Error Estimation of signals via Cesaro-Euler operator of its Fourier-Laguerre series

[1,2]Smita Sonker and [1]Neeraj Devi

[1]Department of Mathematics, National Institute of Technology Kurukshetra,
Haryana 136119, India.
[2]School of Physical Sciences, Jawaharlal Nehru University,
New Delhi- 110067 (India).
smitafma@nitkkr.ac.in, neerajkundu5@gmail.com

ABSTRACT

In this work, we proposed to use the $(C, \beta, \gamma)(E, q)$ composite summation operator with its Fourier-Laguerre series at point $t = 0$ to estimate the error of a function that belongs to the $L[0, \infty)$-class. Our results generalize previous findings by Sonker, who evaluated the degree of approximation by $(C,2)(E,q)$ operator, and Krasniqi, who examined function approximation by $(C,1)(E,q)$ operator. Using this product summation operator, we also presented the error estimation theorem and various graphical interpretations produced by MATLAB program.

Keywords– Fourier-Laguerre approximation, (C, β, γ) operator, (E, q) operator, Cesàro-Euler Composite operator, Error estimation, Graphical Analysis

Introduction And Motivation

It is believed that in the last few decades of the nineteenth century, a well-known Weierstrass theorem served as the foundation for approximation theory of functions. Quantifying the errors generated and figuring out how functions can be best approximated by simpler functions are the two main objectives of approximation theory. The field of signal approximation has been studied extensively and has been the interest of academics. Approximation theory is important tool in the field of robotics [1], applied mathematics [2–4] and operator theory [5]. The theorems of fuzzy numbers and sequence spaces are also covered by summability theory [6]. Product summable operators [7–10] outperform single summable operators [11–13]. Numerous mathematicians studying approximation theory and summability were motivated by this finding. In a study on the error estimate of a function via the $(C, 1)(E, q)$ approach, Krasniqi [14] presented a study that made use of the concept of product summable operators. This paper was further improved in 2014 by Sonker [15]. She approximated the series at $t = 0$ by using the $(C, 2)(E, q)$ operator. In response to these findings, the error in the approximation of the same series function was found in 2015 by Mittal and Singh [16] using the $(T.E_q)$ summable approach. Later, in 2016, Khatri and Mishra [17] calculated the degree of approximation of the Fourier-Laguerre series by using the H^1E^1 product summable operator under appropriate conditions. In 2021, Sharma [18] conducted an investigation into the (T, C_β) approach of the Fourier-Laguerre series, building on earlier work in the field [19]. This motivated us to calculate a function's error at the point $t = 0$ by applying $(C, \beta, \gamma)(E, q)$ composite summation operator of its Fourier- Laguerre series. Our findings are compared to the results given by Krasniqi [14] and Sonker [15] in order to show the efficiency of proposed summation operator. We also introduced the error estimation theorem using composite summation operator along- with some graphical interpretations. Also, using different values of β, γ and q, the existing operators can be derived.

Cesàro-Euler composite summation operator

In this work, we introduced the Cesàro-Euler composite summation operator of order (β, γ, q) as follows:

A function $g(t) \in L(0, \infty)$ is expanded by Fourier-Laguerre method as

$$g(t) \equiv \sum_{m=0}^{\infty} c_m L_m^{\delta}(y), \tag{1}$$

where

$$c_m = \frac{1}{\Gamma(\delta+1)\binom{m+\delta}{m}} \int_0^{\infty} e^{-y}\, y^{\delta} g(y) L_m^{\delta}(y) dy, \tag{2}$$

and $L_m{}^{\delta}(y)$ denotes the m^{th} Laguerre polynomial of order $\delta \geq -1$, defined by generating function

$$\sum_{m=0}^{\infty} L_m{}^{\delta}(t) w^m = (1-w)^{-\delta-1} e^{\left(\frac{-tw}{1-w}\right)},$$

and the integral (2) exists. Also,

$$\sigma(y) = \frac{1}{\Gamma(\beta+1)} e^{-y} y^{\delta}[g(y) - g(0)]. \tag{3}$$

Let the sequence $\{s_m(g;t)\}$ be the m^{th} partial sum of the Fourier-Laguerre series (1) given by

$$s_m(g;t) = \sum_{h=0}^{m} c_h L_h^{\delta}(t),$$

is also known as Fourier-Laguerre polynomial of degree (or order) $\geq m$.

We denote $C_m^{\beta,\gamma}$ or (C, β, γ) the m^{th} Cesàro mean of order (β, γ) with $\beta + \gamma > -1$ of the sequence $\{s_m(g;t)\}$ i.e.

$$C_m^{\beta,\gamma} = \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} s_h,$$

where

$$A_m^{\beta+\gamma} = O\left(m^{\beta+\gamma}\right), \beta + \gamma \succ 1 \; and \,, A_0^{\beta+\gamma} = 1.$$

The Fourier-Laguerre series (1) is said to be (C, β, γ) summable to the definite number s if

$$C_m^{\beta,\gamma} = \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} s_h \to s \; as \; m \to \infty.$$

Also, If

$$E_m^q = \frac{1}{(1+q)^m} \sum_{h=0}^{m} \binom{m}{h} q^v s_h \to s \; as \; m \to \infty,$$

then $s_m(g;y)$ converges to a definite value 's' by E_m^q means (by Hardy [20]), and we write it as,

$$s_m \to s\left(E_m^q\right).$$

We now introduce the Cesàro-Euler product summability mean of order (β,γ,q) as follows.

1. The (C,β,γ) transform of the(E,q) transform defines $(C,\beta,\gamma)(E,q)$ transform of order (β,γ,q) and we shall denote it by $(CE)_m^{q,\beta,\gamma}$. Moreover, if
$$t_m^{CE} = (CE)_m^{q,\beta,\gamma} = \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} E_h^q$$
$$= \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v} q^v s_v \to s \; as \; m \to \infty. \qquad (4)$$
2. The regularity of (C,β,γ) and (E,q) methods implies the regularity of $(CE)_m^{q,\beta,\gamma}$ method.

Useful Lemmas

The proof of the main theorem, we require following lemmas:

Lemma 1: Given by Szegö (1975, p.177, Theorem 7.6.4) [21], let δ is any real number ε are fixed +ve constant. Then

$$L_m^{\delta}(t) = O\left(m^{\delta}\right) \; if \; 0 < t < 1/m, \qquad (5)$$

$$= O(t^{-(2\delta+1)/4} m^{(2\delta-1)/4} \text{ if } 1/m < t < \epsilon, \text{ as } m \to \infty. \qquad (6)$$

Lemma 2 Given by Szegö (1975, p.177, Theorem 7.6.4) [21], let α and δbe an arbitrary real no., and $0 < \chi < 4$and $\varepsilon > 0$, then

$$\max e^{-t/2} t^{\alpha} \mid L_m^{\delta}(t) \mid = O(m^Q)$$

where
$$Q = \max(\alpha - 1/2, \delta/2 - 1/4)\,, \; \epsilon \le t \le (4-\chi)m, \qquad (7)$$
$$= \max(\alpha - 1/3, \delta/2 - 1/4)\,, \; t > m. \qquad (8)$$

Lemma 3: Let $\delta \succ -1$. If $q > 0$, then
$$\frac{1}{(1+q)^m}\sum_{h=0}^{m}\binom{m}{h} q^h h^{(2\delta+1)/4} \le (1+1/q) m^{(2\delta+1)/4} = O\left(m^{(2\delta+1)/4}\right), \; (9)$$

and if $\beta + \gamma \succ -1$, then
$$I = \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma}(1+h)^{\delta} = O\left((1+m)^{\delta}\right). \qquad (10)$$

Proof: The first result is on similar lines as given by Lenski and Szal [22]. Regarding the latter result, A. Zygmund[23][7, Vol. I (1.15) and Theprem [1.17] have stated that
$$A_m^{\beta+\gamma} = \binom{m+\beta+\gamma}{m} \equiv O\left((m+1)^{\delta}\right),$$

is positive for $\beta + \gamma \succ -1$. Moreover, $A_m^{\beta+\gamma}$ is decreasing for $-1 < \beta + \gamma < 0$ and increasing for $\beta + \gamma > 0$. Hence for $\delta < 0$,

$$I = \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{[m/2-1]} A_{m-h}^{\beta-1} A_h^{\gamma}(1+h)^{\delta} + \frac{1}{A_m^{\beta+\gamma}}\sum_{h=[m/2]}^{m} A_{m-h}^{\beta-1} A_h^{\gamma}(1+h)^{\delta}$$

$$= O\left(\frac{(m+1)^{\beta-1}(m+1)^{\gamma}}{(m+1)^{\beta+\gamma}}\right)\sum_{h=0}^{[m/2-1]}(1+h)^{\delta}+ O\big((1+m)^{\delta}\big)\frac{1}{A_m^{\beta+\gamma}}\sum_{h=[m/2]}^{m}A_{m-h}^{\beta-1}A_h^{\delta}$$
$$= O((1+m)^{-1})\sum_{h=0}^{m}(1+h)^{\delta}\int_h^{h+1}dz + O\big((1+m)^{\delta}\big)\frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m}A_{m-h}^{\beta-1}A_h^{\delta}$$
$$= O((1+m)^{-1})\sum_{h=0}^{m}\int_h^{h+1}z^{\delta}\,dz + O\big((1+m)^{\delta}\big)$$
$$= O((1+m)^{-1})\int_0^{m+1}z^{\delta}\,dz + O\big((1+m)^{\delta}\big)$$
$$= O((1+m)^{-1})\frac{(m+1)^{\delta+1}}{\delta+1} + O\big((1+m)^{\delta}\big)$$
$$= O\big((1+m)^{\delta}\big)$$

If $\delta > 0$, the outcome is obvious. Our proof is thus finished.

Also, we will use

$$A_m^{\beta+\gamma}(t) = \frac{L_m^{(\beta+\gamma+1)}}{\Gamma(\beta+\gamma+1)},$$

and also using this we can prove $\beta+\gamma \succ 1$, then

$$\frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m}A_{m-h}^{\beta-1}A_h^{\gamma}\big(h^{(2\delta+1)/4}\big) = O\big(m^{(2\delta+1)/4}\big). \tag{11}$$

Error Estimation Theorem

Let g be a lebesgue integrable function then the error estimation of g at $t = 0$ by the Cesàro-Euler means of order (β,γ,q) with $\beta+\gamma \geq -1$, $q > 0$ of the Fourier-Laguerre series of g is given by

$$\mid (CE)_m^{q,\beta,\gamma}(g;0) - g(0) \mid = o\big(\tau(m)\big) \tag{12}$$

with conditions

$$\sigma(x) = \int_0^x \mid \sigma(y) \mid dy = o\left(x^{\delta+1}\tau(1/x)\right),\ x \to 0, \tag{13}$$

$$\int_{\epsilon}^{m} e^{y/2}\, y^{-(2\delta+3)/4} \mid \sigma(y) \mid dy = o\left(m^{(-2\delta+1)/4}\tau(m)\right), \tag{14}$$

and

$$\int_m^{\infty} e^{y/2}\, y^{-1/3} \mid \sigma(y) \mid dy = o\big(\tau(m)\big), \qquad m \to \infty, \tag{15}$$

where $\tau(x)$ is positive and monotonically increasing signal of x such that $\tau(m) \to$ as $m \to \infty$.

Proof of theorem:

Based on the equality

$$L_m^{\delta}(0) = \binom{m+\delta}{\delta},$$

we obtain

$$s_m(0) = s_m(g;0) = \sum_{h=0}^{m} c_h {L_h}^{\delta}(0)$$
$$= \frac{1}{\Gamma(\delta+1)\binom{m+\delta}{m}} L_m^{\delta}(0)\int_0^{\infty} e^{-y}\, y^{\delta} g(y)\sum_{h=0}^{m}{L_h}^{\delta}(y)dy$$
$$= \frac{1}{\Gamma(\delta+1)}\int_0^{\infty} e^{-y}\, y^{\delta} g(y) L_m^{\delta+1}(y)dy.$$

Now

$$(CE)_m^{q,\beta,\gamma}(g;0) = \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v s_v(0)$$
$$= \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v \frac{1}{\Gamma(\delta+1)}\int_0^\infty e^{-y}\, y^{\delta} g(y) L_v^{\delta+1}(y)dy.$$

Therefore using (3), we have

$$\mid (CE)_m^{q,\beta,\gamma}(g;0) - g(0) \mid = \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v \int_0^\infty \mid \sigma(y) \mid {L_v}^{\delta+1}(y)dy$$
$$= \left(\int_0^{1/m} + \int_{1/m}^{\epsilon} + \int_{\epsilon}^{m} + \int_{m}^{\infty}\right)\frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma}$$
$$\cdot \frac{1}{2^h}\sum_{v=0}^{h}\binom{h}{v} \mid \sigma(y) {L_v}^{\delta+1}(y)dy$$
$$= J\_1 + J\_2 + J\_3 + J\_4. \qquad (16)$$

Using orthogonal property (13), Lemma [1][condition 5] and Lemma [3] we get

$$J_1 = \int_0^{1/m} \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v \mid \sigma(y) \mid {L_v}^{\delta+1}(y)dy$$
$$= \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v O\left(m^{\delta+1}\right)\int_0^{1/m} \mid \sigma(y) \mid dy$$
$$= \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} O\left(m^{\delta+1}\right) o\left(\tau(m)/m^{\delta+1}\right)$$
$$= O\left(m^{\delta+1}\right) o\left(\tau(m)/m^{\delta+1}\right)$$
$$= o\left(\tau(m)\right). \qquad (17)$$

Further using the orthogonal property (14), Lemma [1][condition 6], Lemma (3) and using the argument as in Nigam and Sharma [13] and Krasniqi [14] then integrating by parts, we get

$$J_2 = \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v \int_{1/m}^{\epsilon} \mid \sigma(y) \mid {L_v}^{\delta+1}(y)$$
$$= \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v O\left(v^{(2\delta+1)/4}\right)\int_{1/m}^{\epsilon} \mid \sigma(y) \mid y^{-(2\delta+3)/4}dy$$
$$= \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} O\left(h^{(2\delta+1)/4}\right)\int_{1/m}^{\epsilon} \mid \sigma(y) \mid y^{-(2\delta+3)/4}dy$$
$$= O\left(m^{(2\delta+1)/4}\right)\int_{1/m}^{\epsilon} \mid \sigma(y) \mid y^{-(2\delta+3)/4}dy = o\left(\tau(m)\right). \qquad (18)$$

Using (14), Lemma [2][condition 7] and Lemma [3] we get

$$J_3 = \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v \int_{\epsilon}^{m} \mid \sigma(y) \mid {L_v}^{\delta+1}(y)dy$$
$$\leq \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v \int_{\epsilon}^{m} e^{y/2}\, y^{-(2\delta+3)/4} \mid \sigma(y)| e^{-y/2} y^{(2\delta+3)/4} L_v^{\delta+1}(y)dy$$
$$= \frac{1}{A_m^{\beta+\gamma}}\sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h}\sum_{v=0}^{h}\binom{h}{v}q^v O\left(v^{(2\delta+1)/4}\right)\int_{\epsilon}^{m} e^{y/2}\, y^{-(2\delta+3)/4} \mid \sigma(y) \mid dy$$
$$= O\left(m^{(2\delta+1)/4}\right) o\left(m^{-(2\delta+1)/4}\tau(m)\right) = o\left(\tau(m)\right). \qquad (19)$$

Finally, using (15), Lemma [2][condition 7] and Lemma [3], we get

$$J_4 = \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h} \sum_{v=0}^{h} \binom{h}{v} q^v \int_{m}^{\infty} |\sigma(y)| L_v^{\delta+1}(y) dy$$
$$\leq \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h} \sum_{v=0}^{h} \binom{h}{v} q^v \int_{m}^{\infty} e^{y/2} y^{-(3\delta+5)/6} |\sigma(y)| e^{-y/2} y^{(3\delta+5)/6} L_v^{\delta+1}(y) dy$$
$$= \frac{1}{A_m^{\beta+\gamma}} \sum_{h=0}^{m} A_{m-h}^{\beta-1} A_h^{\gamma} \frac{1}{(1+q)^h} \sum_{v=0}^{h} \binom{h}{v} q^v O\left(m^{(\delta+1)/2}\right) \int_{\epsilon}^{m} e^{y/2} \frac{y^{-1/3}|\sigma(y)|}{y^{(\delta+1)/2}} dy$$
$$= O\left(m^{(\delta+1)/2}\right) o\left(m^{-(\delta+1)/2} \tau(m)\right) = o\left(\tau(m)\right). \tag{20}$$

Combining (16), (17), (18), (19) and (20), we get

$$|\ (CE)_m^{q,\beta,\gamma}(g;0) - g(0)\ | = o\left(\tau(m)\right).$$

Corollary

- If we take $\beta = 1, \gamma = 0$, our findings reduce to the results given by Krasniqi [14].
- If we take $\beta = 2, \gamma = 0$, our findings reduce to the results given by Sonker [15].
- If we take $\beta = 1, \gamma = 0$ *and* $q = 0$, our findings reduce to the results given by Gupta [12].
- If we take $\beta = 0, \gamma = 0$ *and* $q = 1$, our findings reduce to the results given by Nigam and Sharma [13] and many others.

Graphical Analysis

Here, we consider the function

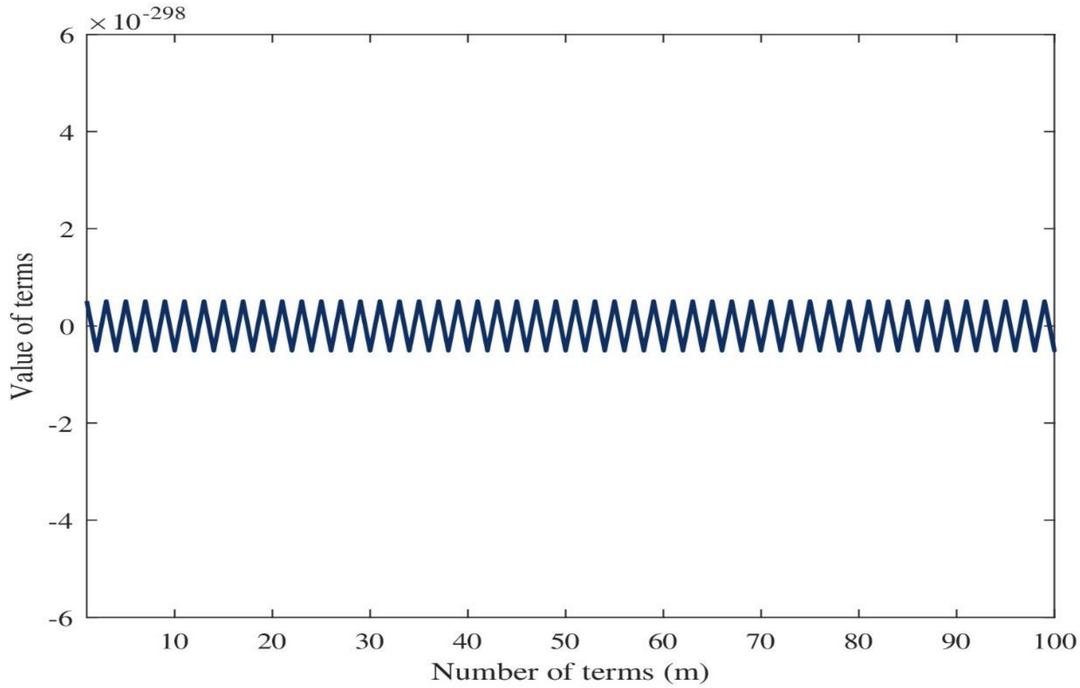
$$g(t) = t^6,$$

with its Fourier-Laguerre series

$$g(t) = \sum_{m=0}^{\infty} (-1)^m \binom{6}{m} \Gamma(7) L_m^{\beta}(t).$$

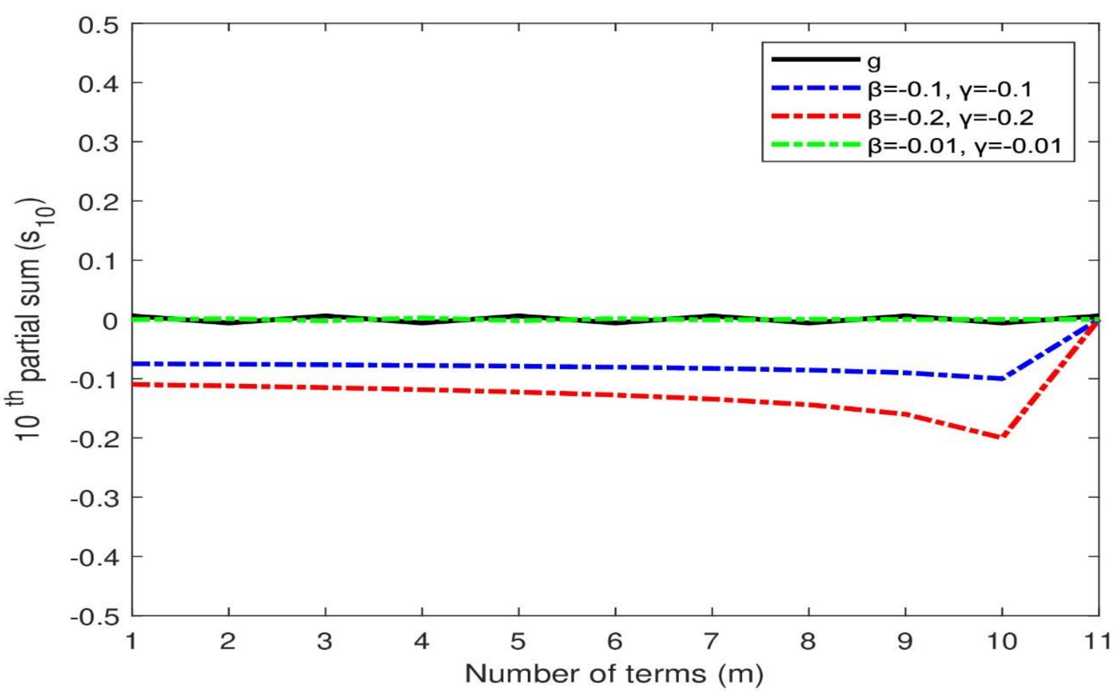
Here, $\{c_m\}$ is the cofficient sequence in the Fourier-Laguerre expansion. $(CE)_m^{q,\beta,\gamma}$ is the proposed mean about the point $t = 0$. We are plotting g and $(CE)_m^{q,\beta,\gamma}$ verses Number of terms.

The Fourier-Laguerre series for above function about the point $t = 0$ is plotted in Figure 1 and we can analyze that oscillations can be seen only for very small values about point $t = 0$.

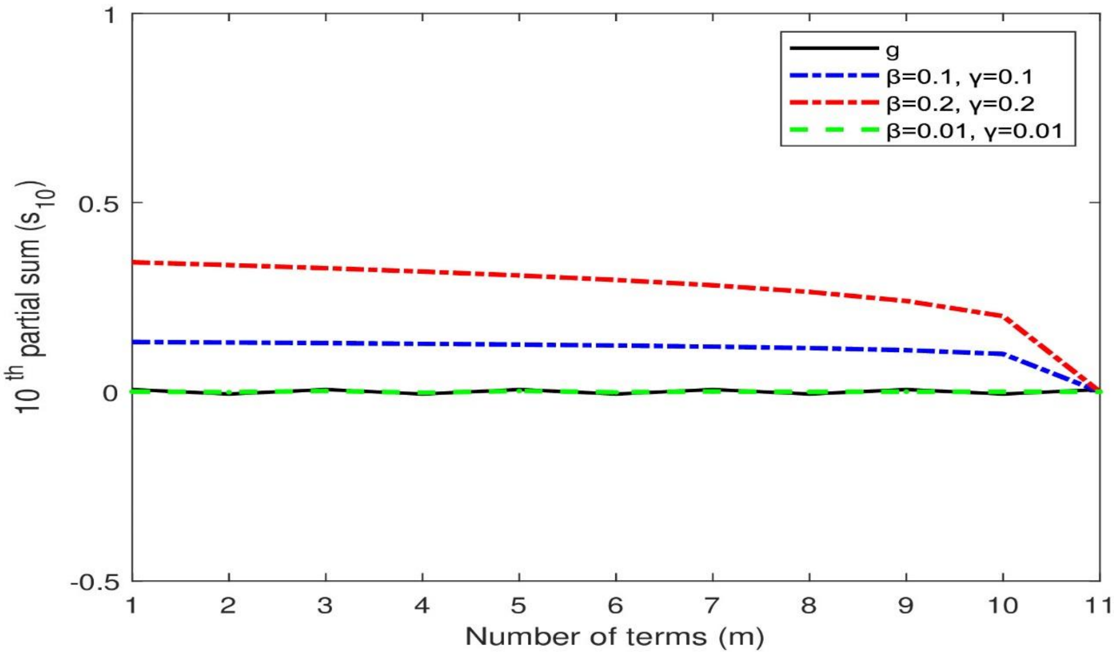


Here, we discuss our results in following cases:

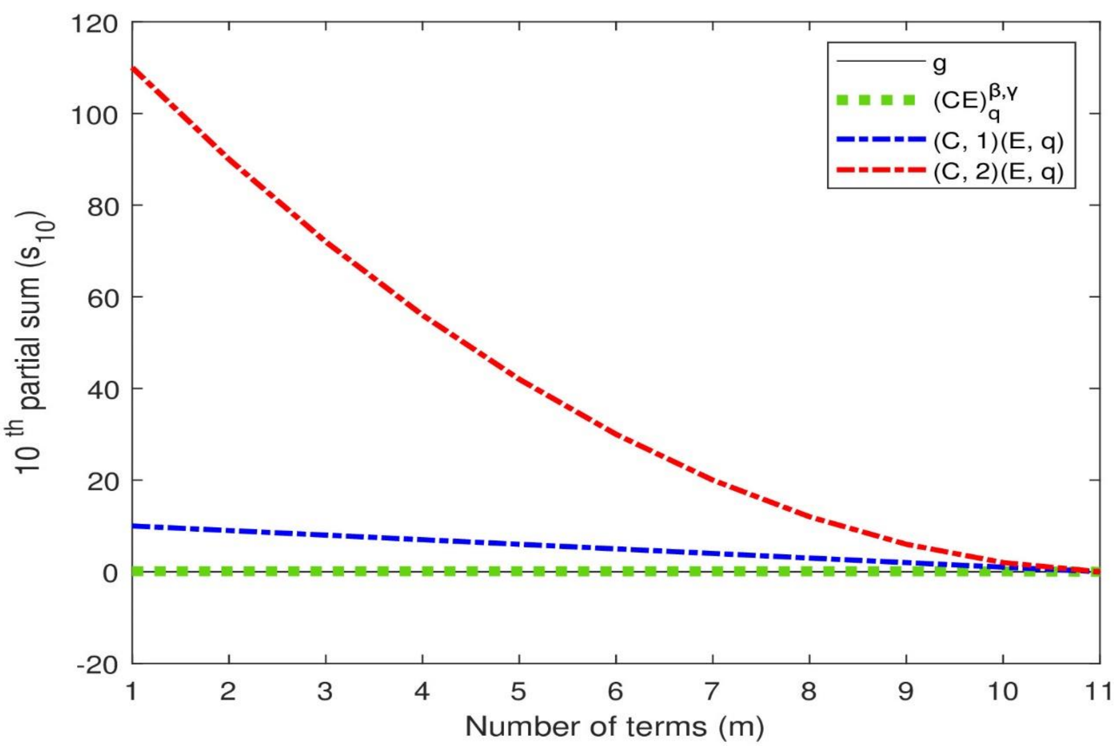
Case 1: When $\beta + \gamma < 0$ and $q > 0$, we interpret that after applying $(CE)_m^{q,\beta,\gamma}$ mean the Fourier-Laguerre polynomial is approximating $g(t)$ from negative side and larger the value of $\beta + \gamma$ better will be the approximation.



Case 2: When $\beta + \gamma > 0$ and $q > 0$, we interpret that after applying $(CE)_m^{q,\beta,\gamma}$ mean is approximating $g(t)$ from positive side and smaller the value of $\beta + \gamma$ better will be the approximation.



Comparison with existing methods: From the graph given below it can be analyzed that the rate of convergence of proposed method is much faster than the existing methods given by Krasniqi [14] and Sonker [15].



From above graphical interpretation, we can say that $(CE)_m^{q,\beta,\gamma}$ product summability method is much efficient. Also, the change in the value of β and γ changes the behavior of approximation.

Conclusion

The use of $(CE)_m^{q,\beta,\gamma}$ product summability of order (β, γ, q) generalized the results discussed in corollary and add flexibility to convergence as with the change in values of β, γ *and* q, changes the behavior of approximation. The rate of convergence is improved with the help of proposed method. Also, using different values of β, γ *and* q, the existing summability methods can be derived. We can infer that our result is much efficient and useful.

Conflict of interest: The authors declare that they have no conflict of interest.

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