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Asymptotically Almost Automorphic Mild Solutions for Second Order Nonautonomous Semilinear Evolution Equations

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Abstract

The aim of this paper is to study the existence of asymptotically almost automorphic mild solution to some classes of second order semilinear evolution equation via the techniques of measure of noncompactness. The investigation is based on a new fixed point result which is a generalization of the well known Darbo's fixed point theorem. Finally examples are given to illustrate the analytical findings.

Key words: Asymptotically almost automorphic, second order nonautonomous differential equations, mild solution, evolution system, Kuratowski measures of noncompactness, fixed point.
AMS Subject Classification : 34G20

1 Introduction

This work is mainly concerned with the existence of asymptotically almost automorphic mild solution for second differential equations. More

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precisely, we will consider the following problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in \mathbb{R}^+ := [0, +\infty), \tag{1}$$

$$y(0) = y_0,\ y'(0) = y_1, \tag{2}$$

where $\{A(t)\}_{t\in\mathbb{R}^+}$ is a family of linear closed operators from E into E that generate an evolution system of linear bounded operators $\{\mathcal{U}(t,s)\}_{(t,s)\in\mathbb{R}^+\times\mathbb{R}^+}$ for $0 \leq s \leq t < +\infty$, $f : \mathbb{R}^+ \times E \to E$ is a Carathéodory function, and $(E, |\cdot|)$ is a real Banach space.

Evolution equations arise in many areas of applied mathematics [2, 37]. This type of equations has received much attention in recent years [1]. There are many results concerning the second-order differential equations, see for example [8, 11, 12, 20, 28, 35]. In recent years there has been an increasing interest in studying the abstract non-autonomous second order initial value problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in [0, T], \tag{3}$$

$$y(0) = y_0,\ y'(0) = y_1. \tag{4}$$

The reader is referred to [10, 19, 22, 36] and the references therein. In the above mentioned works, the existence of solutions to the problem (3)-(4) is related to the existence of an evolution operator $\mathcal{U}(t;s)$ for the homogeneous equation

$$y''(t) = A(t)y(t), \quad \text{for} \quad t \geq 0.$$

For this purpose there are many techniques to show the existence of $\mathcal{U}(t,s)$ which has been developed by Kozak [25].
On the other hand, since Bochner [13] introduced the concept of almost automorphy, the automorphic functions have been applied to many areas including ordinary as well as partial differential equations, abstract differential equations, functional differential equations, integral equations, etc.; see [16, 21, 18, 27, 7]. We also refer the reader to the monographs by N'Guérékata [30, 31] for the basic theory of almost automorphic functions and applications. The concept of asymptotically almost automorphy was introduced by N'Guérékata [29]. Since then, these functions have generated lot of developments and applications, see [39, 14, 24, 17] and the references therein. In the previous works, people have established the existence of asymptotically almost automorphic mild solution of differential equations under the conditions that f satisfies or not the Lipschitz condition.

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In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equation. This technique works fruitfully for both integral and differential equations. More details are found in Aissani and Benchohra [3], Akhmerov *et al.* [4], Alv́ares [5], Banaś and Goebel [9], Olszowy and Wędrychowicz [33], Olszowy [34], and the references therein.

Inspired by the above works,, in this work, using the properties of the analytic semigroups, Kuratowski measure of noncompactness, fixed point theorem, we obtain an existence result without assuming that the nonlinearity f satisfies a Lipschitz type condition.

This work is organized of as follows. In Section 2, we recall some fundamental properties of asymptotically almost automorphic and facts about evolution systems. Section 3 is devoted to establishing some criteria the existence of asymptotically almost automorphic mild solutions to the problem (1)-(2). Furthermore, appropriate examples are provided in section 4 to show the feasibility of our results.

2 Preliminaries and basic results

In this section we recall certain definitions and lemmas to be used subsequently in this paper.
Throughout this paper, we denote by E a Banach space with the norm $|\cdot|$. Let $BC(\mathbb{R}^+,\mathbb{E})$ be the Banach space of all bounded and continuous functions y mapping $\mathbb{R}^+$ into E endowed with the usual supremum norm

$$\|y\|_\infty = \sup_{t\in\mathbb{R}^+} |y(t)|.$$

In what follows, let $\{A(t),\ t\in\mathbb{R}^+\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ which is dense in E and independent of t.

In this work the existence of solution the problem (1)-(2) is related to the existence of an evolution operator $\mathcal{U}(t,s)$ for the following homogeneous problem

$$y''(t) = A(t)y(t) \qquad t\in\mathbb{R}^+. \tag{5}$$

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This concept of evolution operator has been developed by Kozak [25] and recently used by Henríquez *et al.* [22].

Definition 2.1 *A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t,s) : E \to E$, $(t,s) \in \Delta := \{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ : s \leq t\}$, is called an evolution operator of the equation (5) if de following conditions hold:*

(e_1) *For any $x \in E$ the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is continuously differentiable and*

- **(a)** *for each $t \in \mathbb{R}$, $\mathcal{U}(t,t)x = 0, \forall x \in E$,*
- **(b)** *for all $(t,s) \in \Delta$ and for any $x \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x|_{t=s} = -x$.*

(e_2) *For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is of class C^2 and*

- **(a)** $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x = A(t)\mathcal{U}(t,s)x$,
- **(b)** $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x = \mathcal{U}(t,s)A(s)x$,
- **(c)** $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t,s)x|_{t=s} = 0$.

(e_3) *For all $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x$ and*

- **(a)** $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x = A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$.
  Moreover, the map $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$ is continuous,
- **(b)** $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x = \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

Throughout this paper, we will use the following definition of the concept of Kuratowski measure of noncompactness [9].

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Definition 2.2 *The Kuratowski measure of noncompactness α is defined by*

$$\alpha(D) = \inf\{r > 0 : D \text{ has a finite cover by sets of diameter} \leq r\},$$

for a bounded set D in any Banach space E.

Let us recall the basic properties of Kuratowski measure of noncompactness.

Lemma 2.3 *[9] Let E be a Banach space and $C, D \subset E$ be bounded, then the following properties hold:*

- (i_1) $\alpha(D) = 0$ *if only if D is relatively compact,*
- (i_2) $\alpha(\overline{D}) = \alpha(D)$ *; $\overline{D}$ the closure of D,*
- (i_3) $\alpha(C) \leq \alpha(D)$ *when $C \subset D$,*
- (i_4) $\alpha(C+D) \leq \alpha(C)+\alpha(D)$ *where* $C+D = \{x \mid x = y+z; y \in C; z \in D\}$,
- (i_5) $\alpha(aD) = |a|\alpha(D)$ *for any $a \in \mathbb{R}$,*
- (i_6) $\alpha(ConvD) = \alpha(D)$*, where $ConvD$ is the convex hull of D,*
- (i_7) $\mu(C \cup D) = \max(\alpha(C), \alpha(D))$,
- (i_8) $\alpha(C \cup \{x\}) = \alpha(C)$ *for any $x \in E$.*

Denote by $\omega^T(y,\varepsilon)$ the modulus of continuity of y on the interval $[0,T]$ i.e.

$$\omega^T(y,\varepsilon) = \sup\left\{|y(t) - y(s)|\,; t,s \in [0,T], |t-s| \leq \varepsilon\right\}.$$

Moreover, let us put

$$\omega^T(D,\varepsilon) = \sup\left\{\omega^T(y,\varepsilon); y \in D\right\},$$

$$\omega_0^T(D) = \lim_{\varepsilon \to 0} \omega^T(D,\varepsilon).$$

Lemma 2.4 *[15] Let E be a Banach space, $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that*

$$\alpha(D) \leq 2\alpha(D_0).$$

Lemma 2.5 *[23] Let $D = \{y_n\}_{n=0}^{+\infty} \subset C(\mathbb{R}^+, E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $\mathbb{R}^+$, and*

$$\alpha\left\{\int_0^t y_n(s))ds\right\}_{n=0}^{\infty} \leq 2\int_0^t \alpha(D(s))ds, \qquad t \in \mathbb{R}^+.$$

Now, we recall some basic definitions and results on almost automorphic functions and asymptotically almost automorphic functions (for more details, see [13, 31, 38]).

Definition 2.6 *A continuous function $f : \mathbb{R} \to E$ is said to be almost automorphic if for every sequence of real numbers $\{\tau_n'\}$, there exists a subsequence $\{\tau_n\}$ such that*

$$g(t) = \lim_{n\to\infty} f(t + \tau_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - \tau_n) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

Denote by $AA(\mathbb{R}, E)$ the set of all such functions.

Lemma 2.7 *[30] $AA(\mathbb{R}, E)$ is a Banach space with the supremum norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|.$$

Definition 2.8 *A continuous function $f : \mathbb{R} \times E \to E$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $y \in E$ if for every sequence of real numbers $\{\tau_n'\}$, there exists a subsequence $\{\tau_n\}$ such that*

$$\lim_{n\to\infty} f(t + \tau_n, y) = g(t, y)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - \tau_n, y) = f(t, y)$$

for each $t \in \mathbb{R}$ and each $y \in E$. The collection of those functions is denoted by $AA(\mathbb{R} \times E, E)$.

Example 2.9 *[40] The function $f : \mathbb{R} \times E \to E$ given by*

$$f(t, y) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)\cos y$$

is almost automorphic in $t \in \mathbb{R}$ for each $y \in E$, where $E = L^2([0, 1])$.

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The space of all continuous functions $h:\mathbb{R}^+\to E$ such that $\lim_{t\to\infty} h(t)=0$ is denoted by $C_0(\mathbb{R}^+,E)$. Moreover, we denote $C_0(\mathbb{R}^+\times E,E)$; the space of all continuous functions from $\mathbb{R}\times E$ to E satisfying $\lim_{t\to\infty} h(t,y)=0$ in t and uniformly in $y\in E$.

Remark 2.10 *Note that if $\nu(t)\in C_0(\mathbb{R}^+,E)$, then*

$$\int_0^t e^{-(t-s)}\nu(s)ds\in C_0(\mathbb{R}^+,E).$$

Definition 2.11 *A continuous function $f:\mathbb{R}^+\to E$ is said to be asymptotically almost automorphic if it can be decomposed as*

$$f(t)=g(t)+h(t),$$

where

$$g(t)\in AA(\mathbb{R},E),\ \ h(t)\in C_0(\mathbb{R}^+,E).$$

Denote by $AAA(\mathbb{R}^+,E)$ the set of all such functions.

Example 2.12 *The function $f:\mathbb{R}\to\mathbb{R}$ defined by*

$$f(t)=\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)+e^{-t}$$

is an asymptotically almost automorphic function with

$$g(t)=\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\in AA(\mathbb{R},\mathbb{R}),\quad h(t)=e^{-t}\in C_0(\mathbb{R}^+,\mathbb{R}).$$

Lemma 2.13 *[31],[32]. $AAA(\mathbb{R}^+,E)$ is also a Banach space with the norm*

$$\|f\|_\infty=\sup_{t\in\mathbb{R}^+}|f(t)|.$$

Definition 2.14 *A continuous function $f:\mathbb{R}^+\times E\to E$ is said to be asymptotically almost automorphic if it can be decomposed as*

$$f(t,y)=g(t,y)+h(t,y),$$

where

$$g(t,y)\in AA(\mathbb{R}\times E,E),\quad h(t,y)\in C_0(\mathbb{R}^+\times E,E).$$

Denote by $AAA(\mathbb{R}^+\times E,E)$ the set of all such functions.

Example 2.15 *The function* $f : \mathbb{R}^+ \times E \to E$ *given by*

$$f(t, x) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)\cos y + e^{-t}|y|$$

is asymptotically almost automorphic in $t \in \mathbb{R}^+$ *for each* $y \in E$, *where* $E = L^2([0, 1])$.

$$g(t, y) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)\cos y \in AA(\mathbb{R} \times E, E),$$
$$h(t, y) = e^{-t}|y| \in C_0(\mathbb{R}^+ \times E, E).$$

.

Lemma 2.16 *[26]* $f : \mathbb{R} \times E \to E$ *is almost automorphic, and assume that* $f(t, \cdot)$ *is uniformly continuous on each bounded subset* $K \subset E$ *uniformly for* $t \in \mathbb{R}$, *that is for any* $\varepsilon > 0$, *there exists* $\varrho > 0$ *such that* $y, z \in K$ *and* $|y(t) - z(t)| < \varrho$ *imply that* $|f(t, y) - f(t, z)| < \varepsilon$ *for all* $t \in \mathbb{R}$. *Let* $\varphi : \mathbb{R} \to E$ *be almost automorphic. Then the function* $F : \mathbb{R} \to E$ *defined by* $F(t) = f(t, \varphi(t))$ *is almost automorphic.*

Theorem 2.17 *[6] Let* Ω *be a nonempty, bounded, closed and convex subset of a Banach space* E, *and let* $\Gamma : \Omega \to \Omega$ *be a continuous operator satisfying the inequality*

$$\alpha(\Gamma(D)) \leq \Psi(\alpha(D))$$

for any nonempty subset D *of* Ω, *where* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a nondecreasing function such that*

$$\lim_{n \to +\infty} \Psi^n(t) = 0 \text{ for each } t \geq 0.$$

Then Γ *has at least one fixed point in the set* Ω.

3 Main results

Definition 3.1 *A function* $y \in BC(\mathbb{R}^+, E)$ *is said to be a mild solution to the problem (1)-(2) if* y *satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t, 0)y_0 + \mathcal{U}(t, 0)y_1 + \int_0^t \mathcal{U}(t, s)f(s, y(s))ds.$$

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For the proof of our main theorem, we need the following hypotheses:

(H_1) (a) There exists a constant $M \geq 1$ and $\delta > 0$, such that

$$\|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} \leq Me^{-\delta(t-s)} \quad \text{for any } (t,s) \in \Delta$$

and for any sequence of real numbers $\{\tau'_n\}$, we can extract a subsequence $\{\tau_n\}$ and for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|U(t+\tau_n, s+\tau_n) - U(t,s)\|_{\mathcal{B}(E)} \leq \varepsilon e^{-\delta(t-s)},$$

$$\|U(t-\tau_n, s-\tau_n) - U(t,s)\|_{\mathcal{B}(E)} \leq \varepsilon e^{-\delta(t-s)}$$

for each $t, s \in \mathbb{R}$. for all $n > N$, for each $t, s \in \mathbb{R}$, $t \geq s$.

(H_2) There exist a constant $\widetilde{M} \geq 0$ and $\delta > 0$, such that:

$$\left\|\frac{\partial}{\partial s}\mathcal{U}(t,s)\right\|_{\mathcal{B}(E)} \leq \widetilde{M}e^{-\delta(t-s)}, (t,s) \in \Delta.$$

(H_3) The function $f : \mathbb{R}^+ \times E \to E$ is Carathéodory and asymptotically almost automorphic i.e., $f(t,y) = g(t,y) + h(t,y)$ with

$$g(t,y) \in AA(\mathbb{R} \times E, E), \quad h(t,y) \in C_0(\mathbb{R}^+ \times E, E),$$

and $g(t,y)$ is uniformly continuous on any bounded subset $K \subset E$ uniformly for $t \in \mathbb{R}$.
Moreover,

(a) There exist $p \in L^q(\mathbb{R}, \mathbb{R}^+)$, $q \in [1, \infty)$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that for all $t \in \mathbb{R}^+$ and $y \in E$,

$$|g(t,y)| \leq p(t)\psi(|y|) \quad \text{and} \quad \lim_{|y|\to+\infty} \inf \frac{\psi(|y|)}{|y|} = \rho_1.$$

(b) There exist a function $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in \mathbb{R}^+$ and $y \in E$ with $|y| \leq R$,

$$|h(t,y)| \leq \beta(t)\phi(|y|) \quad \text{and} \quad \lim_{R\to+\infty} \inf \frac{\phi(R)}{R} = \rho_2.$$

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(H_4) There exist a locally integrable function $\eta : \mathbb{R} \to \mathbb{R}^+$ and a continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any nonempty bounded set $D \subset E$ we have :

$$\alpha(f(t, D)) \leq \eta(t)\varphi(\alpha(D)) \text{ for a.e } t \in \mathbb{R}^+.$$

Additionally we assume that $\lim_{n\to+\infty} (\psi + \phi)^n(t) = 0$ for a.e $t \in \mathbb{R}^+$. Let $\beta(t)$ be the function involved in the assumption (H_3), then

$$\int_0^t e^{-(t-s)}\beta(s)ds \in C_0(\mathbb{R}^+, \mathbb{R}^+).$$

Put

$$\rho = \sup_{t\in\mathbb{R}^+} \int_0^t e^{-(t-s)}\beta(s)ds.$$

We need the following technical lemma.

Lemma 3.2 *Assume that (H_1) hold. If $\varphi(t) \in AA(\mathbb{R}, E)$, then*

$$\Lambda(t) := \int_{-\infty}^t U(t, s)\varphi(s)\mathrm{ds}, \ \mathrm{t} \in \mathbb{R},$$

belongs to $AA(\mathbb{R}, \mathbb{E})$.

Proof. From (H_1) it is clear that $\Lambda(t)$ is well-defined and continuous on $\mathbb{R}$. Since $\varphi(t) \in AA(\mathbb{R}, E)$, it follows that for every sequence of real numbers $\{\tau_n'\}$, we can extract a subsequence $\{\tau_n\}$ such that

(c_1) $\lim_{n\to\infty} \varphi(t + \tau_n) - \widetilde{\varphi}(t) = 0$ for each $t \in \mathbb{R}$ and,

(c_2) $\lim_{n\to\infty} \widetilde{\varphi}(t - \tau_n) - \varphi(t) = 0$ for each $t \in \mathbb{R}$.

Notes that $\widetilde{\varphi}$ is also bounded on $\mathbb{R}$, and measurable. Define

$$\widetilde{\Lambda}(t) = \int_{-\infty}^t \mathcal{U}(t, s)\widetilde{\varphi}(s)\mathrm{ds}, \quad \mathrm{t} \in \mathbb{R}.$$

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For $t \in \mathbb{R}$, Since $\widetilde{\varphi}$ is measurable, $\widetilde{\Lambda}$ is well-defined.
For $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|\Lambda y)(t+\tau_n) - (\widetilde{\Lambda}y)(t)\right| \\
&= \left|\int_{-\infty}^{t+\tau_n} \mathcal{U}(t+\tau_n, s)\varphi(s)ds - \int_{-\infty}^{t} \mathcal{U}(t,s)\widetilde{\varphi}(s)ds\right| \\
&= \left|\int_{-\infty}^{t} \mathcal{U}(t+\tau_n, s+\tau_n)\varphi(s+\tau_n)ds - \int_{-\infty}^{t} \mathcal{U}(t,s)\widetilde{\varphi}(s)ds\right| \\
&\leq \int_{-\infty}^{t} \|\mathcal{U}(t+\tau_n, s+\tau_n)\|_{\mathcal{B}(E)} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| ds \\
&+ \int_{-\infty}^{t} \|\mathcal{U}(t+\tau_n, s+\tau_n) - \mathcal{U}(t,s))\|_{\mathcal{B}(E)}\widetilde{\varphi}(s)ds \\
&\leq \int_{-\infty}^{t} Me^{-\delta(t-s)} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| ds \\
&+ \int_{-\infty}^{t} \varepsilon e^{-\delta(t-s)} \left|\widetilde{\varphi}(s)\right| ds \\
&\leq M \int_{-\infty}^{t} e^{-\delta(t-s)} ds \sup_{s\in\mathbb{R}} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| \\
&+\varepsilon \int_{-\infty}^{t} e^{-\delta(t-s)} ds \sup_{s\in\mathbb{R}} \left|\widetilde{\varphi}(s)\right| \\
&\leq \frac{M}{\delta} \sup_{s\in\mathbb{R}} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| + \frac{\varepsilon}{\delta} \sup_{s\in\mathbb{R}} \left|\widetilde{\varphi}(s)\right|.
\end{aligned}$$

Using (c_1), we obtain that for $n \to \infty$,

$$\Lambda(t+\tau_n) \to \widetilde{\Lambda}(t).$$

Analogously, one can prove that,

$$\widetilde{\Lambda}(t-\tau_n) \to \Lambda(t) \text{ for each } t \in \mathbb{R} \text{ as } n \to \infty.$$

This we show that

$$\Lambda \in AA(\mathbb{R}, E).$$

Theorem 3.3 *Assume that the hypotheses* $(H_1) - (H_4)$ *are satisfied. If*

$$M\rho_1\|p\|_{L^q} + M\delta^{-1}\rho\rho_2 < 1, \tag{6}$$

and

$$M\max(4\|\eta\|_{L^1}, \|p\|_{L^q}\delta^{-1+\frac{1}{q}}) < 1, \tag{7}$$

then the problem (1)-(2) has a asymptotically almost automorphic mild solution.

Proof. Consider the operator $N : AAA(\mathbb{R}^+, E) \to AAA(\mathbb{R}^+, E)$ defined by

$$(Ny)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds, \qquad (8)$$

where $y \in AAA(\mathbb{R}^+, E)$ with $y = \gamma + \zeta$, γ is the principal term and ζ the corrective term of y. We need to prove that N is weel- defined, that is $N(AAA(\mathbb{R}^+, E)) \subset AAA(\mathbb{R}^+, E)$. Let

$$\sigma(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1,$$

then

$$\begin{aligned} |\sigma(t)| &= |-\tfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1| \\ &\le |\tfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0| + |\mathcal{U}(t,0)y_1| \\ &\le \widetilde{M}e^{-\delta t}|y_0| + Me^{-\delta t}|y_1|. \end{aligned}$$

Since $\delta > 0$, we get $\lim_{t\to+\infty} |(\sigma(t)| = 0$. that is

$$\sigma \in C_0(\mathbb{R}^+, E). \qquad (9)$$

By assumption $f = g + h$ where g is the principal term and h the corrective term. So we can write

$$\begin{aligned} f(t,y(t)) &= g(t,\gamma(t)) + f(t,y(t)) - f(t,\gamma(t)) + h(t,\gamma(t)) \\ &= g(t,\gamma(t)) + H(t,y(t)), \end{aligned} \qquad (10)$$

In view of (10), we have

$$\begin{aligned} W(t) &= \int_0^t \mathcal{U}(t,s)f(s,y(s))ds \\ &= \int_0^t \mathcal{U}(t,s)g(s,\gamma(s))ds + \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &= \int_{-\infty}^t \mathcal{U}(t,s)g(s,\gamma(s))ds - \int_{-\infty}^0 \mathcal{U}(t,s)g(s,\gamma(s))ds \\ &+ \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &= (I_1y)(t) + (I_2y)(t), \end{aligned}$$

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where

$$(I_1y)(t) = \int_{-\infty}^{t} \mathcal{U}(t,s)g(s,\gamma(s))\mathrm{ds},$$

$$\begin{aligned}(I_2y)(t) &= \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &- \int_{-\infty}^{0} \mathcal{U}(t,s)g(s,y(s))ds \\ &= (J_1y)(t) + (J_2y)(t),\end{aligned}$$

where

$$(J_1y)(t) = \int_0^t \mathcal{U}(t,s)H(s,y(s))ds,$$

$$(J_2y)(t) = \int_{-\infty}^{t} \mathcal{U}(t,s)g(s,\gamma(s))\mathrm{ds}.$$

Using (H_3) and Lemma 2.16 , we deduce that $s \to g(s,\gamma(s))$ is in $AA(\mathbb{R},E)$. Thus, by Lemma 3.2 we obtain

$$(I_1y)(t) \in AA(\mathbb{R},E). \tag{11}$$

Let's prove that $J_1 \in C_0(\mathbb{R}^+,E), J_2 \in C_0(\mathbb{R}^+,E)$.
Ideed by definition $H \in C_0(\mathbb{R}^+,E)$, that means given $\varepsilon > 0$, there exists $T > 0$ such that if $t \geq T$, we have $|H(t,y)| \leq \varepsilon$. Therefore if $t \geq T$, we get

$$\begin{aligned}\int_T^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)}|H(s,y(s))|ds &\leq M\varepsilon \int_T^t e^{-\delta(t-s)}ds \\ &\leq \frac{M}{\delta}\varepsilon,\end{aligned}$$

then

$$|(J_1y)(t)| \leq \frac{M}{\delta}\varepsilon \qquad \text{if } t \geq T.$$

So,

$$J_1 \in C_0(\mathbb{R}^+,E). \tag{12}$$

Next, let us show that $J_2 \in C_0(\mathbb{R}^+, E)$.

$$\begin{aligned} |(J_2y)(t)| &\leq \int_{-\infty}^{0} \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} |g(s,y(s))| ds \\ &\leq M \sup_{t\in\mathbb{R}} |g(t,y(t))| \int_0^T e^{-\delta(t-s)} ds \\ &+ M\|g\|_\infty \frac{e^{-\delta(t}}{\delta} \qquad \to 0 \text{ as } \to \infty. \end{aligned}$$

So,

$$J_2 \in C_0(\mathbb{R}^+, E). \tag{13}$$

Finaly combining (9),(11), (12) and (13) proves our claim that $N \in AAA(\mathbb{R}^+, E)$. Next, we will prove that the operator N satisfies all the assumptions of Theorem 2.17. We will break the proof into several steps.
Let

$$B_R = \left\{ y \in AAA(\mathbb{R}^+, E) : \|y\|_\infty \leq R \right\},$$

where R be any positive constant. Then B_R is a bounded, closed and convex subset of $AAA(\mathbb{R}^+, E)$.
Step 1: $N(y) \in B_R$ for any $y \in B_R$.
In fact, if we assume that the assertion is false, then $R < |(Ny)(t)|$. This yields that

$$\begin{aligned} R < |(Ny)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} |g(s,y(s)| ds \\ &+ \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} |h(s,y(s)| ds \\ &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}\, p(s)\psi(|y(s)|) ds \\ &+ \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} \beta(s)\phi(|y(s)|) ds \\ &\leq M\psi(R) \int_0^t e^{-\delta(t-s)} p(s) ds \\ &+ M\ \phi(R) \int_0^t e^{-\delta(t-s)} \beta(s) ds. \end{aligned}$$

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For $t \geq 0$, it follows from the Hölder inequality that

$$R < |(Ny)(t)| \quad \leq \quad M\psi(R)\|p\|_{L^q} + M\rho_2\phi(R).$$

Dividing both sides by R and taking the lim inf as $R \to +\infty$, we have

$$M\rho_1\|p\|_{L^q} + M\delta^{-1}\rho\rho_2 > 1,$$

which contradicts (6). Hence, the operator N transforms the set B_R into itself.

Step 2. N is continuous.

Let $(y_n)_{n\in N}$ be a sequence in B_R such that $y_n \to y$ in B_R.

Case 1. If $t \in [0,T]$; $T > 0$, then, we have

$$|(Ny_n)(t) - (Ny)(t)| \leq M\int_0^t |f(s, y_n(s)) - f(s, y(s))|\, ds.$$

Since the functions f is Carathéodory, the Lebesgue dominated convergence theorem implies that

$$\|Ny_n - Ny\|_\infty \to 0 \quad \text{ as } \; n \to +\infty.$$

Case 2. Since the functions f is Carathéodory, we can see that

$$|f(s, y_n(s)) - f(s, y(s))| \leq \frac{\delta\varepsilon}{M} \quad \text{ for } t \geq T. \tag{14}$$

If $t \in (T, \infty)$, $T > 0$, then (14) and the hypotheses give us that

$$\begin{aligned} |Ny_n(t) - Ny(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} \Big| f(s, y_n(s)) - f(s, y(s)) \Big| ds \\ &\leq M\frac{\delta\varepsilon}{M}\int_0^t e^{-\delta(t-s)}ds \\ &\leq \frac{M}{\delta}\frac{\delta\varepsilon}{M} \\ &\leq \varepsilon. \end{aligned} \tag{15}$$

Then the inequality (15) reduces to

$$\|N(y_n) - N(y)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Now, we conclude that N is continuous from B_R to B_R.

Step 3: $N(B_R)$ is equicontinuous.

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Let $t_1, t_2 \in [0,T]$ with $t_2 > t_1$ and $y \in B_R$. Then, we have

$$\begin{aligned}
&|(N_1y)(t_2) - (N_1y(t_1)| \\
&= \left|\int_0^{t_1} (\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s))g(s,y(s))\right. \\
&+ \left.\int_{t_1}^{t_2} \mathcal{U}(t_2,s)g(s,y(s))ds\right| \\
&+ \left|\int_0^{t_1} (\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s))h(s,y(s))\right. \\
&+ \left.\int_{t_1}^{t_2} \mathcal{U}(t_2,s)h(s,y(s))ds\right| \\
&\leq \int_0^{t_1} \|\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)\|_{B(E)}\ p(s)\psi(|y(s)|)ds \\
&+M \int_{t_1}^{t_2} e^{-\delta(t-s)}p(s)\psi(|y(s)|)ds. \\
&+ \int_0^{t_1} \|\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)\|_{B(E)}\ \beta(s)\phi(|y(s)|)ds \\
&+M \int_{t_1}^{t_2} e^{-\delta(t-s)}\beta(s)\phi(|y(s)|)ds.
\end{aligned}$$

It follows from the Hölder inequality that

$$\begin{aligned}
&|(N_1y)(t_2) - (N_1y(t_1)| \\
&\leq \int_0^{t_1} \|\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)\|_{\mathcal{B}(E)}\ p(s)\psi(|y(s)|)ds \\
&+\frac{M\|p\|_{L^q}\psi(R)}{\delta^{1-\frac{1}{q}}} \left(e^{-\frac{q\delta}{q-1}(t-t_2)} - e^{-\frac{q\delta}{q-1}(t-t_2)}\right)^{1-\frac{1}{q}} \\
&+ \int_0^{t_1} \|\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)\|_{B(E)}\ \beta(s)\phi(|y(s)|)ds \\
&+\frac{M\phi(R)\sup\limits_{t\in\mathbb{R}}\beta(t)}{\delta}(e^{-\delta(t-t_2)} - e^{-\delta(t-t_1)}).
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $t_2 - t_1 \to 0$, which implies that $N(B_R)$ is equicontinuous.
Consider the measure of noncompacteness $\mu(B)$ defined on the family of bounded subsets of the space $AAA(\mathbb{R}^+, E)$ (see [33]) by

$$\mu(B) = \omega_0^T(B) + \sup_{t\in J}\alpha(B(t)) + \lim_{T\to+\infty}\sup\{|y(t)| : t \geq T, y \in E\}.$$

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Step 4: $\mu(N(B)) \leq M \max(4\|\eta\|_{L^1}, \|p\|_{L^q}\delta^{-1+\frac{1}{q}})(\varphi+\psi)(\mu(B))$ for all $B \subset B_R$. For all $B \subset B_R$, $N(B)$ is bounded. Hence, by Lemma 2.4, there exists a countable set $B_1 = \{y\}_{n=1}^{\infty} \subset B$, such that

$$(N(B)) \leq 2\alpha(N(B_1)). \tag{16}$$

Using the properties of α, Lemma 2.4, Lemma 2.5 and assumptions (H_1) and (H_4), we get

$$\begin{aligned} \alpha(NB_1(t)) &\leq \alpha\left(\left\{\int_0^t \mathcal{U}(t,s)f(s,y_n(s))ds\right\}_{n=0}^{\infty}\right) \\ &\leq 2M\int_0^t \{\alpha\left(f(s,y_n(s))ds\right)\}_{n=0}^{\infty} ds \\ &\leq 2M\int_0^t \eta(s)\varphi\left(\{(\alpha(y_n(s))\}_{n=0}^{\infty})\right)) ds \\ &\leq 2M\int_0^t \eta(s)\varphi(\alpha(B(s)))ds. \end{aligned}$$

Form inequality (16), it follows that

$$\alpha(NB(t)) \leq 4M\int_0^t \eta(s)\varphi(\alpha(B(s)))ds,$$

then

$$\alpha(N(B(t)) \leq 4M\|\eta\|_{L^1}\varphi(\sup_{t\in\mathbb{R}^+}\alpha(B(t))).$$

Since

$$\sup_{t\in\mathbb{R}^+}\alpha(B(t)) \leq \sup_{t\in\mathbb{R}^+}\alpha(B(t)) + \lim_{t\to+\infty}\sup\{|y(t)| : t\geq T, y\in E\}),$$

then

$$\alpha(N(B(t)) \leq 4M\|\eta\|_{L^1}\varphi(\sup_{t\in\mathbb{R}^+}\alpha(B(t)) + \lim_{t\to+\infty}\sup\{|y(t)| : t\geq T, y\in E\}). \tag{17}$$

On the other hand, we have

$$\begin{aligned}
|(Ny)(t)| &\leq \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
&+ M\int_{-\infty}^{t} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds + |(I_2y)(t)| \\
&+ M\int_{-\infty}^{T} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds. \\
&+ M\int_{T}^{t} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds + |I_2(t)|. \\
&\leq \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
&+ M\int_{-\infty}^{T} e^{-\delta(t-s)}p(s)ds\psi(\sup_{s\in\mathbb{R}}|\gamma(s)|) \\
&+ M\int_{T}^{t} e^{-\delta(t-s)}p(s)ds\psi(\sup\{|\gamma(t)| : t\geq T, y\in E\}) \\
&+ \sup\{|(I_2y)(t)| : t\geq T, y\in E\}).
\end{aligned}$$

Next, applying the Hölder inequality we derive

$$\begin{aligned}
|(Ny)(t)| &\leq \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
&+ \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}e^{-\delta(t-T)}\psi(\|y\|_\infty). \\
&+ \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}(1-e^{-\frac{q\delta}{q-1}t})^{1-\frac{1}{q}}\psi(\sup\{|y(t)| : t\geq T, y\in E\}) \\
&+ \sup\{|(I_2y)(t)| : t\geq T, y\in E\}).
\end{aligned}$$

Then

$$\begin{aligned}
|(Ny)(t)| &\leq \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
&+ \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}e^{-\delta T}\psi(\|y\|_\infty). \\
&+ \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}\psi(\sup\{|y(t)| : t\geq T, y\in E\}) \\
&+ \sup\{|(I_2y)(t)| : t\geq T, y\in E\}).
\end{aligned}$$

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Since $\delta \geq 0$, $I_2 \in C_0(\mathbb{R}^+, E)$ and

$$\lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\} \leq \sup_{t\in\mathbb{R}} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\},$$

then

$$\begin{array}{l} \lim\limits_{T\to+\infty} \sup\{|(Ny)(t) : t \geq T, y \in E\}) \\ \leq \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}} \psi(\sup\limits_{t\in J} \alpha(B(t)) + \lim\limits_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}). \end{array} \tag{18}$$

Further, combining (17) and (18), we get

$$\begin{array}{l} \sup\limits_{t\in J} \alpha((NB)(t)) + \lim\limits_{T\to+\infty} \sup\{|(Ny)(t) : t \geq T, y \in E\}) \\ \leq 4M\|\eta\|_{L^1} \varphi(\sup\limits_{t\in J} \alpha(B(t)) + \lim\limits_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}) \\ + \frac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}} \psi(\sup\limits_{t\in J} \alpha(B(t)) + \lim\limits_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\} \\ \leq M \max(4\|\eta\|_{L^1}, \frac{\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}})(\varphi + \psi)(\sup\limits_{t\in J} \alpha(B(t)) + \lim\limits_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}). \end{array} \tag{19}$$

From **Step 3** and inequality (19), we conclude that

$$\mu(N(B)) \leq M \max\left(4\|\eta\|_{L^1}, \frac{\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}\right)(\varphi + \psi)(\mu(B)).$$

It follows from Lemma 2.17 that N has at least one fixed point $y \in B_R$, which is just a asymptotically almost automorphic mild solution of problem (1)-(2) on $\mathbb{R}^+$.

4 An Example

Consider the second order differential equation of the form;

$$\begin{cases} \frac{\partial^2}{\partial t^2} z(t,\tau) &= \frac{\partial^2}{\partial \tau^2} z(t,\tau) + 2\sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right)\frac{\partial}{\partial t} z(t,\tau) \\ & + \frac{\sin^2 t}{12\sqrt{1+t^2}} \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)(|z(t,\tau)| + \ln(1+|z(t,\tau)|)) \\ & + \frac{\sin^2 t \quad \sin \pi z(t,\tau)}{15\sqrt{1+t^2}(1+|z(t,\tau)|)}, \qquad t \in \mathbb{R}^+, \quad \tau \in [0,\pi], \\ z(t,0) = & z(t,\pi) = 0, \qquad t \in \mathbb{R}^+, \\ \frac{\partial}{\partial t} z(0,\tau) &= \psi(\tau), \qquad \tau \in [0,\pi]. \end{cases} \tag{20}$$

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Let $E = L^2([0,\pi], \mathbb{R}^+)$ be the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}^+$, and let $H^2([0,\pi], \mathbb{R}^+)$ be the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}^+$, such that $x'' \in L^2([0,\pi], \mathbb{R}^+)$. We consider the operator $A_1 z(\tau) = z''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R}^+, \mathbb{C})$, which is the infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, A_1 has discrete spectrum, the spectrum of A_1 consists of eigenvalues n^2 for $n \in \mathbb{Z}$, with associated eigenvector

$$\omega_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}, n \in \mathbb{Z},$$

the set $\{\omega_n \in \mathbb{Z}\}$ is an orthonormal basis of E. In particular,

$$A_1 x = -\sum_{n=1}^{\infty} n^2 \langle x, w_n \rangle w_n \text{ for } x \in D(A).$$

The cosine function $C(t)$ is given by

$$C(t)x = \sum_{n=1}^{\infty} \cos(nt) \langle x, w_n \rangle w_n \text{ for } x \in D(A), t \in \mathbb{R}^+,$$

form a cosine function on H, with associated sine function

$$S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle x, w_n \rangle w_n \text{ for } x \in D(A), t \in \mathbb{R}^+.$$

From [35], for all $x \in H^2([0,\pi], \mathbb{R}^+), t \in \mathbb{R}^+$, $\|C(t)\|_{B(E)} \leq e^{-t}$ and $\|S(t)\|_{B(E)} \leq e^{-t}$.
Now, we define an operator $A(t) : D(A) \subset H \to H$ by

$$\begin{cases} D(A(t)) = D(A) \\ A(t) = A_1 + b(t,\tau). \end{cases}$$

where $b(t,\tau) = 2\sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)$
Note that $A(t)$ generates an evolutionary process $\mathcal{U}(t,s)$ of the form

$$\mathcal{U}(t,s) = S(t-s)e^{\int_s^t b(t,s)ds}$$

Since $b(t,\tau) = 2\sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right) \leq 2$, we have

$$\mathcal{U}(t,s) = S(t-s)e^{-2(t-s)} \tag{21}$$

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and

$$\|\mathcal{U}\|_{\mathcal{B}(E)} \leq \|S\|_{\mathcal{B}(E)} e^{-2(t-s)} \leq e^{-3(t-s)}$$

We conclude that $\mathcal{U}(t,s)$ is a evolutionary process exponentially stable with $M = 1$ and $\delta = 3$.
It follows from the estimate (21) that $\mathcal{U}(t,s) : E \to E$ is well defined and satisfies the conditions of Definition 2.1.
Hence conditions (H_1) and (H_2) are satisfied.
Now, let

$$z(t)(\tau) = w(t)(\tau),\ t \geq 0,\ \tau \in [0,\pi],$$

$$g(t,z)(\tau) = \frac{\sin^2 t}{12\sqrt{1+t^2}} \sin\Big(\frac{1}{2+\cos t + \cos\sqrt{2}t}\Big)(|z(t,\tau)| + \ln(1+|z(t,\tau)|)),$$

$$h(t,z)(\tau) = \frac{\sin^2 t \sin \pi z(t,\tau)}{15\sqrt{1+t^2}(1+|z(t,\tau)|)}.$$

Then it is easy to verify that $g : \mathbb{R} \times E \times E$ is continuous and

$$g \in AA(\mathbb{R} \times E; E).$$

We can estimate for the functions g:

$$g(t,z)(\tau) \leq \frac{\sin^2 t}{12\sqrt{1+t^2}}(|z(t,\tau)| + \ln(1+|z(t,\tau)|)).$$

Hence conditions $(H_3)(a)$ is satisfied with

$$p(t) = \frac{\sin^2 t}{3\sqrt{1+t^2}}, \quad \psi(t) = \frac{1}{4}(t + \ln(1+t)).$$

Then it is easy to verify that $p \in L^2(\mathbb{R})$ and $\rho_1 = \dfrac{1}{4}$.
On the other hand, it is clear that $h : \mathbb{R}^+ \times E \times E$ is continuous and

$$h \in C_0(\mathbb{R}^+ \times E; E).$$

We can also estimate for the functions h:

$$h(t,z)(\tau) \leq \frac{\pi}{15\sqrt{1+t^2}}|z(t,\tau)|.$$

Hence conditions $(H_3)(b)$ is satisfied with

$$\beta(t) = \frac{\pi}{15\sqrt{1+t^2}}, \quad \phi(R) = R.$$

Then it is easy to verify that $\beta \in C_0(\mathbb{R}^+, \mathbb{R})$, $\rho_2 = 1$ and $\rho \leq \frac{\pi}{15}$.
Furthermore:

$$f(t;z) = g(t;z) + h(t;z) \in AA(\mathbb{R}^+ \times E; E).$$

We can also estimate for the functions f:

$$f(t,z)(\tau) \leq \frac{2\sin^2 t}{\sqrt{1+t^2}} |z(t,\tau)|. \tag{22}$$

By (22), for every $t \in J$, and $B \in D \subset E$, we have

$$\alpha(f(t,D) \leq \frac{\sin^2 t}{12\sqrt{1+t^2}} \alpha(D),$$

Hence conditions $(H4)$ is satisfied with

$$\eta(t) = \frac{1}{6\sqrt{1+t^2}}, \quad \varphi(t) = \frac{\sin^2 t}{2}.$$

Moreover, we have

$$(\psi + \varphi)(t) = \frac{\sin^2 t}{2} + \frac{1}{4}(t + \ln(1+t)) \leq t.$$

We conclude that (see Lemma 2.1. [6])

$$\lim_{n \to +\infty} (\psi + \phi)^n(t) = 0 \text{ for a.e } t \in \mathbb{R}^+.$$

Consequently, can be written in the abstract form (1)-(2) with $A(t)$ and f as defined above. Thus, Theorem 3.3 yields that equation (20) has a asymptotically almost automorphic mild solution.

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