Applying hybrid coupled fixed point theory to the nonlinear hybrid system of second order differential equations

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Abstract

In this work, we apply hybrid fixed point theory method to prove the existence of solution of systems of second order ordinary nonlinear hybrid differential equations with periodic boundaries.

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1 Introduction

Let $X \neq \phi$, and $F : X \times X \to X$ is a mapping. A point $(x, y) \in X \times X$ is said to a coupled fixed point of F in $X \times X$ if $F(x, y) = x$ and $F(y, x) = y$. The notation of coupled fixed point was introduced in 2006 by Baskar and Lshmikantham [1]. It will known that the existence of the fixed point play an important role for showing the existence of solutions of nonlinear integral [2, 3] , differential equations [4, 5] and iterative process [6].

In 1964, Krasnoselskii [7] initiated the idea of study the hybrid fixed point theory for the function which can be written as the sum of two other functions.In 2013, Dhage [2] obtained hybrid fixed point theorems for the operator which can be written as the sum of two other operators using Krasnoselskii fixed point theorem techniques and developed a Krasnoselskii fixed point technique helpful to analyze the existence of solution of nonlinear Volterra fractional integral equations under some conditions.

Recently, in 2015, Dhage and Dhage [8] proved the existence of solutions of the boundary value problems of second order ordinary nonlinear differential equations using the hybrid fixed point theorem which was obtained by themselves [9] .

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More recently, in 2017, Yang et al. [10] introduced the notation of hybrid coupled fixed point theorems and applied this ideas to prove the existence of system of fractional differential equations of order $\alpha, 0 < \alpha < 1$.

In this paper, we apply the hybrid fixed point theorem to study and prove the existence of solution of a certain system of boundary -value problems with periodic boundaries(for short BVPPB) of second order ordinary nonlinear hybrid differential equations.

To this end, the remainder of the article is organized as follows. Section 2, is given some preliminaries and basic definitions. Section3, is established the existence results of coupled nonlinear system of second order differential equations.

2 Preliminaries

Throughout this paper, let E be a nonempty set and $(E, \leq, \|.\|)$ be a partially ordered normed linear space. If $Q : E \to E$ is a mapping. Then Q is said to be monotone nondecreasing if $a \leq b$ the $Q(a) \leq Q(b)$, for all $a, b \in E$. Two elements $a, b \in E$ are said to be comparable if $a \leq b$ or $b \leq a$. If C is nonempty subset of E,C is said to be chain if each two elements $a, b \in C$ are comparable.

Definition 1 *[10] Let $Q : E \to E$ be a mapping. Q is called partially compact if for each chain C subset of E, $Q(C)$ is reltively compact subset of E.*

Definition 2 *[2, 10] Let $Q : E \to E$ be a mapping. Given an element $a \in E$. Define orbit $\Gamma(a; Q)$ as:*

$$\Gamma(a; Q) = \{a, Qa, Q^2a, Q^3a, ..., Q^na, ...\}.$$

If for any sequence $\{a_n\} \in \Gamma(a; Q)$ such that: $a_n \to a^*$ as $n \to \infty$ then $Qa_n \to Qa^*$, for each $a \in E$, then Q is called $\Gamma-$ orbitally continuous in E. Furthermore, $(E, \leq, \|.\|)$ is said to be $\Gamma-$ orbitally complete if each sequence $\{a_n\} \in \Gamma(a; Q)$ converges to an element $a^* \in E$.

Definition 3 *[2, 10] A mapping $\phi : E \to E$ is said to be $D-$ function if it is upper semi continuous and monotone nondecreasing such that: $\phi(0) = 0$.*

Definition 4 *[2, 10] A mapping $\Upsilon : E \to E$ is called partially nonlinear $D-$ contraction in E , if for each comparable elements $a, b \in E$, there exist a $D-$ function $\phi : \Re^+ \to \Re^+$ such that the following conditions are satisfied:*

(i) $\|\Upsilon a - \Upsilon b\| \leq \phi(\|a - b\|)$, *and*

(ii) $\phi(r) < r$ *for all* $r > 0$.

Definition 5 *[10] Let $(E, \leq, \|.\|)$ be a $\Gamma-$ orbitally complete linear space. The positive cone $K \in E$ is defined as: $K = \{x \in E : x \geq 0\}$*

The following theorem will be used as tool to prove the main results.

Theorem 6 *[10] Let E is a partially ordered $\Gamma-$ orbitally complete normed linear space and K is the positive cone of E. Let K is normal and C be a nonempty closed subset of E. Consider $P, Q : C \to C$ are two monotone nondecreasing mappings such that the following are satisfied:* (1) *P is $\Gamma-$ orbitally continuous and a partially nonlinear $D-$ contraction,*

(2) *Q is $\Gamma-$ orbitally continuous and a partially compact,*

(3) *there exist an element $a \in C$ such that: $a \leq Pa + Qy$ for all $y \in C$ and*

(4) *every pair of elements in C has an upper and lower bounded.*

Then $F(x, y) = Px + Qy$ has a coupled fixed point in $E \times E$.

Now, we give some notations and definitions about the problem which we will study the existence of its solution.

Definition 7 *Consider $I = [0, b] \in \Re$ where $b > 0$. The periodic boundary value problem of second-order nonlinear differential equation can be written as:*

$$\frac{d^2x(t)}{dt^2} = f(t, x(t) + h(t, x(t)), \qquad x(0) = x(b), x'(0) = x'(b) \tag{1}$$

for all $t \in I$, where $f, h : I \times \Re \to \Re$ are continuous function. The solution of the differential equation (1) is the function $x \in C^2(I, \Re)$ that satisfies equation (1), where by $x \in C^2(I, \Re)$ we mean the space of twice continuously differentiable real-valued functions on I.

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Definition 8 *the space $C(I,\Re)$ is the space of all continuous real-value function defined on I. It is easy to prove that: $C(I,\Re)$ is a Banach space with the norm:*

$$\|x\| = \sup_{t\in I} |x(t)|. \tag{2}$$

Therefore, it is also clear that $C(I,\Re)$ is partially ordered with respect to the partially order relation:

$$x \leq y \ \text{ if and only if } \quad x(t) \leq y(t) \quad \text{for all} \quad t \in I. \tag{3}$$

Also $C(I,\Re)$ is a partially ordered $\Gamma-$ orbitally complete linear space with normal cone $K_C = \{x \in C(I,\Re) : x(t) \geq 0, t \in I\}$ [10].

Lemma 9 *[11] Consider $\sigma, g \in L^1(I,\Re)$, then x is a solution of the differential equation:*

$$x''(t) + \sigma(t)x(t) = g(t), \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

if and only if x is a solution of the integral equation:

$$x(t) = \int_0^b G_\sigma(t,s)g(s)ds,$$

where $G_\sigma(t,s)$ is a Green's function associated with the differential equation:

$$x''(t) + \sigma(t)x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

Remark 10 *The Green function G_σ is continuous an nonnegative on $I \times I$ and there exist the number :*

$$M_\sigma = \max\{|G_\sigma(t,s)| : t,s \in [0,b]\}$$

for all $\sigma \in L^1(I,\Re^+)$.

Lemma 11 *[8] Consider the differential equations (1). Let $F(t,x) = f(t,x) + \lambda x$, such that:*

$(P1)$ The function $f : I \times \Re \to \Re$ is continuous and there exist a constant $k_1 > 0$ such that$|\ F(t,x)| \leq k_1$ for all $t \in I$,

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$(P2)$ The function $h : I \times \Re \to \Re$ is continuous and exist a constant $k_2 > 0$ such that$|\ h(t,x)| \leq k_2$ for all $t \in I$.

Then a function $v \in C(I,\Re)$ is a solution of the differential equation:

$$\frac{d^2x(t)}{dt^2} + \lambda x(t) = F(t, x(t) + h(t, x(t)), \qquad x(0) = x(b), x'(0) = x'(b) \tag{4}$$

if and only if v is the solution of the integral equation:

$$x(t) = \int_0^b G(t,s)F(s,x(s))ds + \int_0^b G(t,s)h(s,x(s))ds, \tag{5}$$

for all $t \in I$, where $G(t,s)$ is a Green's function associated with the differential equation:

$$x''(t) + \lambda x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b)\ \ ,\ \ x'(0) = x'(b),$$

3 Main results

Now , consider the system of nonlinear differential equations:

$$\begin{aligned}
&\tfrac{d^2x(t)}{dt^2} = f(t, x(t) + h(t, y(t)), \quad t \in I,\\
&\tfrac{d^2y(t)}{dt^2} = f(t, y(t) + h(t, x(t)), \quad t \in I,\\
&x(0) = x(b) \quad , \quad x'(0) = x'(b),\\
&y(0) = y(b) \quad , \quad y'(0) = y'(b).
\end{aligned} \tag{6}$$

Then the hybrid system (6) can be written as:

$$\begin{aligned}
&\tfrac{d^2x(t)}{dt^2} + \lambda x(t) = F(t, x(t) + h(t, y(t)), \quad t \in I,\\
&\tfrac{d^2y(t)}{dt^2} + \lambda y(t) = F(t, y(t) + h(t, x(t)), \quad t \in I,\\
&x(0) = x(b) \quad , \quad x'(0) = x'(b),\\
&y(0) = y(b) \quad , \quad y'(0) = y'(b),
\end{aligned} \tag{7}$$

where $F(t, x(t)) = f(t, x(t)) + \lambda x(t)$. By applying Lemma 11, we have the following Lemma .

Lemma 12 *Consider the hybrid system of differential equations (7), such that the conditions (P1) and (P2) hold.*

Then a function $(v_1, v_2) \in C(I, \Re) \times C(I, \Re)$ *is a solution of the differential equation the system (7) :*

if and only if (v_1, v_2) *is the solution of the system of nonlinear integral equation:*

$$\begin{aligned} x(t) &= \int_0^b G(t,s)F(s,x(s))ds + \int_0^b G(t,s)h(s,y(s))ds, \\ y(t) &= \int_0^b G(t,s)F(s,y(s))ds + \int_0^b G(t,s)h(s,x(s))ds \end{aligned} \tag{8}$$

for all $t \in I$*, where* $G(t,s)$ *is a Green's function associated with the differential equation:*

$$x''(t) + \lambda x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b) \ , \ \ x'(0) = x'(b),$$

By using the continuity of the integrals, we can define the two mappings: $T_1 : C(I, \Re) \to C(I, \Re)$ and $T_2 : C(I, \Re) \to C(I, \Re)$ as:

$$T_1(x(t)) = \int_0^b G(t,s)F(s,x(s))ds,$$

and

$$T_2(x(t)) = \int_0^b G(t,s)h(s,x(s))ds.$$

Then we can define the following :

$$T(x(t), y(t)) = T_1 x(t) + T_2 y(t)$$

The coupled fixed point of the operator T is the solution of the system (7).

With the two conditions (P1) and (P2) consider the following set of assumptions:

$(P3)$ There exists $\lambda > 0$ and $\varepsilon > 0$ such that:

$$0 \leq [f(t,x) + \lambda x] - [f(t,y) + \lambda y] \leq \varepsilon(x - y),$$

for all $t \in I$ and $x, y \in \Re$ such that: $x \geq y$.

$(P4)$ The function $h(t,x)$ nondecreasing.

$(P5)$ There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) \leq f(t, v(t)) + h(t, y(t)),$$

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$$v(0) \leq v(b) \quad , \quad v'(0) \leq v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$.

Lemma 13 *Assume that (P1) and (P3) holds. Let $\varepsilon Mb < 1$. Then the operator T_1 is $\Gamma-$ orbitally continuous, nondecreasing and a partially nonlinear $D-$ contraction.*

Proof. Let $E = C(I, \Re)$. The proof will be done in 3 steps.

Step 1: To prove that T_1 is nondecreasing.

Let $x, y \in E$, such that: $x \geq y$ Then by (P3), we have that:

$$T_1x(t) = \int_0^b G(t,s)F(s,x(s))ds \geq \int_0^b G(t,s))F(s,y(s))ds \geq T_1y(t),$$

for all $t \in I$. Thus T_1 is nondecreasing operator.

Step 2: To prove that T_1 is $\Gamma-$ orbitally continuous.

Take a sequence $\{x_n\} \in \Gamma(x; T_1)$ for any $x \in E$ with $x_n \to x^*$ as $n \to \infty$. Since T_1 is continues by (P1) again, then we have that:

$$\begin{aligned} \lim_{n\to\infty}(T_1x_n)(t) &= \textstyle\int_0^b G(t,s)\lim_{n\to\infty} F(s,x_n(s))ds \\ &= \textstyle\int_0^b G(t,s)F(s,x^*(s))ds \\ &= (T_1x^*)(t), \end{aligned} \tag{9}$$

for all $t > 0$. Hence , T_1 is $\Gamma-$ orbitally continuous.

Step 3: To prove that T_1 is partially nonlinear $D-$ contraction.

For any comparable element $x, y \in E$ such that:$x \geq y$. For $t \in I$, we have that:

$$\begin{aligned} \mid (T_1x)(t) - (T_1y)(t) \mid &= \mid \textstyle\int_0^b G(t,s)[F(s,x(s)) - F(s,y(s))]ds \mid \\ &\leq \textstyle\int_0^b G(t,s) \mid [F(s,x(s)) - F(s,y(s))] \mid ds \\ &\leq M\varepsilon b\|x-y\|, \end{aligned} \tag{10}$$

Let $\tau = M\varepsilon b$. New define the mapping $\Theta(l) = \tau l$, for each $l \in \Re^+$. Thus $\Theta(l) < l$ and $\Theta(0) = 0$. Therefore, we proved that:$\|T_1x - T_1y\| \leq \Theta(\|x - y\|)$. Hence T_1 is partially nonlinear $D-$ contraction.

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Lemma 14 *Assume that (P2) and (P4) holds. Then the operator* T_2 *is* $\Gamma-$ *orbitally continuous, nondecreasing and a partially compact in* $E = C(I, \Re$ *.*

Proof. The proof is also done in 3 steps.

Step 1: To prove that: T_2 is $\Gamma-$ orbitally continuous.

Consider for any $x \in E$ with $x_n \to x^*$ as $n \to \infty$. Since T_2 is continues by (P2), then we have that:

$$\begin{aligned} \lim_{n\to\infty}(T_2x_n)(t) &= \int_0^b G(t,s)\lim_{n\to\infty} h(s,x_n(s))ds \\ &= \int_0^b G(t,s)h(s,x^*(s))ds \\ &= (T_2x^*)(t), \end{aligned} \tag{11}$$

for all $t > 0$. Hence , T_2 is $\Gamma-$ orbitally continuous.

Step 2: To prove that: T_2 is nondecreasing.

Let $x, y \in E$, such that: $x \geq y$ Then by (P2), we have that:

$$T_2x(t) = \int_0^b G(t,s)h(s,x(s))ds \geq \int_0^b G(t,s))h(s,y(s))ds \geq T_2y(t),$$

for all $t \in I$. Thus T_2 is nondecreasing operator.

Step 3: To prove that T_2 is partially compact in $E = C(I, \Re)$. Let C be an arbitrary chain in E. We show $T_2(C)$ is uniformly bounded and equicontinuous set in E. Consider $x \in C$ be arbitrary, then we get that:

$$\begin{aligned} \mid T_2x(t) \mid &= \mid \int_0^b G(t,s)h(s,x(s))ds \mid \\ &\leq \int_0^b G(t,s) \mid h(s,x(s)) \mid ds \\ &\leq Mk_2b, \end{aligned} \tag{12}$$

Thus , we have that: $\|T_2x\| \leq \tau$, where $\tau = Mk_2b$. Hence, $T_2(C)$ is uniformly bounded subset in E.

To prove $T_2(C)$ is an equicontinuous : let $t_1, t_2 \in I$ such that: $t_1 - t_2 < 0$. We have that:

$$\begin{aligned} \mid T_2x(t_2) - T_2x(t_1) \mid &= \mid \int_0^b [G(t_1,s) - G(t_2,s)]h(s,x(s))ds \mid \\ &\leq \int_0^b \mid G(t_1,s) - G(t_2,s) \mid h(s,x(s))ds \\ &\leq \int_0^b \mid G(t_1,s) - G(t_2,s) \mid k_2 ds \to 0, \end{aligned} \tag{13}$$

as $t_1 \to t_2$. Hence $T_2(C)$ is a compact subset of E . Therefore, T_2 is partially compact in $E = C(I, \Re)$.

Theorem 15 *Let the hypotheses (P1), (P2), (P3), (P4) , (P5) and* $\varepsilon Mb < 1$ *hold. Then the hybrid system (6) has a coupled solution on* I.

Proof. By (P5) There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) \leq f(t, v(t) + h(t, y(t)),$$

$$v(0) \leq v(b) \quad , \quad v'(0) \leq v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$. Then , we get that: There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) + \lambda v(t) \leq F(t, v(t)) + h(t, y(t)),$$

$$v(0) \leq v(b) \quad , \quad v'(0) \leq v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$. Applying Theorem 6, we get that : the mapping $T(x, y) = T_1x + T_2y$ has a coupled fixed point. This coupled fixed point is the solution of the hybrid system (6).

Corollary 16 *Let the hypotheses (P1), (P2), (P3), (P4) and* $\varepsilon Mb < 1$ *hold. Consider there exists an element* $v \in C^2(I, \Re)$ *such that:*

$$v''(t) \geq f(t, v(t) + h(t, y(t)),$$

$$v(0) \geq v(b) \quad , \quad v'(0) \geq v'(b),$$

for all $t \in I$ *and* $y \in C(I, \Re)$.

Then the hybrid system (6) has a coupled solution on I.

Example 17 *Let* $I = [0, 1]$. *Suppose we have the following system of differential equations:*

$$\begin{aligned} &x''(t) = tan^{-1}x(t) - x(t) + h(t, y(t)), \quad t \in I, \\ &y''(t) = tan^{-1}y(t) - y(t) + h(t, x(t)), \quad t \in I, \\ &x(0) = x(1) \quad , \quad x'(0) = x'(1), \\ &y(0) = y(1) \quad , \quad y'(0) = y'(1), \end{aligned} \tag{14}$$

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and $t \in [0,1]$. *Consider* $g : I \times \Re \to \Re$ *such that:*

$$g(t,x) == \begin{cases} 1, & \text{if } x \leq 1 \\ \frac{2x}{1+x}, & \text{if } x > 1. \end{cases} \tag{15}$$

It is clear that $f(t,x) = \tan^{-1} x(t) - x(t)$ *and* f *and* g *are continues functions.Define* $F(t,x) = \tan^1 x(t) - x(t)$. *Then we get that:* $|F(t,x)| < \frac{1}{\pi} = k_1$. *Therefore ,* $f(t,x)$ *is satisfied condition (P1) . Also, since we have that:*

$$0 \leq \tan^1 x - \tan^1 y \leq \frac{1}{1+\kappa^2}(x-y) \quad \forall x, y \in \Re, \;\; x > y, \;\; x > \kappa > y.$$

Thus f *is satisfied condition (P3) with* $\lambda = 1$. *It is also clear that:* $\lambda > \varepsilon = \frac{1}{1+\kappa^2}$ *, where* $x > \kappa > y$. *Also,* $g(t,x)$ *is nondecreasing in* x *for all* $t \in I$ *and bounded . Thus* g *is satisfied the conditions (P2) and (P4). Finally , the function :*

$$v(t) = -2\int_0^1 G(t,s)ds + \int_0^1 G(t,s))ds,$$

is satisfied the condition (P5). Then the hybrid system has (14) a coupled solution.

4 Declaration of conflicting interests

The author declared no potential conflicts on interest with respect to the research, authorship, and/or publication of this article.

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