

Solution of Higher Order Fractional Delay Differential Equations Using Darbo Type Fixed Point Theorem

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Abstract

In this work, we combine measure non-compactness and generalised operators in the setting of partial order Banach spaces to provide some generalised Darbo-type fixed point theorems. We further demonstrate how our findings could potentially applied in reality by establishing that higher-order fractional delay differential equations have solutions. We back up our results with numerical estimations based on a realistic scenario. This work explores fixed point theory in the context of partial order Banach spaces and shows how it can possibly used in real-world situations through realistic examples and facts.

Keywords: Darbo fixed point theorem, Fractional Dealy Differential Equation, Adomian Decomposition Method.

MSC: 47H08, 47H09, 47H10

1 Introduction

1.1 Partially Ordered Spaces

When we combine a complete normed vector space and a partial order relation, we get a sophisticated mathematical structure called a partially ordered Banach space. This framework, which visually conforms to the fundamental vector space operations, allows for the identifying

comparison of designated items inside the space via a "greater than" or "less than" connection. More specifically, we have a field of scalars, which might be real or complex numbers, and a Banach space, X , built on top of it. Put X under a partial order \leq . Certain criteria must be satisfied before this partial order \leq could possibly be considered compatible with the vector space structure and norm: If $x, y, z, w \in E$ and $\lambda \in \mathbb{R}$ are valid, the relation \leq on E is termed partial order. Let E represent any real linear vector space.

1. (Addition to Preserve Order) For any element z in X , if $x \leq y$, then $x + z \leq y + z$ holds true.
2. (Order-Preserving Scalar Multiplication) It follows that $\alpha x \leq \alpha y$ given $x \leq y$ and a positive scalar α .
3. (Positive Cone) All $x \in X$ for which $x \geq 0$ make up the positive cone, represented by the symbol X_+ . This cone must have the characteristics of being pointed, closed, and convex.

This framework enables the detailed examination of ordered relationships by providing a structured approach to analyze ordered systems within normed vector spaces. The partial order relation \leq is defined on the real linear space E . For any two elements $x, y \in E$, they are said to be comparable if either $x \leq y$ or $y \leq x$. A partially ordered normed space is a partially ordered set E equipped with a norm $\|\cdot\|$.

A normed linear space E is called complete if the metric d induced by the norm $\|\cdot\|$ is defined on all of E . If a nonincreasing (or nondecreasing) sequence $\{x_n\}$ satisfies $x_n \leq x^*$ (or $x_n \geq x^*$) for every $n \in \mathbb{N}$ and converges to x^* , the space E is termed regular.

Banas and Goebel, through their work in nonlinear analysis and its applications, introduced the following concepts:

If $T(E)$ is a relatively compact subset of E , then an operator T mapping E onto itself is said to be compact. Similarly, if $T(S)$ is a relatively compact subset of E for every bounded subset S of E , the operator T is considered totally bounded. An operator T is termed completely continuous if it is both continuous and totally bounded on E .

The set \mathbb{R} of real numbers with the norm defined by the absolute value function and the usual order relation \leq clearly demonstrates this characteristic. Similar to this, every partially compact subset of the space $C(J, \mathbb{R})$ with the usual standard supremum norm $\|\cdot\|$ determined by $\|x(t)\| = \sup_{t \in J} |x(t)|$ is compatible with the conventional order relation known by $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in J$. To get fresh conclusions, we substitute the cost of monotone and boundedness for the operator's bounds, closure, and convexity conditions.

The mathematical expression of $C = \{v \in E : v \geq 0\}$ is provided by taking the Banach space E with norm $\|\cdot\|$ with a positive cone. $(E, \|\cdot\|)$ is a partially ordered Banach space; let \sqsubseteq be the ordered relation induced by cone C .

1.2 Delay Differential Equation

Delay differential equations have diverse applications across disciplines like chemistry, biology, engineering, economics, and physics. For example, they describe neural communication in the brain, where signal transmission experiences temporal delays [7]. In ecology, DDEs model species growth influenced by reproductive and migratory delays. In chemistry, they analyze reactions where transit times between reaction sites affect outcomes [9]. Engineering applications include modeling feedback loops and delays in signal processing [21, 16].

To address higher-order DDEs, we apply generalized fixed point theorems. For example, the fractional delay differential equation:

$$D^\alpha Y(t) = G(t, Y(t), Y(\beta(t))), \tag{1.1}$$

where $t \in [0, 1]$, $n - 1 < \alpha \leq n$, with initial conditions $Y^r(0) = d_r$ for $r = 0, 1, \dots, n - 1$. Here, $Y(t)$ is the dependent variable, G represents the governing function, and $\beta(t)$ denotes the delayed argument.

These generalizations demonstrate the existence of solutions for complex DDEs. By leveraging advanced fixed point theorems in partially ordered Banach spaces, this framework expands the utility of fixed point theory in solving intricate mathematical problems associated with delay equations.

1.3 Some concepts of fractional derivative and integral

First, we review the basic concepts of fractional integrals and derivatives along with their properties. Caputo’s definition, which is extensively applied in various branches of applied mathematics, is also incorporated here [22, 23].

Definition 1.1. A real-valued function $R(y)$, for $y > 0$, is said to belong to the space D_α if $\alpha \in \mathbb{R}$, and there exists a real number $\beta > \alpha$, such that $R(y) = y^\beta R_1(y)$, where $R_1(y) \in D[0, \infty]$. Furthermore, it is in the space \mathcal{D}^m if $R^m \in D_\alpha$ for $m \in \mathbb{N} \cup \{0\}$.

Definition 1.2. The Riemann-Liouville fractional integral operator of order $\kappa \geq 0$, for a function $R \in D_\alpha$, $\alpha \geq -1$, is defined as:

$$J_y^\kappa R(y) = \frac{1}{\Gamma(\kappa)} \int_0^y (y - \xi)^{\kappa-1} R(\xi) d\xi, \quad \kappa > 0, y > 0,$$

and

$$J_y^0 R(y) = R(y).$$

The following properties of the operator J^κ are established in [20]. For $R(y) \in D_\alpha$, $\alpha \geq -1$, $\kappa, \lambda \geq 0$, and $\mu > -1$, we have:

- i) $J_y^\kappa J_y^\lambda R(y) = J_y^{\kappa+\lambda} R(y).$
- ii) $J_y^\kappa J_y^\lambda R(y) = J_y^\lambda J_y^\kappa R(y).$
- iii) $J_y^\kappa y^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\kappa+\mu+1)} y^{\kappa+\mu}.$

To address certain limitations of the Riemann-Liouville derivative, a modified fractional derivative, known as Caputo’s derivative, is introduced [22].

Definition 1.3. *In the sense of Caputo’s derivative, the fractional derivative of $R(y)$ is defined as:*

$$D_y^\kappa R(y) = J_{y, m-\kappa} D_y^m R(y) = \frac{1}{\Gamma(m-\kappa)} \int_0^y (y-\xi)^{m-\kappa-1} R_m(\xi) d\xi,$$

for $m - 1 < \kappa \leq m, m \in \mathbb{N}, y > 0, R \in D_y^m$, and $D_y^\kappa P = 0$.

Definition 1.4. *The Caputo time-fractional derivative operator of order $\kappa > 0$ is defined for any smallest integer $m > \kappa$ as:*

$$D_t^\kappa V(y, t) = \frac{\partial^\kappa V(y, t)}{\partial t^\kappa} = \begin{cases} \frac{1}{\Gamma(m-\kappa)} \int_0^t (t-s)^{m-\kappa-1} \frac{\partial^m V(y,s)}{\partial s^m} ds, & m - 1 < \kappa < m, \\ \frac{\partial^m V(y,t)}{\partial t^m}, & \kappa = m. \end{cases}$$

The following lemma is helpful for solving problems involving these operators.

Lemma 1.1. *If $m - 1 < \kappa < m, m \in \mathbb{N}$, and $R \in D_y^m, \alpha \geq -1$, then:*

$$D_y^\kappa J_y^\kappa R(y) = R(y),$$

and

$$J_y^\kappa D_y^\kappa R(y) = R(y) - \sum_{p=0}^{m-1} R^p(0^+) \frac{y^p}{p!}, \quad y > 0.$$

Definition 1.5. *The Mittag-Leffler function, denoted by $M_\kappa(x)$, is defined as:*

$$M_\kappa(x) = \sum_{j=0}^\infty \frac{x^j}{\Gamma(1 + \kappa j)}, \quad \kappa \in \mathbb{C}, \text{Re}(\kappa) > 0, x \in \mathbb{C}.$$

If $\kappa = 1$, the Mittag-Leffler function reduces to the exponential function:

$$\sum_{j=0}^\infty \frac{x^j}{\Gamma(1 + j)} = \sum_{j=0}^\infty \frac{x^j}{j!}.$$

Lemma 1.2. For $\kappa > 0$, the general solution to the homogeneous equation $D_{0^+}^\kappa \psi(y) = 0$ is given by:

$$\psi(y) = c_0 + c_1y + c_2y^2 + c_3y^3 + \dots + c_{m-1}y^{m-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, m - 1$, and $m = \lfloor \kappa \rfloor + 1$.

Lemma 1.3. For $\kappa > 0$, we have:

$$J_{0^+}^\kappa D_{0^+}^\kappa \psi(y) = \psi(y) + c_0 + c_1y + c_2y^2 + c_3y^3 + \dots + c_{m-1}y^{m-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, m - 1$, and $m = \lfloor \kappa \rfloor + 1$.

2 Fixed point theorems

The term "measure of non-compactness" is a popular mathematical approach for describing the extent to which a set, operator, or bounded set lacking compactness. This idea is especially relevant to fixed-point theorems, stability studies and partially ordered Banach spaces for nonlinear equations [17, 13, 1, 18, 8, 19]. Kuratowski introduced the concept of non-compactness measures in [13].

Definition 2.1. If E is a Banach space, then Kuratowski's MNC for $A \in E$ is the map $\alpha : M(E) \rightarrow \mathbb{R}_+$.

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ finite number of sets with diameter } < \epsilon, \text{ can cover it}\}.$$

The following formulation recalls the concept of MNC that was axiomatically provided in [13].

Definition 2.2. If α is represented by the symbol Ξ , then following characteristics are true:

1. $\ker \Xi = \{C \in M(E) | \Xi(C) = 0\} \neq \emptyset$ and $\ker \Xi \in N(E)$.
2. If $C \subseteq D$ then $\Xi(C) \leq \Xi(D)$ for all $C, D \in E$.
3. $\Xi(\overline{\text{conv}(C)}) = \Xi(\text{conv}(C))$ for all $C \in E$, where \bar{C} denotes the closure of C .
4. $\Xi(C \cup D) = \max\{\Xi(C), \Xi(D)\}$ for all $C, D \in E$.
5. $\Xi(\lambda_1 C + \lambda_2 D) \leq \lambda_1 \Xi(C) + \lambda_2 \Xi(D)$ if $\lambda_1 + \lambda_2 = 1$ and $\lambda_1, \lambda_2 \geq 0$ for all $C, D \in E$.
6. If C_n is decreasing sequence of nonempty closed, bounded subset of E and $\lim_{n \rightarrow \infty} \Xi(C_n) = 0$, then $C_\infty = \bigcap_{n \geq 1} C_n$ is nonempty compact.

Definition 2.3. [6] Let A be a nonempty, convex, and closed set in Banach space E . Let $Y : A \rightarrow A$ be a continuous map such that, given a nonempty set $S \subseteq A$, where Ξ is a MNC, $k \in [0, 1)$ with

$$\Xi(Y(S)) \leq k\Xi(S).$$

Then, at least one fixed point in A is admitted by Y .

To expand the finding of [19] in a partly ordered Banach space, we now review the definition below. The consequence of [19] is further refined in a partially ordered Banach space by recalling the definition below.

Theorem 2.1. Consider $\langle E, \|\cdot\|, \Xi \rangle$ be the partially ordered Banach space having a positive cone C is normal. Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing and continuous function and that $\mathfrak{S} : E \rightarrow E$ is a monotonic increasing and continuous operator. There is also $0 < \eta(p, q) < 1$ such that for $0 < p < q < \infty$ with

$$p \leq \Delta(Y) + \varphi(\Delta(Y)) \leq q \Rightarrow \Delta(\mathfrak{S}Y) + \varphi(\Delta(\mathfrak{S}Y)) \leq \bar{\eta}(p, q) \Delta(Y) + \varphi(\Delta(Y)), \tag{2.1}$$

for all non-empty subset Y of E , where Δ being essentially the measure of noncompactness. Then \mathfrak{S} has a fixed point v^* and $\xi_0 \Xi \mathfrak{S}\xi_0$ for $\xi_0 \in E$, with the sequence $\langle \mathfrak{S}^n \xi_0 \rangle$ converges monotonically to v^* .

The consequence that follows may be obtained if we recognise the nondecreasing and continuous function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as $\varphi(t) \equiv 0$ in the theorem (2.1).

Proposition 2.1. Consider $\langle E, \|\cdot\|, \Xi \rangle$ be the partially ordered Banach space having a positive cone C is normal. Let $\mathfrak{S} : E \rightarrow E$ is a monotonic increasing and continuous operator. If there is $0 < \eta(p, q) < 1$ for $0 < p < q < \infty$ with

$$p \leq \Delta(Y) \leq q \Rightarrow \Delta(\mathfrak{S}Y) \leq \eta(p, q) \Delta(Y), \tag{2.2}$$

for all non-empty subset Y of E , where Δ being essentially the measure of noncompactness. Then \mathfrak{S} has a fixed point v^* and $\xi_0 \Xi \mathfrak{S}\xi_0$ for $\xi_0 \in E$, with the sequence $\langle \mathfrak{S}^n \xi_0 \rangle$ converges monotonically to v^* .

If we take $\Delta(X) = \text{diam}(X)$ in Proposition 1.1 we get next result.

Proposition 2.2. Consider $\langle E, \|\cdot\|, \Xi \rangle$ be the partially ordered Banach space having a positive cone C is normal. Let $\mathfrak{S} : E \rightarrow E$ is a monotonic increasing and continuous operator. If there is $0 < \eta(p, q) < 1$ for $0 < p < q < \infty$ with

$$p \leq \text{diam}(Y) \leq q \Rightarrow \text{diam}(\mathfrak{S}Y) \leq \eta(p, q) (\text{diam}(Y)), \tag{2.3}$$

for all non-empty subset Y of E . Then \mathfrak{S} has a fixed point v^* and $\xi_0 \Xi \mathfrak{S}\xi_0$ for $\xi_0 \in E$, with the sequence $\langle \mathfrak{S}^n \xi_0 \rangle$ converges monotonically to v^* .

Proof. Here is a \mathfrak{S} -invariant nonempty convex closed subset B and $diam(B_\infty) = 0$, with respect to Propositions (2.1) and 3.2 [12]. This indicates that B_∞ is a singleton set, therefore we have a fixed point of $\mathfrak{S} \neq \phi$.

We assume that there are two different fixed points in order to demonstrate uniqueness. We may define the set $X = \{\iota, \kappa\}$ after $\iota, \kappa \in Y$. In this instance,

$$diam(X) = diam(\mathfrak{S}X) = \|\iota - \kappa\| > 0.$$

Then using (2.4) we get

$$p \leq diam(Y) \leq q \Rightarrow diam(\mathfrak{S}Y) \leq \eta^-(p, q)(diam(Y)).$$

Since $diam(X) = diam(\mathfrak{S}X)$ results to contradiction to the property of function $\eta(p, q) < 1$ and hence $\iota = \kappa$. □

Proposition 2.3. Consider $(E, \|\cdot\|, \sqsubseteq)$ be the partially ordered Banach space having a positive cone C is normal. Let $\mathfrak{S} : E \rightarrow E$ is a monotonic increasing and continuous operator. If there is $0 < \eta(p, q) < 1$ for $0 < p < q < \infty$ with

$$p \leq \|\vartheta_1 - \vartheta_2\| \leq q \Rightarrow \|\mathfrak{S}\vartheta_1 - \mathfrak{S}\vartheta_2\| \leq \eta(p, q)(\|\vartheta_1 - \vartheta_2\|), \tag{2.4}$$

for all $\vartheta_1, \vartheta_2 \in E$. If $\xi_0 \sqsubseteq \mathfrak{S}\xi_0$ for $\xi_0 \in E$ then \mathfrak{S} has unique fixed point v^* and sequence $\langle \mathfrak{S}^n \xi_0 \rangle$ converges monotonically to v^* .

Proof. We consider $\Delta : M \rightarrow R_+$ by the rule $\Delta(X) = diam(X)$, where

$$diam(X) = \sup \{\|\vartheta_1 - \vartheta_2\|, \vartheta_1, \vartheta_2 \in D\},$$

specifies the set X 's diameter. In the sense of Definition 2.2, it is obvious from this formulation that Δ is a MNC. Now, by the virtue of equation (2.4). If

$$p \leq \sup_{\vartheta_1, \vartheta_2 \in X} [\|\vartheta_1 - \vartheta_2\|] \leq q,$$

then

$$\begin{aligned} \sup_{\vartheta_1, \vartheta_2 \in X} [\|\mathfrak{S}\vartheta_1 - \mathfrak{S}\vartheta_2\|] &\leq \sup_{\vartheta_1, \vartheta_2 \in X} \|\mathfrak{S}\vartheta_1 - \mathfrak{S}\vartheta_2\| \\ &\leq \eta(p, q) \sup_{\vartheta_1, \vartheta_2 \in X} [\|\vartheta_1 - \vartheta_2\|] \\ &\leq \eta(p, q) \sup_{\vartheta_1, \vartheta_2 \in X} \|\vartheta_1 - \vartheta_2\|. \end{aligned} \tag{2.5}$$

Therefore, applying the concept of set diameter, we obtain

$$p \leq diam(X) \leq q \Rightarrow diam(\mathfrak{S}X) \leq \eta(p, q)(diam(X)).$$

Therefore, from the viewpoint of Proposition 2.2, \mathfrak{S} has a fixed point. □

3 Solution of Delay Differential Equation

The existence of a solution to the non-homogenous higher order delay differential equation (??) in the space of bounded and continuous functions defined on the interval $[0, \delta]$, represented by $BC[0, \delta]$, is examined in this section. In order to examine the problem (1.1), we must take into account the following presumptions:

1. The non-negative real number λ and the monotonic increasing functions $G : [0, \delta] \times BC[0, \delta] \times BC[0, \delta] \rightarrow \mathbb{R}_+$ possess the following inequality:

$$|G(u, Y_1(u), Y_1(\beta(u))) - G(u, Y_2(u), Y_2(\beta(u)))| \leq \lambda_1 |Y_1(u) - Y_2(u)| + \lambda_2 |Y_1(\beta(u)) - Y_2(\beta(u))|.$$

2. There is real number M_G^* which is not negative with

$$M_G^* = \max \{G(u, 0, 0, \dots, 0); u \in [0, 1]\}.$$

3. There exist a positive real number ρ , with inequality

$$\sum_{r=1}^{n-1} \frac{\partial Y(0^+) \eta^r}{\partial Y^r r!} + \frac{((\lambda_1 + \lambda_2)\rho + M_G^*) \eta^\alpha}{\Gamma(\alpha + 1)} \leq \rho.$$

Theorem 3.1. Under the assumptions (1) – (3) the delay differential equation (1.1) admits at least one solution $Y \in BC([0, \delta])$ provided $(\lambda_1 + \lambda_2) \delta^\alpha < \Gamma(\alpha + 1)$.

Proof. It needs to be notice that the space $\langle BC[0, \delta], \|\cdot\|, \sqsubseteq \rangle$ is a partially order Banach space with a normal positive cone $C = \{u \in E : u \geq 0\}$. The closed and bounded ball $B_\rho = \{\vartheta \in [0, \delta] : \|\vartheta\| \leq \rho\}$ should also be taken into account. Next, define the mapping $Q : BC[0, \delta] \times BC[0, \delta] \rightarrow \mathbb{R}$. By using the inversed differential operator on (1.1), we obtain

$$Q(Y(u)) = \sum_{r=1}^{n-1} \frac{\partial Y(0^+) u^r}{\partial Y^r r!} + I^\alpha (G(u, Y(u), Y(\beta(u)))) \tag{3.1}$$

We prove the operator Q is bounded. Consider $\nu \in [0, \delta]$ then we have

$$\begin{aligned}
 |Q(Y(\nu))| &\leq \sum_{r=1}^{\infty} \frac{\partial^r Y(0^+) \nu^r}{\partial Y^r r!} + |I^\alpha (G(\nu, Y(\nu), Y(\beta(\nu))))| \\
 &\leq \sum_{r=1}^{\infty} \frac{\partial^r Y(0^+) \nu^r}{\partial Y^r r!} + |I^\alpha (G(\nu, Y(\nu), Y(\beta(\nu))) - G(\nu, 0, 0) + G(\nu, 0, 0))| \\
 &\leq \sum_{r=1}^{\infty} \frac{\partial^r Y(0^+) \nu^r}{\partial Y^r r!} + |I^\alpha (|G(\nu, Y(\nu), Y(\beta(\nu))) - G(\nu, 0, 0)| + |G(\nu, 0, 0)|)| \\
 &\leq \sum_{r=1}^{\infty} \frac{\partial^r Y(0^+) \nu^r}{\partial Y^r r!} + I^\alpha \lambda_1 |Y(\nu)| + \lambda_2 |Y(\beta(\nu))| + M_G^* \\
 &\leq \sum_{r=1}^{\infty} \frac{\partial^r Y(0^+) \nu^r}{\partial Y^r r!} + \frac{(\lambda_1 + \lambda_2) \|Y\| + M_G^* \eta^\alpha}{\Gamma(\alpha + 1)},
 \end{aligned} \tag{3.2}$$

where $M^* = \sup\{\chi(\vartheta, 0, 0, 0, 0) : \vartheta \in [0, \delta]\}$. Next we prove Q satisfies the assumptions of Proposition 2.3. Since, the function G is monotonic increasing this leads to Q being also a non-decreasing and continuous mapping on $BC[0, \delta]$.

Now, from the equation (3.2) and assumption (3) we conformed that Q non-decreasing and continuous mapping on B_ρ .

Next, we prove that Q satisfies the inequality (2.4). Consider $X, Y \in B_\rho$, we obtain

$$\begin{aligned}
 |Q(X(\nu)) - Q(Y(\nu))| &\leq \sum_{r=1}^{\infty} \frac{\partial^r X(0^+) \nu^r}{\partial X^r r!} + I^\alpha (G(\nu, X(\nu), X(\beta(\nu)))) \\
 &\quad - \sum_{r=1}^{\infty} \frac{\partial^r Y(0^+) \nu^r}{\partial Y^r r!} + I^\alpha (G(\nu, Y(\nu), Y(\beta(\nu)))) \\
 &\leq I^\alpha (|G(\nu, X(\nu), X(\beta(\nu))) - G(\nu, Y(\nu), Y(\beta(\nu))))|) \\
 &\leq I^\alpha (\lambda_1 |X(\nu) - Y(\nu)| + \lambda_2 |X(\beta(\nu)) - Y(\beta(\nu))|).
 \end{aligned} \tag{3.3}$$

This inequality is true for all $\nu \in [0, \delta]$. Therefore we obtain

$$\begin{aligned}
 \sup_{\nu \in [0, \delta]} |Q(X(\nu)) - Q(Y(\nu))| &\leq I^\alpha \left(\lambda_1 \sup_{\nu \in [0, \delta]} |X(\nu) - Y(\nu)| \right. \\
 &\quad \left. + \lambda_2 \sup_{\nu \in [0, \delta]} |X(\beta(\nu)) - Y(\beta(\nu))| \right).
 \end{aligned} \tag{3.4}$$

Hence

$$\|QX - QY\| \leq \eta(p, q) \|X - Y\|. \tag{3.5}$$

where $\eta(p, q) = \frac{(\lambda_1 + \lambda_2)\delta^\alpha}{\Gamma(\alpha + 1)}$. All the conditions of proposition 2.3 are satisfied. Hence the operator Q has a fixed point which is the solution of (1.1). □

4 Numerical Illustrations

Example 4.1.

$$D^\alpha Y(u) = -Y(u) - Y(u - 0.3) + e^{-u+0.3}, \quad 2 < \alpha \leq 3 \tag{4.1}$$

with $Y(0) = 1, Y^{(1)}(0) = -1, Y^{(2)}(0) = 1$ and $Y(u) = e^{-u}, u \leq 0$.

If we take

$$G(u, Y(u), Y(\beta(u))) = -Y(u) - Y(u - 0.3) + e^{-u+0.3},$$

in equation (1.1) we get equation (4.1).

Also,

$$\begin{aligned} &|G(u, Y_1(u), Y_1(\beta(u))) - G(u, Y_2(u), Y_2(\beta(u)))| \\ &\leq |-Y_1(u) - Y_1(u - 0.3) + Y_2(u) + Y_2(u - 0.3)| \\ &\leq |Y_1(u) - Y_2(u)| + |Y_1(u - 0.3) - Y_2(u - 0.3)| \\ &\leq \lambda_1 |Y_1(u) - Y_2(u)| + \lambda_2 |Y_1(u - 0.3) - Y_2(u - 0.3)|, \end{aligned}$$

where $\lambda_1 = \lambda_2 = 1$.

Further

$$\begin{aligned} \sum_{r=1}^{\infty} \frac{\partial^r Y(0^+) u^r}{\partial Y^r} + \frac{(\lambda_1 + \lambda_2)\rho + M_G^* \eta^\alpha}{\Gamma(\alpha + 1)} &= 1 - 1 + \frac{1}{2} + \frac{2\rho}{\Gamma(2.5 + 1)} \\ &= \frac{1}{2} + \frac{2\rho}{\Gamma(2.5 + 1)}. \end{aligned}$$

Now in the view of inequality (3.5) we have

$$\frac{1}{2} + \frac{2\rho}{\Gamma(2.5 + 1)} \leq \rho \Rightarrow \frac{1}{2} \leq \rho \left(1 - \frac{2}{\Gamma(2.5 + 1)}\right).$$

Hence we get

$$\frac{\Gamma(2.5 + 1)}{2\Gamma(2.5 + 1) - 2} \leq \rho.$$

Moreover,

$$\frac{(\lambda_1 + \lambda_2)\delta^\alpha}{\Gamma(\alpha + 1)} = \frac{2}{\Gamma(\alpha + 1)} < 1 \text{ for } 2 < \alpha \leq 3.$$

All the conditions of Theorem 3.1 are satisfied, hence equation (4.1) has exactly one solution.

Numerical Estimation

The solution of differential equation is obtained by method introduced in [20] as follows,

$$Y(u) = Y(0) + uY'(0) + \frac{u^2}{2}Y''(0) + I_u^\alpha [-Y(u) - Y(u - 0.3) + e^{-u+0.3}] .$$

Applying the initial conditions we get,

$$Y(u) = Y(0) + uY'(0) + \frac{u^2}{2}Y''(0) + I_u^\alpha [-Y(u) - Y(u - 0.3) + e^{-u+0.3}] = 1 - u + \frac{u^2}{2} + I_u^\alpha e^{-\frac{u}{\tau}} + I_u^\alpha [-Y(u) - Y(u - \frac{u}{\tau})] .$$

Plot of the approximate and exact solution of (4.1) for u and $Y(u)$.

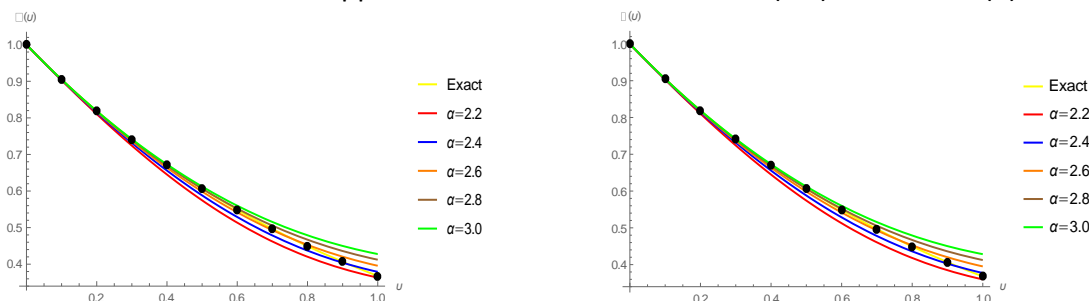


Figure 1: $\tau = 2$ & $\alpha = 2.2, 2.4, 2.6, 2.8, 3.0$

Figure 2: $\tau = 3$ & $\alpha = 2.2, 2.4, 2.6, 2.8, 3.0$

Error Estimation for solution of (4.1) for u and $Y(u)$ for $\tau = 2$.					
$\alpha = 2.2$			$\alpha = 2.4$		
t	Exact	Approx	Error	Approx	Error
0.0	1.0000	1.0000	0.0000	1.0000	0.0000
0.2	0.818731	0.810334	0.008397	0.814268	0.004463
0.4	0.670320	0.645015	0.025305	0.655771	0.014549
0.6	0.548812	0.513409	0.035403	0.528865	0.019946
0.8	0.449329	0.420269	0.029059	0.437045	0.012284
0.10	0.367879	0.363394	0.004485	0.379058	0.011179
$\alpha = 2.2$			$\alpha = 2.4$		
t	Exact	Approx	Abs. Error	Approx	Abs. Error
0.0	1.0000	1.0000	0.0000	1.0000	0.0000
0.2	0.818731	0.81032	0.008411	0.814264	0.004467
0.4	0.670320	0.64478	0.025540	0.655681	0.014639
0.6	0.548812	0.512416	0.036395	0.528422	0.020389
0.8	0.449329	0.417978	0.031351	0.435919	.013409
0.10	0.367879	0.359838	0.008042	0.377237	0.009357

$\alpha = 2.6$		$\alpha = 2.8$		$\alpha = 3.0$	
Approx	Error	Approx	Error	Approx	Error
1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
0.816656	0.002075	0.818077	0.000654	0.818908	0.000177
0.663568	0.006752	0.669054	0.001266	0.672821	0.002501
0.541703	0.007109	0.551934	0.003122	0.559816	0.011004
0.452820	0.003491	0.466895	0.008858	0.478919	0.029589
0.395687	0.027807	0.412200	0.044320	0.427757	0.059877

Table 1: Error Estimation for solution of (4.1) for u and $Y(u)$ for $\tau = 2$.

$\alpha = 2.6$		$\alpha = 2.8$		$\alpha = 3.0$	
Approx	Abs. Error	Approx	Abs. Error	Approx	Abs. Error
1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
0.816655	0.002075	0.818077	0.000654	0.818908	0.000177
0.663539	0.006752	0.669048	0.001266	0.672823	0.002501
0.541544	0.007109	0.551906	0.003122	0.559839	0.011004
0.452412	0.003491	0.466874	0.008858	0.479073	0.029589
0.395125	0.027807	0.412425	0.044320	0.428391	0.059877

Table 2: Error Estimation for solution of (4.1) for u and $Y(u)$ for $\tau = 3$.

Example 4.2.

$$D^\alpha (Y(u)) = Y \frac{u}{\tau} - u^2 + u, \tag{4.2}$$

with initial conditions

$$Y(0) = 0, Y'(0) = -\frac{1}{2}, \text{ \& } Y''(0) = 1,$$

and exact solution

$$y(u) = -\frac{1}{2}(u^2 - u).$$

The assumptions of Theorem 3.1 can be easily verified, hence the differential equation (4.2) has solution. This solution can be obtained by the numerical scheme introduced in [20].

$$Y(u) = u + u^2 + J^\alpha (-u^2 + u) + J^\alpha Y \frac{u}{\tau}$$

$$Y_0 = -\frac{u}{2} + u^2 + I_u^\alpha (-u^2 + u)$$

$$Y_{i+1} = I_u^\alpha Y_i \frac{u}{\tau} + Y_i(u) .$$

Plot of the approximate and exact solution of (4.2) for u and $Y(u)$.

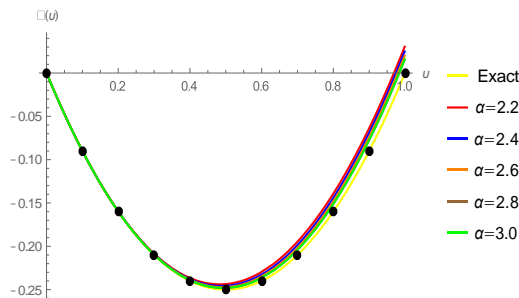
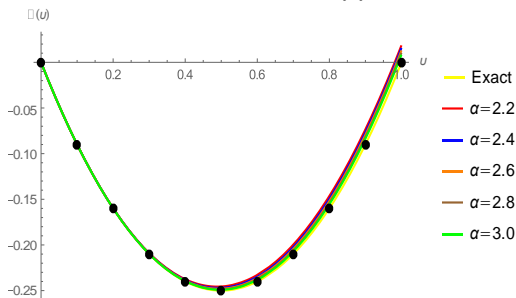


Figure 3: $\tau = 2$ & $\alpha = 2.2, 2.4, 2.6, 2.8, 2.0$

Figure 4: $\tau = 3$ & $\alpha = 2.2, 2.4, 2.6, 2.8, 2.0$

		$\alpha = 2.2$		$\alpha = 2.4$	
u	Exact	Approx	Error	Approx	Error
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	-0.16000	-0.15968	0.00032	-0.15982	0.00018
0.4	-0.24000	-0.23754	0.00245	-0.23841	0.00159
0.6	-0.24000	-0.23280	0.00719	-0.23486	0.00514
0.8	-0.16000	-0.146412	0.01359	-0.14948	0.01052
0.10	0.00000	0.01859	0.01859	0.01577	0.01577
		$\alpha = 2.6$		$\alpha = 2.8$	
Approx	Error	Approx	Error	Approx	Error
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.159901	0.00011	-0.15995	0.00005	-0.15997	0.00003
-0.23898	0.00102	-0.23935	0.00064	-0.23959	0.00041
-0.236383	0.00361	-0.23748	0.00251	-0.23827	0.001723
-0.151989	0.00801	-0.15399	0.00601	-0.15556	0.00444
0.0130281	0.01303	0.01052	0.01052	0.00834	0.00834

Table 3: Absolute Error of (4.2) for $u \in [0, 1]$ for $\tau = 2$ by ADM.

		$\alpha = 2.2$		$\alpha = 2.4$	
u	Exact	Approx	Abs. Error	Approx	Abs. Error
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	-0.16000	-0.15956	0.00043	-0.15976	0.00024
0.4	-0.24000	-0.23658	0.00342	-0.23779	0.00221
0.6	-0.24000	-0.22962	0.01038	-0.23263	0.00737
0.8	-0.16000	-0.13927	0.02073	-0.14413	0.01587
0.10	0.00000	0.03145	0.03145	0.02594	0.02594

$\alpha = 2.6$		$\alpha = 2.8$		$\alpha = 3.0$	
Approx	Abs. Error	Approx	Abs. Error	Approx	Abs. Error
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.15986	0.00013	-0.15993	0.00007	-0.15993	0.00007
-0.23859	0.00141	-0.23911	0.00089	-0.23911	0.00089
-0.23483	0.00516	-0.23642	0.00357	-0.23642	0.00357
-0.14803	0.01197	-0.15111	0.00889	-0.15111	0.00889
0.02095	0.02095	0.01661	0.00890	0.01661	0.01661

Table 4: Absolute Error of (4.2) for $u \in [0, 1]$ for $\tau = 3$ by ADM.

Table 1 to 6 shows the numerical values of the exact and approximate solutions of problems (4.1) and (4.2) for the delay term $\tau = 2$ and $\tau = 3$. Moreover we had estimated the error

5 Conclusion

This study marks a significant breakthrough in extending fixed point theory within the framework of partially ordered Banach spaces. By employing the generalized operators presented in [18], we have established a comprehensive Darbo-type fixed point theorem. This theorem encompasses a broad spectrum of fixed point results along with their subsequent implications, forming an extensive set of fixed point theorems.

Our findings have been effectively applied to higher-order fractional delay differential equations, underscoring the practical significance of this research. Within the realm of mathematical analysis and its applications, this work contributes to both the practical and the theoretical advancement of fixed point theory in partially ordered Banach spaces.

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References

[1] A. Das, B. Hazarika, & P. Kumam, Some new generalization of Darbo’s fixed point theorem and its application on integral equations. Mathematics, 7(3) (2019), 214, doi:[org/10.3390/math7030214](https://doi.org/10.3390/math7030214).

- [2] B. C. Dhage, Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations. *Tamkang Journal of Mathematics*, 45(4) (2014), 397-426, doi:[org/10.5556/j.tkjm.45.2014.1512](https://doi.org/10.5556/j.tkjm.45.2014.1512).
- [3] B. Ermentrout, & A. Mahajan, Simulating, analyzing, and animating dynamical systems: a guide to XPPAUT for researchers and students. *Appl. Mech. Rev.*, 56(4) (2003), B53-B53, doi:[org/10.1115/1.1579454](https://doi.org/10.1115/1.1579454).
- [4] D. Guo, V'. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Analysis, Theory, Methods and Applications*, 11 (5) (1987), 623-632, doi:[org/10.1016/0362-546X\(87\)90077-0](https://doi.org/10.1016/0362-546X(87)90077-0).
- [5] E. Keeler, & A. Meir, A theorem on contraction mappings. *J. Math. Anal. Appl.*, 28 (1969), 326-329, doi:[org/10.1016/0022-247X\(69\)90031-6](https://doi.org/10.1016/0022-247X(69)90031-6).
- [6] G. Darbo, Punti uniti in trasformazioni a codominio non compatto. *Rendiconti del Seminario matematico della Università di Padova*, 24 (1955), 84-92, http://dml.mathdoc.fr/item/RSMUP_1955_24_84_0/
- [7] G. P. Boswell, Modelling mycelial networks in structured environments. *Mycological research*, 112(9) (2008), 1015-1025, doi:[org/10.1016/j.mycres.2008.02.006](https://doi.org/10.1016/j.mycres.2008.02.006).
- [8] H. K. Nashine, R. W. Ibrahim, R. P. Agarwal, & N. H. Can, Existence of local fractional integral equation via a measure of non-compactness with monotone property on Banach spaces. *Advances in Difference Equations*, 1 (2020), 1-13, doi:[org/10.1186/s13662-020-03153-3](https://doi.org/10.1186/s13662-020-03153-3).
- [9] H.L.,Smith, Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems. *American Mathematical Soc* **41** (1995), 239-358,<https://api.semanticscholar.org/CorpusID:56671641>.
- [10] I. Altun, & D. Turkoglu, A Fixed Point theorem for mapping satisfying a general contractive condition of operator type. *Journal of Computational Analysis & Applications*, 9(1) (2007), 9-14, <https://avesis.gazi.edu.tr/activitycitation/index/1/35c9c75b-9b49-4eb7-9663-46852d02473b>.
- [11] J. Garcia-Falset, & K. Latrach, On Darbo-Sadovskii's fixed point theorems type for abstract measures of (weak) noncompactness. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 22(5) (2015), 797-812, doi:[org/10.36045/bbms/1450389249](https://doi.org/10.36045/bbms/1450389249).
- [12] J. K. Hale, Functional differential equations. In *Analytic Theory of Differential Equations: The Proceedings of the Conference at Western Michigan University, Kalamazoo*, 183 (2006), doi:[org/10.1007/BFb0060406](https://doi.org/10.1007/BFb0060406).

- [13] J. Banaś, On measures of noncompactness in Banach spaces. *Commentationes Mathematicae Universitatis Carolinae*, 21(1) (1980), 131-143, doi:[eudml.org/doc/17021](https://doi.org/eudml.org/doc/17021).
- [14] P. Krishnapriya, M. Pitchaimani, & T. M. Witten, Mathematical analysis of an influenza A epidemic model with discrete delay. *Journal of computational and Applied Mathematics*, 324 (2017), 155-172, doi:[org/10.1016/j.cam.2017.04.030](https://doi.org/10.1016/j.cam.2017.04.030).
- [15] R.R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, B. N. Sadovskii, The Fixed-Point Index of Condensing Operators. In: *Measures of Noncompactness and Condensing Operators. Operator Theory: Advances and Applications*, Birkhäuser, Basel, 55 (1992), doi:[org/10.1007/978-3-0348-5727-7_3](https://doi.org/10.1007/978-3-0348-5727-7_3).
- [16] S. I. Niculescu, C. E. de Souza, L. Dugard, & J. M. Dion, Robust exponential stability of uncertain systems with time-varying delays. *IEEE Transactions on Automatic Control*, 43(5) 1998, 743-748, doi:[10.1109/9.668851](https://doi.org/10.1109/9.668851).
- [17] T. Abdeljawad, E. Karapınar, & K. Taş, Existence and uniqueness of a common fixed point on partial metric spaces. *Applied Mathematics Letters*, 24(11) (2011), 1900-1904, doi:[org/10.1016/j.aml.2011.05.014](https://doi.org/10.1016/j.aml.2011.05.014).
- [18] V. Nikam, A.K. Shukla, & D. Gopal, Existence of a system of fractional order differential equations via generalized contraction mapping in partially ordered Banach space. *International Journal of Dynamics and Control*, 12 (2024), 125-135, doi:[org/10.1007/s40435-023-01245-y](https://doi.org/10.1007/s40435-023-01245-y).
- [19] V. Parvaneh, M. Khorshidi, M. De La Sen, H. Işık, & M. Mursaleen, Measure of noncompactness and a generalized Darbo's fixed point theorem and its applications to a system of integral equations. *Advances in Difference Equations*, 243 (2020) (1), 243, doi:[10.1186/s13662-020-02703-z](https://doi.org/10.1186/s13662-020-02703-z).
- [20] V. Daftardar-Gejji, & H. Jafari, Adomian decomposition: a tool for solving a system of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 301(2) (2005), 508-518, doi:[org/10.1016/j.jmaa.2004.07.039](https://doi.org/10.1016/j.jmaa.2004.07.039).
- [21] W. K. Ling, *Control of chaos in nonlinear circuits and systems*, World Scientific, 64 (2009), <http://www.worldscibooks.com/chaos/6717.html>.
- [22] Z. Odibat, & S. Momani, Numerical methods for nonlinear partial differential equations of fractional order. *Applied Mathematical Modelling*, 32(1) (2008), 28-39, doi:[org/10.1016/j.apm.2006.10.025](https://doi.org/10.1016/j.apm.2006.10.025).
- [23] Z. Odibat, & S. Momani, The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics. *Computers & Mathematics with Applications*, 58(11-12) (2009), 2199-2208, doi:[org/10.1016/j.camwa.2009.03.009](https://doi.org/10.1016/j.camwa.2009.03.009).