Solution of Higher Order Fractional Delay Differential Equations Using Darbo Type Fixed Point Theorem

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Abstract

In this work, we combine measure non-compactness and generalized operators in the setting of partial order Banach spaces to provide some generalized Darbo-type fixed point theorems. We further demonstrate how our findings could potentially apply in reality by establishing that higher-order fractional delay differential equations have solutions. We back up our results with numerical estimations based on a realistic scenario. This work explores fixed point theory in the partial order Banach spaces and shows how it can be used in real-world situations through realistic examples and facts.

Keywords: Darbo fixed point theorem, Fractional Dealy Differential Equation, Adomian Decomposition Method.
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1 Introduction

1.1 Partially Ordered Spaces

When we combine a complete normed vector space and a partial order relation, we get a sophisticated mathematical structure called a partially ordered Banach space. This framework, which visually conforms to the fundamental vector space operations, allows for the identifying comparison of designated items inside the space via a "greater than" or "less than" connection. More specifically, we have a field of scalars, which might be real or complex numbers, and a Banach space, X, built on top of it. Put X under a partial order $\leq$. Certain criteria must be satisfied before this partial order $\leq$ could possibly be considered compatible with the vector space structure and norm: If $x, y, z, w \in E$ and $\lambda \in \mathbb{R}$ are valid, the relation $\leq$. on E is termed partial order. Let E represent any real linear vector space.

- (Addition to Preserve Order) For any element z in X, if $x \le y$, then $x+z \le y+z$ holds true.
- (Order-Preserving Scalar Multiplication) It follows that $\alpha x \le \alpha y$ given $x \le y$ and a positive scalar α.
- (Positive Cone) All $x \in X$ for which $x \ge 0$ make up the positive cone, represented by the symbol $X+$. This cone must have the characteristics of being pointed, closed, and convex.

This framework enables the detailed examination of ordered relationships by providing a structured approach to analyse ordered systems within normed vector spaces. The partial order relation $\le$. is defined on the real linear space E. For any two elements $x, y \in E$, they are said to be comparable if either $x \le y$ or $y \le x$. A partially ordered normed space is a partially ordered set E equipped with a norm $\|\cdot\|$.

A normed linear space E is called complete if the metric d is induced by the norm $\|\cdot\|$ is defined on all of E. If a nonincreasing (or nondecreasing) sequence $\{x_n\}$satisfies $x_n \le x^* (or x_n \ge x^*)$for every $n \in \mathbb{N}$ and converges to x^*, the space E is termed regular. Banas and Goebel introduced the following concepts through their work in nonlinear analysis and its applications: If T(E) is a relatively compact subset of E, then an operator T mapping E onto itself is considered compact. Similarly, if T(S) is a relatively compact subset of E for every subset S of E, the operator T is considered totally bounded. An operator T is termed completely continuous if it is continuous and totally bounded on E.

The set $\mathbb{R}$ of real numbers with the norm defined by the absolute value function and the usual order relation $\le$ clearly demonstrates this characteristic. Similar to this, every partially compact subset of the space $C(J, \mathbb{R})$ with the usual standard supremum norm $\|\cdot\|$ determined by $\|x(t)\| = \sup_{t \in j} |x(t)|$ is compatible with the conventional order relation known by $x \le y$ if and only if $x(t) \le y(t)$for all $t \in J$. To get fresh conclusions, we substitute the cost of monotone and boundedness for the operator's bounds, closure, and convexity conditions.

The mathematical expression of $\mathbb{C} = \{\upsilon \in \mathbb{E} : \upsilon \geqslant 0\}$ is provided by taking the Banach space $\mathbb{E}$ with norm ||.|| with a positive cone. $\langle \mathbb{E}, \|.\| \rangle$ is a partially ordered Banach space; let $\sqsubseteq$ be the ordered relation induced by cone $\mathbb{C}$.

1.2 Delay Differential Equation

Integral and differential calculus play vital roles in applied sciences and engineering [1,8,7,14]. Among these, delay differential equations (DDEs) are significant as they incorporate delays in the dependent variable or its derivatives. Unlike ordinary differential equations, where the system's current state determines the derivative, DDEs include the influence of past states. This characteristic makes DDEs suitable for modelling memory-dependent or highly responsive systems. Dealy differential equations have diverse applications across disciplines like chemistry, biology, engineering, economics, and physics. For example, they describe neural communication in the brain, where signal transmission experiences temporal delays [7]. In ecology, DDEs model species growth influenced by reproductive and migratory delays. In chemistry, they analyse reactions where transit times between reaction sites affect outcomes [9]. Engineering applications include modelling feedback loops and delays in signal processing [21, 16].

To address higher-order DDEs, we apply generalized fixed-point theorems. For example, the fractional delay differential equation:

$$D^{\alpha}\mathcal{Y}(t) = G(t, \mathcal{Y}(t), \mathcal{Y}(\beta(t))), \tag{1.1}$$

where $t \in [0,1]$, $n-1 < \alpha \le n$, with initial conditions $\mathcal{Y}^{r}(0) = d_r$ *for* $r = 0,1,\ldots,n-1$. Here, $\mathcal{Y}(t)$ is the dependent variable, $\mathcal{G}$ represents the governing function, and $\beta(t)$ denotes the delayed argument.

These generalizations demonstrate the existence of solutions for complex DDEs. By leveraging advanced fixed point theorems in partially ordered Banach spaces, this framework expands the utility of fixed point theory in solving intricate mathematical problems associated with delay equations.

1.3 Some concepts of fractional derivative and integral

First, we review the basic concepts of fractional integrals and derivatives and their properties. Caputo's definition, extensively applied in various branches of applied mathematics, is also incorporated here [22, 23]. A real-valued function $\mathcal{R}(y)$, *for* $y > 0$, is said to belong to the space $\mathcal{D}_{\alpha} if \alpha \in \mathbb{R}$, and there exists a real number $\beta > \alpha$, such that $\mathcal{R}(y) = y^{\kappa}\mathcal{R}_1(y)$,where $\mathcal{R}_1(y) \in \mathcal{D}[0,\infty]$. Furthermore, it is in the space $\mathcal{D}_{\alpha}^{m}$ if $\mathcal{R}^{m} \in \mathcal{D}_{\alpha}$ for $m \in \mathbb{N} \cup \{0\}$.

Definition:1.2 The Riemann-Liouville fractional integral operator of order $\kappa \ge 0$, for a function $\mathcal{R} \in \mathcal{D}_{\alpha}, \alpha \ge -1$, is defined as:

$$\mathcal{J}_y^{\kappa}\mathcal{R}(y) = \frac{1}{\Gamma(\kappa)}\int_0^y (y-\xi)^{\kappa-1}\,\mathcal{R}(\xi)d\xi, \quad \kappa > 0, y > 0, \text{ and } \mathcal{J}_y^{0}\mathcal{R}(y) = \mathcal{R}(y). \tag{1.2}$$

The following properties of the operator $\mathcal{J}_y^{\kappa}$ are established in [20]. For $\mathcal{R}(y) \in \mathcal{D}_{\alpha}$, $\alpha \ge -1, \kappa, \lambda \ge 0$, and $\mu > -1$, we have:

i. $\mathcal{J}_y^{\kappa}\mathcal{J}_y^{\lambda}\mathcal{R}(y) = \mathcal{J}_y^{\kappa+\lambda}\mathcal{R}(y)$.
ii. $\mathcal{J}_y^{\kappa}\mathcal{J}_y^{\lambda}\mathcal{R}(y) = \mathcal{J}_y^{\lambda}\mathcal{J}_y^{\kappa}\mathcal{R}(y)$.
iii. $\mathcal{J}_y^{\kappa}y^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\kappa+\mu+1)}y^{\kappa+\mu}$.

To address certain limitations of the Riemann-Liouville derivative, a modified fractional derivative, known as Caputo's derivative, is introduced [22].

Dfinition:1.3 In the sense of Caputo's derivative, the fractional derivative of $\mathcal{R}(y)$ is defined as:

$$\mathcal{D}_y^{\kappa}\mathcal{R}(y) = \mathcal{J}_y^{m-\kappa}\mathcal{D}_y^{m}\mathcal{R}(y) = \frac{1}{\Gamma(m-\kappa)}\int_0^y (y-\xi)^{m-\kappa-1}\,\mathcal{R}^{m}(\xi)d\xi \tag{1.3}$$

for $m-1 < \kappa \le m, m \in \mathbb{N}$, $y > 0, \mathcal{R} \in \mathcal{D}_{-1}^{m}$, and $\mathcal{D}_y^{\kappa}\mathcal{P} = 0$.

Defintion:1.4 The Caputo time-fractional derivative operator of order $\kappa > 0$ is defined for any smallest integer $m > \kappa$ as:

$$\mathcal{D}_t^{\kappa}\mathcal{V}(y,t) = \frac{\partial^{\kappa}\mathcal{V}(y,t)}{\partial t^{\kappa}} = \begin{cases} \frac{1}{\Gamma(m-\kappa)}\int_0^t (t-s)^{m-\kappa-1}\frac{\partial^m \mathcal{V}(y,s)}{\partial s^m}ds, & m-1<\kappa<m, \\ \frac{\partial^m \mathcal{V}(y,t)}{\partial t^m}, & \kappa = m. \end{cases} \quad (1.4)$$

The following lemma is helpful for solving problems involving these operators.

Lemma:1.1 If $m-1<\kappa<m, m \in \mathbb{N}$, and $\mathcal{R} \in \mathcal{D}_{\alpha}^m, \alpha \geq -1$, then:

$$\mathcal{D}_y^{\kappa}\mathcal{J}_y^{\kappa}\mathcal{R}(y) = \mathcal{R}(y), \quad \text{and } \mathcal{J}_y^{\kappa}\mathcal{D}_y^{\kappa}\mathcal{R}(y) = \mathcal{R}(y) - \sum_{p=0}^{m-1}\mathcal{R}^p\,(0^+)\frac{y^p}{p!}, \quad y>0.$$

Definition:1.5 The Mittag-Leffler function, denoted by $\mathbb{M}_{\kappa}(x)$, is defined as:

$\mathbb{M}_{\kappa}(x) = \sum_{j=0}^{\infty}\frac{x^j}{\Gamma(1+\kappa j)}, \quad \kappa \in \mathbb{C}, \mathrm{Re}(\kappa) > 0, x \in \mathbb{C}.$ If $\kappa = 1$, the Mittag-Leffler function reduces to the exponential function: $\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(1+j)} = \sum_{j=0}^{\infty}\frac{x^j}{j!}$.

Lemma 1.2 For $\kappa > 0$, the general solution to the homogeneous equation $\mathcal{D}_{0^+}^{\kappa}\psi(y) = 0$ is given by:

$$\psi(y) = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots + c_{m-1}y^{m-1}, \quad (1.5)$$

where $c_i \in \mathbb{R}, i = 0,1,2,\ldots,m-1$, and $m = \lfloor\kappa\rfloor + 1$.

Lemma: 1.3 For $\kappa > 0$, we have:

$$\mathcal{J}_{0^+}^{\kappa}\mathcal{D}_{0^+}^{\kappa}\psi(y) = \psi(y) + c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots + c_{m-1}y^{m-1}, \quad (1.6)$$

where $c_i \in \mathbb{R}, i = 0,1,2,\ldots,m-1$, and $m = \lfloor\kappa\rfloor + 1$.

2. Fixed Point Theorem

The term "measure of non-compactness" is a popular mathematical approach for describing how much a set, operator, or bounded set lacks compactness. This idea is especially relevant to fixed-point theorems, stability studies and partially ordered Banach spaces for nonlinear equations [1, 8, 13,17, 18, 19]. compactness measures in [13].

Definition: 2.1 If $\mathbb{E}$ is a Banach space, then Kuratowski's $\mathfrak{MNC}$ for $\mathcal{A} \in \mathbb{E}$ is the map $\alpha : \mathfrak{M}(\mathbb{E}) \to \mathbb{R}_+$.

$\alpha(\mathcal{A}) = inf\{\epsilon > 0$: A finite number of sets with diameter $< \epsilon$, can cover it$\}$.

The following formulation recalls the concept of $\mathfrak{MNC}$ that was axiomatically provided in [13].

Definition: 2.2 If α is represented by the symbol Ξ, then the following characteristics are true:

a) ker $\Xi = \{\mathcal{C} \in \mathfrak{M}(\mathbb{E}) \mid \Xi(\mathcal{C}) = 0\} \neq \phi$ and ker $\Xi \in \mathfrak{N}(\mathbb{E})$.
b) If $\mathcal{C} \subseteq \mathcal{D}$ then $\Xi(\mathcal{C}) \leq \Xi(\mathcal{D})$ for all $\mathcal{C}, \mathcal{D} \in \mathbb{E}$.
c) $\Xi(conv(\overline{\mathcal{C}})) = \Xi(conv(\mathcal{C}))$ for all $\mathcal{C} \in \mathbb{E}$, where $\overline{\mathcal{C}}$ denotes the closure of $\mathcal{C}$.
d) $\Xi(\mathcal{C} \cup \mathcal{D}) = \max\{\Xi(\mathcal{C}), \Xi(\mathcal{D})\}$ for all $\mathcal{C}, \mathcal{D} \in \mathbb{E}$.

e) $\Xi(\lambda_1\mathcal{C}+\lambda_2\mathcal{D}) \le \lambda_1\Xi(\mathcal{C})+\lambda_2\Xi(\mathcal{D})$ *if* $\lambda_1+\lambda_2=1$ and $\lambda_1,\lambda_2 \ge 0$ for all $\mathcal{C},\mathcal{D}\in\mathbb{E}$.

f) If $\mathcal{C}_n$ is a decreasing sequence of nonempty closed, bounded subsets of $\mathbb{E}$ and $\lim_{n\to\infty}\Xi(\mathcal{C}_n)=0$ then $\mathcal{C}_\infty=\bigcap_{n\geqslant 1}\mathcal{C}_n$ is nonempty compact.

Definition: 2.2 [6] Let $\mathcal{A}$ be a nonempty, convex, and closed set in Banach space $\mathbb{E}$. Let $\Upsilon:\mathcal{A}\to\mathcal{A}$ be a continuous map such that, given a non-empty set $\mathcal{S}\subseteq\mathcal{A}$, where Ξ is a $\mathfrak{MNC}, k\in[0,1)$ with

$$\Xi(\Upsilon(S)) \le k\Xi(S) \tag{2.1}$$

Then, at least one fixed point in $\mathcal{A}$ is admitted by Υ.

We now review the definition below to expand the finding of [19] in a partly ordered Banach space. The consequence of [19] is further refined in a partially ordered Banach space by recalling the definition below.

Theorem:2.1 Consider $\langle\mathbb{E},\|.\|,\sqsubseteq\rangle$ be the partially ordered Banach space having a positive cone $\mathbb{C}$ is normal. Let $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing and continuous function and that $\mathfrak{I}:\mathbb{E}\to\mathbb{E}$ is a monotonic increasing and continuous operator. There is also $0<\eta(p,q)<1$ such that for $0<p<q<\infty$ with

$$p \leqslant \Delta(\mathcal{Y})+\varphi(\Delta(\mathcal{Y})) \leqslant q \Rightarrow \Delta(\mathfrak{I}\mathcal{Y})+\varphi(\Delta(\mathfrak{I}\mathcal{Y})) \leqslant \bar{\eta}(p,q)\Delta(\mathcal{Y})+\varphi(\Delta(\mathcal{Y})), \tag{2.2}$$

for all non-empty subset $\mathcal{Y}$ of $\mathbb{E}$, where Δ being essentially the measure of noncompactness. Then $\mathfrak{I}$ has a fixed point ν^* and $\xi_0 \sqsubseteq \mathfrak{I}\xi_0$ for $\xi_0\in\mathbb{E}$, with the sequence $\langle\mathfrak{I}^n\xi_0\rangle$ converges monotonically to ν^*.

The consequence that follows may be obtained if we recognise the nondecreasing and continuous function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ *as* $\varphi(t)\equiv 0$ in the theorem (1.1).

Proposition:2.1 Consider $\langle\mathbb{E},\|.\|,\sqsubseteq\rangle$ be the partially ordered Banach space having a positive cone $\mathbb{C}$ is normal. Let $\mathfrak{I}:\mathbb{E}\to\mathbb{E}$ is a monotonic increasing and continuous operator. If there is $0<\eta(p,q)<1$ *for* $0<p<q<\infty$ with

$$p \leqslant \Delta(\mathcal{Y}) \leqslant q \Rightarrow \Delta(\mathfrak{I}\mathcal{Y}) \leqslant \eta(p,q)\Delta(\mathcal{Y}), \tag{2.3}$$

for all non-empty subset $\mathcal{Y}$ of $\mathbb{E}$, where Δ being essentially the measure of noncompactness. Then $\mathfrak{I}$ has a fixed point ν^* and $\xi_0 \sqsubseteq \mathfrak{I}\xi_0$ for $\xi_0\in\mathbb{E}$, with the sequence $\langle\mathfrak{I}^n\xi_0\rangle$ converges monotonically to ν^*.

If we take $\Delta(\mathcal{X})=diam(\mathcal{X})$ in Proposition 2.1 we get next result.

Proposition: 2.2 Consider $\langle\mathbb{E},\|.\|,\sqsubseteq\rangle$ be the partially ordered Banach space having a positive cone $\mathbb{C}$ is normal. Let $\mathfrak{I}:\mathbb{E}\to\mathbb{E}$ is a monotonic increasing and continuous operator. If there is

$0<\eta(p,q)<1$ for $0<p<q<\infty$ with

$$p \leqslant diam(\mathcal{Y}) \leqslant q \ \Rightarrow diam(\Im\mathcal{Y}) \leqslant \eta(p,q)\big(diam(\mathcal{Y})\big), \tag{2.4}$$

for all non-empty subset $\mathcal{Y}$ of $\mathbb{E}$. Then $\Im$ has a fixed point ν^* and $\xi_0 \sqsubseteq \Im\xi_0$ for $\xi_0 \in \mathbb{E}$, with the sequence $\langle \Im^n \xi_0 \rangle$ converges monotonically to ν^*.

Proof: Here is a $\Im$-invariant nonempty convex closed subset $\mathcal{B}$ and $diam(\mathcal{B}_\infty)=0$, with respect to Propositions (2.1) and 3.2 [12]. This indicates that $\mathcal{B}_\infty$ is a singleton set, therefore we have a fixed point of $\Im \neq \phi$.

We assume that there are two different fixed points in order to demonstrate uniqueness. We may define the set $X=\{\iota,\kappa\}$ after $\iota,\kappa \in Y$. In this instance, $diam(X)=diam(\Im X)=\|\iota-\kappa\|>0.$ Then using (2.4) we get

$$p \le diam(\mathcal{Y}) \le q \Rightarrow diam(\Im\mathcal{Y}) \le \tilde{\eta}(p,q)\big(diam(\mathcal{Y})\big) \tag{2.5}$$

Since $diam(\mathcal{X}) = diam(\Im\mathcal{X})$results to contradiction to the property of the function $\eta(p,q) < 1$and hence $\iota=\kappa$.

Proposition: 2.3 Consider $\langle \mathbb{E}, \|.\|, \sqsubseteq \rangle$ be the partially ordered Banach space having a positive cone $\mathbb{C}$ is normal. Let $\Im:\mathbb{E}\to\mathbb{E}$ is a monotonic increasing and continuous operator. If there is $0<\eta(p,q)<1$ for $0<p<q<\infty$ with

$$p \le \|\vartheta_1 - \vartheta_2\| \le q \Rightarrow \|\Im\vartheta_1 - \Im\vartheta_2\| \le \eta(p,q)(\|\vartheta_1-\vartheta_2\|), \tag{2.6}$$

for all $\vartheta_1,\vartheta_2 \in \mathbb{E}$. If $\xi_0 \sqsubseteq \Im\xi_0$ for $\xi_0 \in \mathbb{E}$ then $\Im$ has unique fixed point ν^* and sequence $\langle \Im^n \xi_0 \rangle$ converges monotonically to ν^*.

Proof: We consider $\Delta: M \to \mathbb{R}_+$ by the rule $\Delta(\mathcal{X}) = diam(\mathcal{X})$, where

$$diam(\mathcal{X}) = sup\{\|\vartheta_1-\vartheta_2\|, \vartheta_1,\vartheta_2 \in \mathcal{D}\}, \tag{2.7}$$

specifies the set $\mathcal{X}^s$ diameter. In the sense of Definition (1.1), it is obvious from this formulation that Δ is a $\mathfrak{MNC}$. Now, by the virtue of equation (2.2). If

$p \le \sup_{\vartheta_1,\vartheta_2\in X} [\|\vartheta_1-\vartheta_2\|] \le q$,then

$$\begin{aligned}\sup_{\vartheta_1,\vartheta_2\in X}[\|\Im\vartheta_1-\Im\vartheta_2\|] &\le \sup_{\vartheta_1,\vartheta_2\in X}\|\Im\vartheta_1-\Im\vartheta_2\| \\ &\le \eta(p,q)\sup_{\vartheta_1,\vartheta_2\in X}[\|\vartheta_1-\vartheta_2\|] \\ &\le \eta(p,q)\left(\sup_{\vartheta_1,\vartheta_2\in X}\|\vartheta_1-\vartheta_2\|\right).\end{aligned}$$

Therefore, applying the concept of set diameter, we obtain

$$p \leqslant diam(\mathcal{X}) \leqslant q \ \Rightarrow diam(\Im\mathcal{X}) \leqslant \eta(p,q)\big(diam(\mathcal{X})\big). \tag{2.8}$$

Therefore, from the viewpoint of Proposition 2.1, $\Im$ has a fixed point.

3. Solution of Delay Differential Equation

The existence of a solution to the non-homogeneous higher order delay differential equation (1.1) in the space of bounded and continuous functions defined on the interval $[0,\delta]$, represented by $\mathcal{BC}[0,\delta]$, is examined in this section. In order to examine the problem (1.1), we must take into account the following presumptions:

1) The non-negative real number λ and the monotonic increasing functions $G:[0,\delta]\times BC[0,\delta]\times BC[0,\delta]\to\mathbb{R}_+$ possess the following inequality:

$$\left|G\left(v,\mathcal{Y}_1(v),\mathcal{Y}_1(\beta(v))\right)-G\left(v,\mathcal{Y}_2(v),\mathcal{Y}_2(\beta(v))\right)\right| \\ \leq \lambda_1|\mathcal{Y}_1(v)-\mathcal{Y}_2(v)|+\lambda_2|\mathcal{Y}_1(\beta(v))-\mathcal{Y}_2(\beta(v))|. \tag{3.1}$$

2) There is real number $\mathcal{M}_{\mathcal{G}}*$ which is not negative with

$$M_G^*=max\{G(v,0,0,\ldots,0);v\in[0,1]\}. \tag{3.2}$$

3) There exist a positive real number ρ, with inequality

$$\sum_{r=1}^{n-1}\frac{\partial^r\mathcal{Y}(0^+)}{\partial\mathcal{Y}^r}\frac{\eta^r}{r!}+\frac{\left((\lambda_1+\lambda_2)\rho+M_G^*\right)\eta^\alpha}{\Gamma(\alpha+1)}\leq\rho. \tag{3.3}$$

Theorem: 3.1 Under the assumptions (1) - (3) the delay differential equation (1.1) admits at least one solution $\mathcal{Y}\in\mathcal{BC}([0,\delta])$ provided $(\lambda_1+\lambda_2)\delta^\alpha<\Gamma(\alpha+1)$.

Proof: It needs to be notice that the space $\langle\mathcal{BC}[0,\delta],\|.\|,\sqsubseteq\rangle$ is a partially order Banach space with a normal positive cone $\mathbb{C}=\{v\in\mathbb{E}:v\geqslant 0\}$. The closed and bounded ball $B_\rho=\{\theta\in[0,\delta]:\|\theta\|\leq\rho\}$ should also be taken into account.

Next, define the mapping $\mathcal{Q}:\mathcal{BC}[0,\delta]\times\mathcal{BC}[0,\delta]\to\mathbb{R}$. By using the inverses differential operator on (1.1), we obtain

$$Q(\mathcal{Y}(v))=\sum_{r=1}^{n-1}\frac{\partial^r\mathcal{Y}(0^+)}{\partial\mathcal{Y}^r}\frac{v^r}{r!}+I^\alpha\left(G\left(v,\mathcal{Y}(v),\mathcal{Y}(\beta(v))\right)\right). \tag{3.4}$$

We prove the operator $\mathcal{Q}$ is bounded. Consider $v\in[0,\delta]$ then we have

$$\begin{aligned}|Q(\mathcal{Y}(v))| &\leq\left|\sum_{r=1}^{n-1}\frac{\partial^r\mathcal{Y}(0^+)}{\partial\mathcal{Y}^r}\frac{v^r}{r!}\right|+\left|I^\alpha\left(G\left(v,\mathcal{Y}(v),\mathcal{Y}(\beta(v))\right)\right)\right|\\ &\leq\left|\sum_{r=1}^{n-1}\frac{\partial^r\mathcal{Y}(0^+)}{\partial\mathcal{Y}^r}\frac{v^r}{r!}\right|+\left|I^\alpha\left(G\left(v,\mathcal{Y}(v),\mathcal{Y}(\beta(v))\right)-G(v,0,0)+G(v,.0,0)\right)\right|\\ &\leq\left|\sum_{r=1}^{n-1}\frac{\partial^r\mathcal{Y}(0^+)}{\partial\mathcal{Y}^r}\frac{v^r}{r!}\right|+\left|I^\alpha\left(\left|G\left(v,\mathcal{Y}(v),\mathcal{Y}(\beta(v))\right)-G(v,0,0)\right|+|G(v,0,0)|\right)\right|\\ &\leq\left|\sum_{r=1}^{n-1}\frac{\partial^r\mathcal{Y}(0^+)}{\partial\mathcal{Y}^r}\frac{v^r}{r!}\right|+\left|I^\alpha\lambda_1|\mathcal{Y}(v)|+\lambda_2|\mathcal{Y}(\beta(v))|+M_G^*\right|\end{aligned}$$

$$\leq \left|\sum_{r=1}^{n-1} \frac{\partial^r \mathcal{Y}(0^+)}{\partial \mathcal{Y}^r}\frac{v^r}{r!}\right| + \frac{\left((\lambda_1+\lambda_2)\|\mathcal{Y}\| + M_G^*\right)\eta^\alpha}{\Gamma(\alpha+1)},$$

$$\left|Q\big(\mathcal{Y}(v)\big)\right| \leq \left|\sum_{r=1}^{n-1} \frac{\partial^r \mathcal{Y}(0^+)}{\partial \mathcal{Y}^r}\frac{v^r}{r!}\right| + \frac{\left((\lambda_1+\lambda_2)\|\mathcal{Y}\| + M_G^*\right)\eta^\alpha}{\Gamma(\alpha+1)}, \tag{3.5}$$

where $\mathcal{M}^* = sup\{\chi(\theta,0,0,0,0): \theta \in [0,\delta]\}$.

Next we prove $\mathcal{Q}$ satisfies the assumptions of Proposition 2.3. Since, the function $\mathcal{G}$ is monotonic increasing this leads to $\mathcal{Q}$ being also a non-decreasing and continuous mapping on $\mathcal{BC}[0,\delta]$.

Now, from equation (3.5) and assumption (3) we confirmed that $\mathcal{Q}$ non-decreasing and continuous mapping on $\mathcal{B}_\rho$.

Next, we prove that $\mathcal{Q}$ satisfies the inequality (2.7). Consider $\mathcal{X},\mathcal{Y} \in \mathcal{B}_\rho$, we obtain

$$\left|Q\big(\mathcal{X}(v)\big) - Q\big(\mathcal{Y}(v)\big)\right| \leq \left|\begin{array}{l}\sum_{r=1}^{n-1} \frac{\partial^r \mathcal{X}(0^+)}{\partial \mathcal{X}^r}\frac{v^r}{r!} + I^\alpha\left(G\left(v,\mathcal{X}(v),\mathcal{X}\big(\beta(v)\big)\right)\right)\\ -\sum_{r=1}^{n-1} \frac{\partial^r \mathcal{Y}(0^+)}{\partial \mathcal{Y}^r}\frac{v^r}{r!} + I^\alpha\left(G\left(v,\mathcal{Y}(v),\mathcal{Y}\big(\beta(v)\big)\right)\right)\end{array}\right|$$

$$\leq I^\alpha\left(\left|G\left(v,\mathcal{X}(v),\mathcal{X}\big(\beta(v)\big)\right) - G\left(v,Y(v),Y\big(\beta(v)\big)\right)\right|\right)$$

$$\leq \mathcal{I}^\alpha\left(\lambda_1|\mathcal{X}(v)-\mathcal{Y}(v)| + \lambda_2\left|\mathcal{X}\big(\beta(v)\big)-\mathcal{Y}\big(\beta(v)\big)\right|\right).$$

This inequality is true for all $v \in [0,\delta]$. Therefore we obtain

$$\sup_{v\in[0,\delta]}\left|Q\big(\mathcal{X}(v)\big) - Q\big(\mathcal{Y}(v)\big)\right| \leq \mathcal{I}^\alpha\left(\begin{array}{c}\lambda_1\left(\sup_{v\in[0,\delta]}|\mathcal{X}(v)-\mathcal{Y}(v)|\right)\\ +\lambda_2\left(\sup_{v\in[0,\delta]}\left|\mathcal{X}\big(\beta(v)\big)-\mathcal{Y}\big(\beta(v)\big)\right|\right)\end{array}\right).$$

Hence

$$\|Q\mathcal{X} - Q\mathcal{Y}\| \leq \eta(p,q)\|\mathcal{X}-\mathcal{Y}\|. \tag{3.6}$$

where $\eta(p,q) = \frac{(\lambda_1+\lambda_2)\delta^\alpha}{\Gamma(\alpha+1)}$. All the conditions of proposition 2.3 are satisfied. Hence, the operator $\mathcal{Q}$ has a fixed point, which is the solution of (1.1).

4. Numerical Illustrations

Example:4.1

$$D^\alpha \mathcal{Y}(v) = -\mathcal{Y}(v) - \mathcal{Y}(v-0.3) + e^{-v+0.3},\ \ 2 < \alpha \leq 3 \tag{4.1}$$

with $\mathcal{Y}(0) = 1, \mathcal{Y}^{(1)}(0) = -1, \mathcal{Y}^{(2)}(0) = 1$ and $\mathcal{Y}(v) = e^{-v}, v \leq 0$. (4.2)

If we take

$$G\left(v,\mathcal{Y}(v),\mathcal{Y}\big(\beta(v)\big)\right) = -\mathcal{Y}(v) - \mathcal{Y}(v-0.3) + e^{-v+0.3}, \tag{4.3}$$

in equation (1.1) we get equation (4.1).

Also,

$$\left|G\left(v,\mathcal{Y}_1(v),\mathcal{Y}_1\big(\beta(v)\big)\right) - G\left(v,\mathcal{Y}_2(v),\mathcal{Y}_2\big(\beta(v)\big)\right)\right|$$

$$\leq |-\mathcal{Y}_1(v) - \mathcal{Y}_1(v-0.3) + \mathcal{Y}_2(v) + \mathcal{Y}_2(v-0.3)|$$

$$\leq |\mathcal{Y}_1(v) - \mathcal{Y}_2(v)| + |\mathcal{Y}_1(v-0.3) - \mathcal{Y}_2(v-0.3)|$$

$$\leq \lambda_1|\mathcal{Y}_1(\upsilon)-\mathcal{Y}_2(\upsilon)|+\lambda_2|\mathcal{Y}_1(\upsilon-0.3)-\mathcal{Y}_2(\upsilon-0.3)|,$$

where $\lambda_1=\lambda_2=1$.

Further

$$\sum_{r=1}^{n-1}\frac{\partial^r\mathcal{Y}(0^+)}{\partial\mathcal{Y}^r}\frac{\upsilon^r}{r!}+\frac{((\lambda_1+\lambda_2)\rho+M_G^*)\eta^\alpha}{\Gamma(\alpha+1)}=1-1+\frac{1}{2}+\frac{2.\rho}{\Gamma(2.5+1)}=\frac{1}{2}+\frac{2.\rho}{\Gamma(2.5+1)}. \tag{4.4}$$

Now in view of inequality (3.5) we have

$$\frac{1}{2}+\frac{2.\rho}{\Gamma(2.5+1)}\leq\rho\Rightarrow\frac{1}{2}\leq\rho\left(1-\frac{2}{\Gamma(2.5+1)}\right). \tag{4.5}$$

Hence we get

$$\frac{\Gamma(2.5+1)}{2.\Gamma(2.5+1)-2}\leq\rho.\ \text{Moreover}\ \frac{(\lambda_1+\lambda_2)\delta^\alpha}{\Gamma(\alpha+1)}=\frac{2}{\Gamma(\alpha+1)}<1\,\text{for}\,2<\alpha\leq 3. \tag{4.6}$$

All the conditions of Theorem 3.1 are satisfied. Hence equation (4.6) has exactly one solution.

4.1 Numerical Estimation

The solution of the differential equation is obtained by the method introduced in [20] as follows,

$$\mathcal{Y}(\upsilon)=\mathcal{Y}(0)+\upsilon\mathcal{Y}'(0)+\frac{\upsilon^2}{2}\mathcal{Y}''(0)+I_\upsilon^\alpha\left(-\mathcal{Y}(\upsilon)-\mathcal{Y}(\upsilon-0.3)+e^{-\upsilon+0.3}\right). \tag{4.7}$$

Applying the initial conditions we get,

$$\begin{aligned}\mathcal{Y}(\upsilon)&=\mathcal{Y}(0)+\upsilon\mathcal{Y}'(0)+\frac{\upsilon^2}{2}\mathcal{Y}''(0)+I_\upsilon^\alpha\left(-\mathcal{Y}(\upsilon)-\mathcal{Y}(\upsilon-0.3)+e^{-\upsilon+0.3}\right)\\&=1-\upsilon+\frac{\upsilon^2}{2}+I_\upsilon^\alpha\left(e^{-\frac{\upsilon}{\tau}}\right)+I_\upsilon^\alpha\left(-\mathcal{Y}(\upsilon)-\mathcal{Y}\left(\frac{\upsilon}{\tau}\right)\right).\end{aligned} \tag{4.8}$$

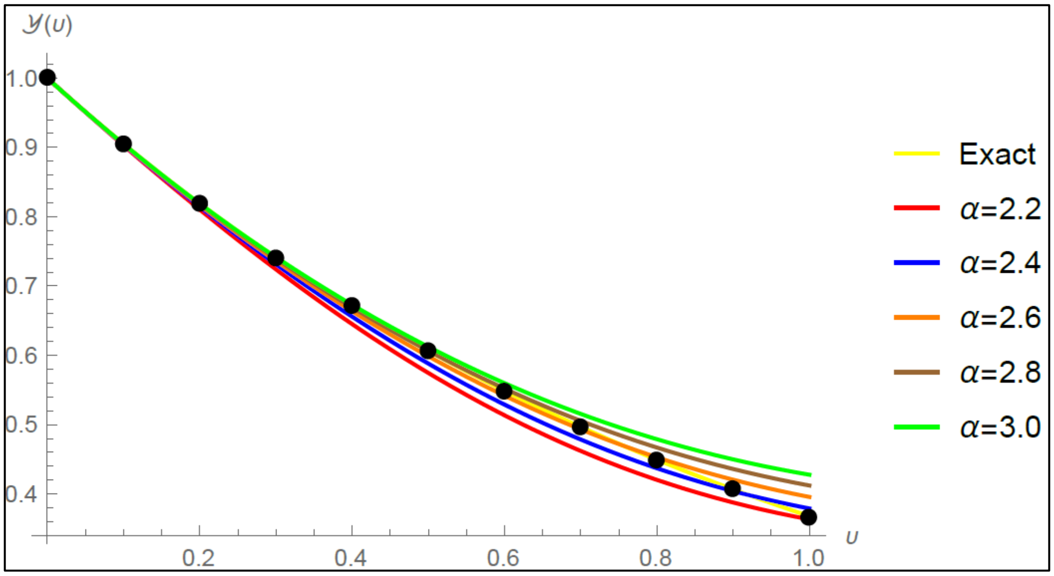

Fig: 4.1 Plot of the approximate and exact solution of (4.1) for υ and $\mathcal{Y}(\upsilon)$.

$\tau = 2$ & $\alpha = 2.2, 2.4, 2.6, 2.8, 3.0$

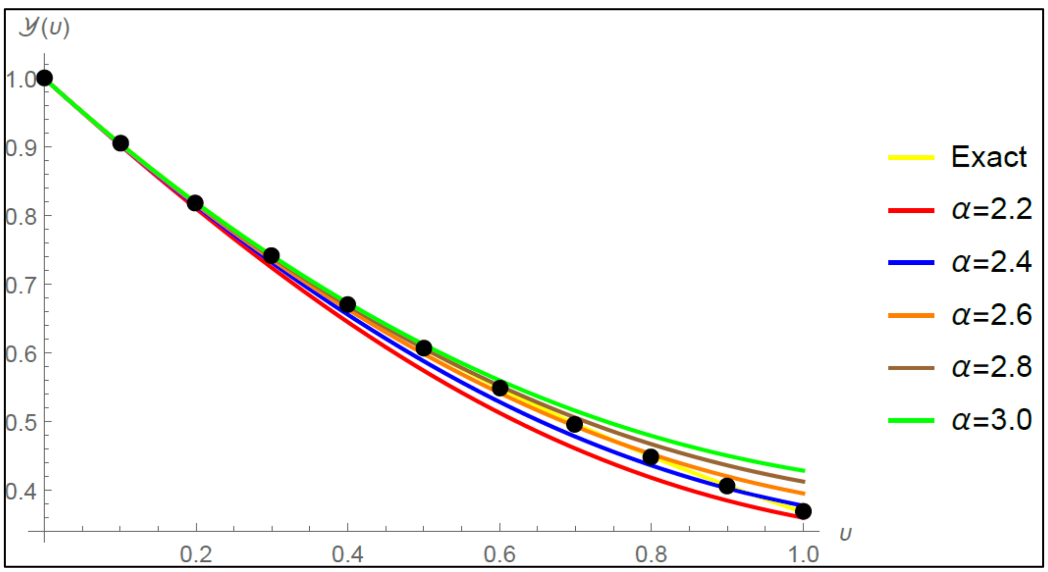

Fig: 4.2 Plot of the approximate and exact solution of (4.1) for υ and $\mathcal{Y}(\upsilon)$.

$\tau = 3$ & $\alpha = 2.2, 2.4, 2.6, 2.8, 3.0$

Table: 4.1 Error Estimation for the solution of (4.1) for $\upsilon \& \mathcal{Y}(\upsilon)$ for $\tau = 2$

t	Exact Sol	$\alpha = 2.2$		$\alpha = 2.4$	
		Approx Sol	**Error**	**Approx Sol**	**Error**
0.0	1.0000	1.0000	0.0000	1.0000	0.0000
0.2	0.818731	0.810334	008397	814268	0.004463
0.4	0.670320	0.645015	0.025305	0.655771	0.014549
0.6	0.548812	0.513409	0.035403	0.528865	0.019946
0.8	0.449329	0.420269	0.029059	0.437045	0.012284
1.0	0.367879	0.363394	0.004485	0.379058	0.011179
$\alpha = 2.6$		$\alpha = 2.8$		$\alpha = 2.4$	
Approx Sol	**Error**	**Approx Sol**	**Error**	**Approx Sol**	**Error**
1.0000	0.00000	1.0000	0.0000	1.0000	0.0000
0.816656	0.002075	0.818077	0.000654	0.818908	0.000177
0.663568	0.006752	0.669054	0.001266	0.672821	0.002501
0.541703	0.007109	0.551934	0.003122	0.559816	0.011004
0.452820	0.003491	0.466895	0.008858	0.478919	0.029589
0.395687	0.027807	0.412200	00.044320	0.427757	0.059877

Table: 4.2 Error Estimation for the solution of (4.1) for $\upsilon \& \mathcal{Y}(\upsilon)$ for $\tau = 3$

t	Exact Sol	$\alpha = 2.2$		$\alpha = 2.4$	
		Approx Sol	**Error**	**Approx Sol**	**Error**
0.0	1.0000	1.0000	0.0000	1.0000	0.0000
0.2	0.818731	0.81032	0.008411	0.814264	0.004467
0.4	0.670320	0.64478	0.02554	0.655681	0.014639
0.6	0.548812	0.512416	0.036395	0.52842	0.020389
0.8	0.449329	0.417978	0.031351	0.435919	0.013409
1.0	0.367879	0.359838	0.009357	0.009357	0.009357
$\alpha = 2.6$		$\alpha = 2.8$		$\alpha = 2.4$	
Approx Sol	**Error**	**Approx Sol**	**Error**	**Approx Sol**	**Error**
1.0000	0.00000	1.0000	0.0000	1.0000	0.0000
0.816655	0.002075	0.818077	0.000654	0.818908	0.000177
0.663539	0.006752	0.669048	0.001266	0.672823	0.002501
0.541544	0.007109	0.551906	0.003122	0.559839	0.011004
0.452412	0.003491	0.466874	0.008858	0.479073	0.029589
0.395125	0.027807	0.412425	0.044320	0.428391	0.059877

Example: 4.2

$$\mathcal{D}^{\alpha}\left(\mathcal{Y}(\upsilon)\right)=\mathcal{Y}\left(\frac{\upsilon}{\tau}\right)-\upsilon^{2}+\upsilon, \tag{4.9}$$

with initial conditions $\mathcal{Y}(0)=0,\mathcal{Y}'(0)=-\frac{1}{2},\mathcal{Y}''(0)=1,$ (4.10)

and exact solution $\mathcal{Y}(\upsilon)=\frac{1}{2}(\upsilon^{2}-\upsilon).$ (4.11)

The assumptions of Theorem 3.1 can be easily verified. Hence the differential equation (4.2) has a solution. This solution can be obtained by the numerical scheme introduced in [20].

$$\mathcal{Y}(\upsilon)=\upsilon+\upsilon^{2}+I_{\upsilon}^{\alpha}\left(-\upsilon^{2}+\upsilon\right)+I_{\upsilon}^{\alpha}\left(\mathcal{Y}\left(\frac{\upsilon}{\tau}\right)\right).$$

$$\mathcal{Y}_{0}=-\frac{\upsilon}{2}+\upsilon^{2}+I_{\upsilon}^{\alpha}\left(-\upsilon^{2}+\upsilon\right)$$

$$\mathcal{Y}_{i+1}=I_{\upsilon}^{\alpha}\left(Y_{i}\left(\frac{\upsilon}{\tau}\right)+Y_{i}(\upsilon)\right). \tag{4.12}$$

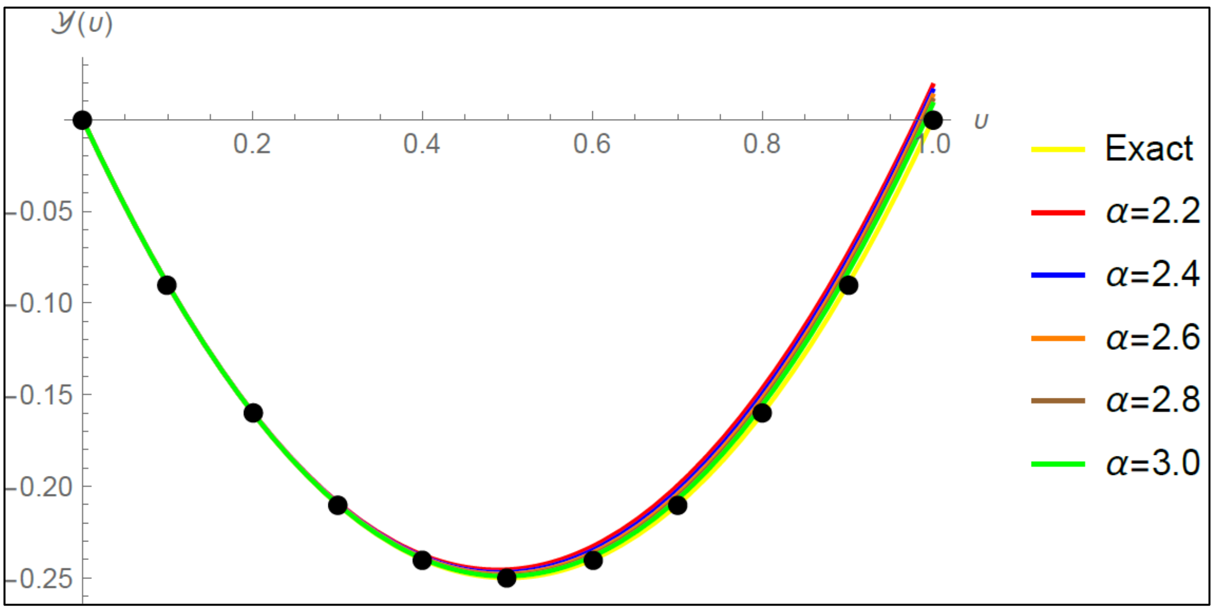

Fig: 4.3 Plot of the approximate and exact solution of (4.2) for υ and $\mathcal{Y}(\upsilon)$.
$\tau=2$,$\alpha=2.2,2.4,2.6,2.8,2.0$

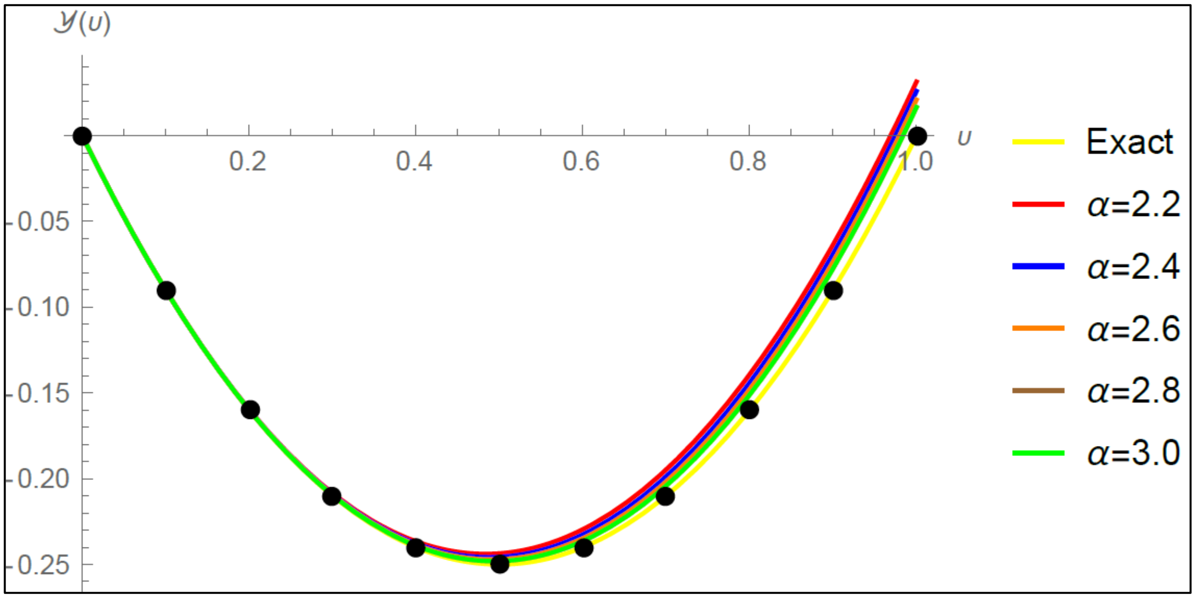

Fig: 4.4 Plot of the approximate and exact solution of (4.2) for υ and $\mathcal{Y}(\upsilon)$.
$\tau = 3$ & $\alpha = 2.2, 2.4, 2.6, 2.8, 2.0$

Table: 4.3 Error Estimation for the solution of (4.2) for υ & $\mathcal{Y}(\upsilon)$ for $\tau = 2$

t	Exact Sol	$\alpha = 2.2$		$\alpha = 2.4$	
		Approx Sol	**Error**	**Approx Sol**	**Error**
0.0 &	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	-0.16000	-0.15968	0.00032	-0.15982	0.00018
0.4	-0.24000	-0.23754	0.00245	-0.23841	0.00159
0.6	-0.24000	-0.23280	0.00719	-0.2348	0.00514
0.8	-0.16000	-0.146412	0.01359	-0.14948	0.01052
1.0	& 0.00000	0.01859 &	0.01859	0.01577	0.01577

$\alpha = 2.6$		$\alpha = 2.8$		$\alpha = 3.0$	
Approx Sol	**Error**	**Approx Sol**	**Error**	**Approx Sol**	**Error**
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.159901	0.00011	-0.15995	0.00005	-0.15997	0.00003
-0.23898	0.00102	-0.23935	0.00064	-0.23959	0.00041
-0.236383	0.00361	-0.23748	0.00251	-0.23827	0.001723
-0.151989	0.00801	-0.15399	0.00601	-0.15556	0.00444
0.0130281	0.01303	0.01052	0.01052	0.00834	0.00834

Table: 4.4 Error Estimation for the solution of (4.2) for υ & $\mathcal{Y}(\upsilon)$ for $\tau = 3$

t	Exact Sol	$\alpha = 2.2$		$\alpha = 2.4$	
		Approx Sol	**Error**	**Approx Sol**	**Error**
0.0 &	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	-0.16000	-0.15956	0.00043	-0.15976	0.00024
0.4	-0.24000	-0.23658	0.00342	-0.23779	0.00221
0.6	-0.24000	-0.22962	0.01038	-0.2326	0.00737
0.8	-0.16000	-0.13927 &	0.02073	-0.14413	0.01587
1.0	0.00000	0.03145	0.03145	0.02594	0.02594

$\alpha = 2.6$		$\alpha = 2.8$		$\alpha = 3.0$	
Approx Sol	**Error**	**Approx Sol**	**Error**	**Approx Sol**	**Error**
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.15986	0.00013	0.15993	0.00007	-0.15993	0.00007
-0.23859	0.00141	-0.23911	0.00089	-0.23911	0.00089
-0.23483	0.00516	-0.23642	0.00357	-0.23642	0.00357
-0.14803	0.01197	-0.15111	0.00889	-0.15111	0.00889
0.02095	0.02095	0.01661	0.00890	0.01661	0.01661

Table 4.1, 4.2, 4.3 and 4.4 shows the numerical values of the exact and approximate solutions of problems (4.1) and (4.2) for the delay term $\tau = 2$ and $\tau = 3$. Moreover, we estimated the errors in these tables.

5 Conclusion

This study marks a significant breakthrough in extending fixed point theory within the framework of partially ordered Banach spaces. We have established a comprehensive Darbo-type fixed point theorem by employing the generalized operators presented in [18]. This theorem encompasses a broad spectrum of fixed-point results and their subsequent implications, forming an extensive set of theorems.

Our findings have been effectively applied to higher-order fractional delay differential equations, underscoring the practical significance of this research. Within the realm of mathematical analysis and its applications, this work contributes to both the practical and the theoretical advancement of fixed point theory in partially ordered Banach spaces.

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