} = \frac{a + x_n x_{n-k}}{x_n + x_{n-k}}$, Comput. Math. Appl., 56, (2008), 1172–1175.
37. You-Hui Suab & Wan-Tong Lia, Global attractivity of a higher order nonlinear difference equation, J. Difference Equ. Appl., 11(10), (2005), 947-958.
38. E. M. E. Zayed, M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-l} + \alpha_2 x_{n-k}}{\beta_0 x_n + \beta_1 x_{n-l} + \beta_2 x_{n-k}}$, Mathematica Bohemica, 3, (2010), 319-336.
39. M. M. El-Dessoky, On the Difference equation $x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{dx_{n-s} - e}$, Math. Methods Appl. Sci., Vol. 40(3), (2017), 535–545.