New Way For Extending Ideals of Ternary Semigroups to Fuzzy Setting

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Abstract. Point-wise definitions of fuzzy interior ideals and fuzzy quasi-ideals in the context of ternary semigroups are systematically derived. Their eqivalence with the set-theoretic formulations is also established. Fuzzy left (lateral, right)ideals, fuzzy ideals, fuzzy bi-ideals, fuzzy interior ideals, and fuzzy quasi-ideals are explored within ternary semigroups under the framework of Tom Head's metatheorem. It is demonstrated that the classes of ternary fuzzy subsemigroups, along with various fuzzy ideals, are projection closed. Additionally, alternative proofs for several results are furnished using metatheorem-based approaches, which eliminate the need for explicit calculations. These proofs are more concise and straightforward, offering calculation-free solutions. Furthermore, regular, intra-regular, and completely regular semigroups are characterized in terms of different types of fuzzy ideals.

Keywords. Ternary semigroups; fuzzy interior ideals; fuzzy quasi-ideals; metatheorem: projection closed.

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1. Introduction

The concept of ternary algebraic systems was initially introduced by Lehmer [8] in 1932. S. Banach describes a ternary semigroup as an algebraic framework ordinary where the ternary operation does not necessarily reduce to that of a semigroup. J. Los [10] demonstrated that any ternary semigroup can be embedded within an ordinary semigroup, such that the ternary operation in the former extends the binary operation of the later. Lehmer's research [8] delves deeper into specific triple systems, referred to as triplexes, which are characterized as commutative ternary groups. Sioson [18] contributed to the study of ideal theory within ternary semigroupsand provided definitions of ideals, introducing the notion of regular ternary semigroups and characterizing them through quasi- ideals. Santiago [16] further advanced the theory of ternary semigroups and semiheapswhlie Neumann in 1936 [12] explored the notion of regularity. Hewitt and Zuckerman [5] explored ternary operations within semigroups.

Dutta and Kar [3] developed the theory of regular ternary semigroups, while Kar and Sarkar [6, 7] extended this work by investigating fuzzy ideals in ternary semigroups. Shabir and Bano [17] defined prime bi-ideals in the context of ternary semigroups.

This study introduces point-wise formulations of fuzzy interior ideals and fuzzy quasi-ideals in the context of ternary semigroups. The results concerning various types of ideals in ternary semigroups are extended to the fuzzy domain through the application of a novel technique known as the 'metatheorem'. The metatheorem provides a structured framework for the extension of classical algebraic results to fuzzy systems. This methodological approach has demonstrated its capability in broadening algebraic structures into fuzzy contexts and has been extensively utilized in the investigation of semigroups and semirings in [13, 14]. Furthermore, we demonstrate that various classes of fuzzy ideals are projection closed. Moreover, the paper provides characterizations of several forms of regularity-namely, regular, intra-regular, and completely regular through the utilization of fuzzy ideals, including left, lateral, right, bi-ideals, interior ideals, quasi-ideals, and semiprime ideals in ternary semigroups via metatheorem approach.

2. Preliminaries

For the definition and basic results of ternary semigroup. We refer[1-3, 11, 17]. Throughout this paper. J will stands for [0,1). Also $C_{ss,}$ C_l $(C_{lt}, C_r, C_i, C_b, C_{in}, C_q, C_s)$ denotes the classes of crisp subsemigroups, left (lateral, right, two sided, bi-, interior ideals, quasi-, semiprime) ideals of a ternary semigroup S and $\mathcal{C}_{ss}$, $\mathcal{C}_l$ $(\mathcal{C}_{lt}, \mathcal{C}_r, \mathcal{C}_i, \mathcal{C}_b, \mathcal{C}_{in}, \mathcal{C}_q, \mathcal{C}_s)$ denotes their fuzzy classes respectively.

Lemma 2.1. [11] A non-empty subset A of a ternary semigroup is a

(i) subsemigroup (lateral Ideal) of $S \Leftrightarrow A^3 \subseteq A(SAS \subseteq A)$

(ii) left(right) ideal of S if $S \Leftrightarrow SSA \subseteq A(ASS \subseteq A)$

We recalled the definition of fuzzy set μ of a non-empty set X definedbyZadeh [19] as a mapping $\mu : X \rightarrow [0, 1]$. Let S be ternary semigroup and F(S) stands for the set of fuzzy subsets of X.

Definition 2.2. [6] $\mu \in F(S)$ is called a fuzzy ternary subsemigroup of a ternary semigroup S if $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} \ \forall x, y, z \in S.$

Definition 2.3. [6] $\mu \in F(S)$ is called a

(1) fuzzy left (lateral, right) ideal of S if $\mu(xyz) \geq \mu(z)$ $(\mu(xyz) \geq \mu(y),\ \mu(xyz) \geq \mu(z))$ $\forall\ x, y, z \in S.$

(2) fuzzy ideal of S $\mu(xyz) \geq \mu(z),\ \mu(xyz) \leq \mu(y)$ and $\mu(xyz) \geq \mu(x)\ \forall\ x, y, z \in S.$

(3) fuzzy semiprime ideal if $\mu(x) = \mu(x^3)\ \forall x \in S.$

Definition 2.4. [15,9, 7] $\mu \in F(S)$ is called a

(i) fuzzy bi-ideal of S if $\mu \in \mathcal{C}_{ss}$ and $\mu(x_1 s_1 x_2 s_2 x_3) \geq \min\{\mu(x_1), \mu(x_2), \mu(x_3)\}$ $\forall x_1, x_2, x_3, s_1, s_2 \in S$

(ii) fuzzy interior ideal of S if $\mu \in \mathcal{C}_{ss}$ and $S \circ \mu \circ S \circ \mu \circ S \subseteq \mu.$

(iii) fuzzy quasi-ideal of S if $\mu \circ S \circ S \cap (S \circ \mu \circ S \cup S \circ \mu \circ S \circ S) \cap S \circ S \circ \mu \subseteq \mu.$

(iv) fuzzy semiprime if $\mu(x) \geq \mu(x^3)\ \forall x \in S.$

Example 1 Let $S = M_2(N_0) = 2 \times 2$ matrices over the non-negative integers. The operations are matrix addition and multiplication. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\ E = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \in M_2(N_0)$$

Define: $\mu : M_2(N_0) \rightarrow [0, 1]$ suchthat $\mu(M) = \dfrac{\det(M)}{1 + \det(M)}$. Then

$\mu(A)=1/2,\ \mu(B)=4/5,\ \mu(C)=9/10=6/7,\ \ \mu(D)=16/17,\ \mu(E)=25/26$

$ABCDE=\begin{pmatrix}120 & 0\\ 0 & 120\end{pmatrix}$. Therefore, $\mu(ABCDE)=14400/14401$

Fuzzy Subsemigroup: $\mu(ABC)=36/37$ and $\min\{\mu(A),\ \mu(B),\ \mu(C)\}=1/2$. Therefore $\mu(ABC)\geq\min\{\mu(A),\ \mu(B),\ \mu(C)\}$.

Fuzzy Left (right, lateral) ideal: $\mu(ABC)=36/37$. Clearly $\mu(ABC)\geq\mu(C)$, $\mu(ABC)\geq\mu(B)$ and $\mu(ABC)\geq\mu(A)$.

Fuzzy Bi-Ideal: $\min\{\mu(A),\ \mu(C),\ \mu(E)\}=1/2$. Therefore, $\mu(ABCDE)\geq\min\{\mu(A),\ \mu(C),\ \mu(E)\}$.

Now we briefly study 'metatheorem' formulated by Tom Head [3] in the year 1995. Let S be a ternary semigroup and P(S) and C(S) denotes the set of all subsets and characteristic subsets of S. The mapping $Chi: P(S)\rightarrow C(S)$ defined by $Chi(A)=\chi_A$ is a bijection.

Proposition 2.5. $P(S)\cong C(S)$ under the isomorphism *Chi*.

Proposition 2.6. The mapping $Chi: P(S)\rightarrow C(S)$ commutes with the finite intersection and product of sets in ternary semigroup S.

Definition 2.7 [4] Consider a ternary semigrop S, $\mu\in F(S)$ and $r\in J=[0,1)$. Then the functioln Rep : $F(S)\rightarrow C(S)^J$ is defined by

$$\text{Rep}(\mu)(r)(x)=\begin{cases}1 & \text{if } \mu(x)>r\\ 0 & \text{if } \mu(x)\leq r\end{cases}$$

Proposition 2.8 [4] The function Rep is injective and Rep $\left(\bigcap_{i=1}^{k}\mu_i\right)=\bigcap_{i=1}^{k}\text{Rep}(\mu_i)$ and $\left(\bigcup_{i\in M}\mu_i\right)=\bigcup_{i\in M}\text{Rep}(\mu_i)$, where $\mu_i\in F(S)$ and M is an index set.

Proposition 29. [4] Rep is an order isomorphism of $F(S)$ onto $I(S)$, where $I(S)$ denores the image of the Rep function.

Definition 2.10. The product $\mu_1*\mu_2*\mu_3$ of fuzzy sets $\mu_1,\ \mu_2,\mu_3$ in a ternary semigroup S is defined as follows:

$$(\mu_1*\mu_2*\mu_2)(w)=\begin{cases}\sup\limits_{w=x*y*z}[\min\{\mu_1(x),\ \mu_2(y),\mu_3(z)\}] & \\ 0 & if\ x \text{ not expressed as } w=x*y*z\end{cases}$$

Tom Head [4] defined above binary operation * on F(S) as the convolutional extension of the binary operation * on S.

Proposition 2.11. [4] For $\mu_1,\ \mu_2 \in F(S),\ \text{Rep}(\mu_1 * \mu_2) = \text{Rep}(\mu_1) * \text{Rep}(\mu_2)$.

Definition 2.12. [4] Consider $\mathcal{C}$ as a collection of fuzzy sets within a ternary semigroup S. We defined $\mathcal{C}$ as being projection closed if every $\mu \in \mathcal{C}$ and for any $r \in J$, the Rep(μ)(r) remains an element of $\mathcal{C}$.

Proposition 2.13. [4] $C_1, C_2 (\mathcal{C}_1, \mathcal{C}_2)$ be the classes of crisp (fuzzy) subsets of a tertnarysemigroup S. Then $\mathcal{C}_1 \subseteq \mathcal{C}_2 (\mathcal{C}_1 = \mathcal{C}_2) \Leftrightarrow C_1 \subseteq C_2 (C_1 = C_2)$.

Metatheorem 2.14. [4] $(S, *)$ be a ternary semigroup. Consider the algebra $(F(S), \inf, \sup, *)$. Let $L(a_1, a_2, ..., a_m)$ and $M(a_1, a_2, ..., a_m)$ be two expression defined over the variables set $\{a_1, a_2, ..., a_m\}$ and operations set $\{\inf, \sup, *\}$ on $P(S)$. Let $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_m$ are projection closed classes of fuzzy sets of *S* and $D_1, D_2, ..., D_m$be their corresponding crisps classes. Then

$$L(\mu_1, \mu_2, ..., \mu_m)\ \text{REL}\ M(\mu_1, \mu_2, ..., \mu_m)$$

Hold $\forall \mu_1\ in\ \mathcal{D}_1, ..., \mu_m\ in\ \mathcal{D}_m \Leftrightarrow$ it holds $\forall \mu_1\ in\ D_1, ..., \mu_m\ in\ D_m$ where REL represent one of the three symbols ≤ , = or ≥.

3. Fuzzy Interior Ideals and Fuzzy Quasi-Ideals in ternary semigroups

Now we define both fuzzy interior ideal and fuzzy quasi-ideal of a ternary semigroup point wise as:

Definition 3.1. Let S be a ternary semigroup. Then, $\mu \in \mathcal{C}_{ss}$ is called a fuzzy interior ideal of S if $\mu(x_1 x_1 s_2 x_2 s_3) \geq \min\{\mu(x_1),\ \mu(x_2))\} \forall x_1, x_2, s_1, s_2, s_3 \in S$.

Definition 3.2. Let S be a ternary semigroup. Then $\mu \in F(S)$ is called a fuzzy quasi-ideal if $\forall z \in S,$

$$\mu(z) \geq \min\left\{ \sup_{x=s_1 s_2 x_3} \mu(x_3),\ \max\left\{ \sup_{x=s_1 x_2 s_3} \mu(x_2),\ \sup_{x=s_1 s_2 x_3 s_4 s_5} \mu(x_3) \right\},\ \sup_{x=x_1 s_2 s_3} \mu(x_1) \right\}$$

Example 3. In Example 1, we have already seen that μ is a ternary fuzzy subsemigroup of S. Also $\mu(ABCDE) = 14400/14401$ and $\min\{\mu(B), \mu(D)\} = 4/5$. Therefore, $\mu(ABCDE) \geq \min\{\mu(B),\ \mu(D)\}$.

Now we provide the equivalence between the Definitions 2.4 and 3.1.

Theorem 3.3. Let $\mu \in \mathcal{C}_{ss}$. Then $\mu \in \mathcal{C}_{in} \Leftrightarrow if\ S \circ \mu \circ S \circ \mu \circ S \subseteq \mu$

Proof: Let $z \in S$. Then,

$$S \circ \mu \circ S \circ \mu \circ S \subseteq \mu$$

$$\Leftrightarrow \quad (S \circ \mu \circ S \circ \mu \circ S \circ \mu)(z) \leq \mu(z)$$

$\Leftrightarrow \quad \sup\limits_{z=s_1x_1s_2x_2s_3} \{\min[S(s_1), \mu(x_1), S(s_2), \mu(x_2), S(s_3)]\} \leq \mu(z)$

$\Leftrightarrow \quad \sup\limits_{z=s_1x_1s_2x_2s_3} \{\min[\mu(x_1), \mu(x_2)]\} \leq \mu(z)$

$\Leftrightarrow \quad \mu(s_1x_1s_2x_2s_3) \geq \{\min[\mu(x_1), \mu(x_2)]\} \; \forall \; s_1, s_2, s_3, x_1, x_2 \in S$

$\Leftrightarrow$ μ is a fuzzy interior ideal of S.

Next we provide the equivalence between the Definitions 2.4 and 3.2.

Theorem 3.4. Let $\mu \in F(S)$. Then $\mu \in C_q$ if and only if $\mu \circ S \circ S \cap (S \circ \mu \circ S \cup S \circ S \circ \mu \circ S \circ S\} \cap S \circ S \circ \mu \subseteq \mu$.

Proof: Let $z \in S$. Then,

$S \circ S \circ \mu \cap (S \circ \mu \circ S \cup S \circ S \circ \mu \circ S \circ S\} \cap \mu \circ S \circ S \subseteq \mu$

$\Leftrightarrow (S \circ S \circ \mu \cap (S \circ \mu \circ S \cup S \circ S \circ \mu \circ S \circ S) \cap \mu \circ S \circ S)(z) \leq \mu(z)$

$\Leftrightarrow \min\{(S \circ S \circ \mu)(z), \, (S \circ \mu \circ S \cup S \circ S \circ \mu \circ S \circ S)(z), (\mu \circ S \circ S)(z)\} \leq \mu(z)$

$\Leftrightarrow \min\{ \sup\limits_{z=s_1s_2x_3} [\min(S(s_1), S(s_2), \mu(x_3))], \lim\limits_{z=x_1s_2s_3} [\min(\mu(x_1), S(s_2), S(s_3))],$

$\max\{ \sup\limits_{z=s_1x_2s_3} [\min S(s_1), \, \mu(x_2), \, S(s_3))], \sup\limits_{z=z_1}[\min(S(s_1), S(s_2), \mu(x_3), S(s_4), S(s_5))]\}\} \leq \mu(z),$

where $z_1 = s_1s_2x_3s_4s_5$

$\Leftrightarrow \min\{ \sup\limits_{z=s_1s_2s_3} \mu(x_3), \max\{ \sup\limits_{z=s_1x_2s_3} \mu(x_2), \sup\limits_{z=s_1s_2x_3s_4s_5} \mu(x_3)\}, \sup\limits_{z=x_1s_2s_3} \mu(x_1)\} \leq \mu(z)$

$\Leftrightarrow \mu$ is a fuzzy quasi-ideal of S.

4. Projection closed classes in Ternary semigroups

Theorem 41. C_{ss} is projection closed.

Proof. Consider $\mu \in C_{ss}$. Therefore $\mu(xyz) \geq \min\{\mu(x), \, \mu(y), \, \mu(z)\} \; \forall \; x, y, z \in S$. To show that $\text{Rep}\,(\mu)(r) \in C_{ss} \;\; \forall \; r \in J.$ Equivalently, $\text{Rep}\,(\mu)(r)(xyz) \geq \min\{\text{Rep}\,(\mu)(r)(x), \, \text{Rep}(\mu(r)(y), \, \text{Rep}(\mu)(r)(z)\} \forall \; x, y, z \in S$. Let $x, y, z \in S$ and $r \in J$. Suppose $\min\{\text{Rep}\,(\mu)(r)(x), \, \text{Rep}\,(\mu)(r)(y), \, \text{Rep}\,(\mu)(r)(z)\} = 1$. Therefore, $\text{Rep}\,(\mu)(r)(x) = 1 = \text{Rep}\,(\mu)(r)(y) = 1 = \text{Rep}\,(\mu)(r)(z)$. This implies that $\mu(x) > r, \; \mu(y) > r$ and $\mu(z) > r.$ Thus $\mu(xyz) \geq \min\{\mu(x), \, \mu(y), \, \mu(z)\} > \text{r}$ and hence If $\min\{\text{Rep}\,(\mu)(r)(x), \text{Rep}\,(\mu)(r)(y), \, \text{Rep}\,(\mu)(r)(z)\} = 0$, then the inequality holds trivially.

Theorem 4.2 C_i, C_{lt} , C_r , C_i and C_s are projection closed.

Theorem 4.3. C_{in} is projection closed.

Proof. Consider $\mu \in C_{in}$. Then $\mu \in C_{ss}$ and $\mu(s_1x_1s_2x_2s_3) \geq \min\{\mu(x_1), \mu(x_2)\}$ $\forall\, x_1, x_2, s_1, s_2, s_3 \in S$. In view of Theorem 4.1, it is sufficient to show that $\text{Rep}(\mu)(r)(s_1x_1s_2x_2s_3) \geq \min\{\text{Rep}(\mu)(r)(x_1), \text{ Rep}(\mu)(r)(x_2)\}$ $\forall\, x_1, x_2, s_1, s_2, s_3$ and $\forall\, r \in J$. Let $x_1, x_2, s_1, s_2, s_3 \in S$ and $\forall\, r \in J$. Suppose that $\min\{\text{Rep}(\mu)(r)(x_1),$ $\text{Rep}(\mu)(r)(x_2)\} = 1$. Then, $\text{Rep}(\mu)(r)(x_1) = 1 = \text{Rep}(\mu)(r)(x_2)$. This implies that $\mu(x_1) > r < \mu(x_2)$. Therefore, $\mu(s_1x_1s_2x_2s_3) \geq \min\{\mu(x_1),\ \mu(x_2)\} > r$. Thus $\text{Rep}(\mu)(r)(s_1x_1s_2x_2s_3) = 1$. If $\min\{\text{Rep}(\mu)(r)(x_1), \text{ Rep}(\mu)(r)(x_2)\} = 0$. Then the inequality holds trivially.

Theorem 4.4. C_b is projection closed.

Proof. Consider $\mu \in C_b$. Then $\mu \in C_{ss}$ and $\mu(x_1s_1x_2, s_2x_3) \geq \min\{\mu(x_1), \mu(x_2),\ \mu(x_3)\}$ $\forall x_1, x_2, x_3, s_1, s_2 \in S$. We get the result by proceeding same as in Theorem 4.1.

Theorem 4.5, C_q is projection closed.

Proof: Consider $\mu \in C_q$. Therefore,

$$\mu(z) \geq \min\{\sup_{z=s_1s_2x_3} \mu(x_3), \max\{\sup_{z=s_1x_2s_3} \mu(x_2), \sup_{z=s_1s_2x_3s_4s_5} \mu(x_3)\}, \sup_{z=x_1s_2s_3} \mu(x_1)\}$$

$\forall\, z \in S$. To show that $\text{Rep}(\mu)(r)(z) \in C_q$. Let $z \in S$ and $r \in J$. Suppose that

$$\min\{\sup_{z=s_1s_2x_3} \text{Rep}(\mu)(r)(x_3), \max\{\sup_{z=s_1x_2s_3} \text{Rep}(\mu)(r)(x_2), \sup_{z=z_1} \text{Rep}(\mu)(r)(x_3)\}, \sup_{z=x_1s_2s_3} \text{Rep}(\mu)(r)(x_1)\}$$

=1, where $z_1 = s_1s_2x_3s_4s_5$. Then, $\sup_{z=s_1s_2x_3} \text{Rep}(\mu)(r)(x_3) = 1 = \max\{\sup_{z=s_1x_2s_3}$ $\text{Rep}(\mu)(r)(x_2), \sup_{z=z_1} \text{Rep}(\mu)(r)(x_3)\} = \sup_{z=x_1s_2s_3} \text{Rep}(\mu)(r)(x_1),$ where $z_1 = s_1s_2x_3s_4s_5$,

Now, $\sup_{z=s_1s_2x_3} \text{Rep}(\mu)(r)(x_3) = 1$ implies that there exist, $s_1, s_2, x_3 \in S$ satisfying $z = s_1s_2x_3$ in S such that $\text{Rep}(\mu)(r)(x_3) = 1$. Therefore, $\mu(x_3) > r$. Thus $\sup_{z=s_1s\ x_3} \mu(x_3) > r$. Similarly $\sup_{z=x_1s_2s_3} \text{Rep}(\mu)(r)(x_1) = 1$ implies that $\sup_{z=x_1s_2s_3} \mu(x_1) > r$ and $\max\{\sup_{z=s_1x_2s_3} \text{Rep}(\mu)(r)(x_2), \sup_{z=s_1s_2x_3s_4s_5} \text{Rep}(\mu)(r)(x_3) = 1$ implies that either $\sup_{z=s_1x_2s_2} f(x_2) > r$ or $\sup_{z=z_1} f(x_3) > r,$ where $z_1 = s_1s_2x_3s_4s_3$.

Thus in any case $\mu(z) > r$ and $\text{Rep}(\mu)(r)(z) = 1$ .If $\min\{\sup_{z=s_1s_2x_3} \text{Rep}(\mu)(r)(x_3)$ $\max\{\sup_{z=s_1x_2s_3} \text{Rep}(\mu)(r)(x_2)\ \sup_{z=z_1} \text{Rep}(\mu)(r)(x_3)\}, \sup_{z=x_1s_2s_3} \text{Rep}(\mu)(r)(x_1)\} = 0$ then the inequality holds trivially where $z_1 = s_1s_2x_3s_4s_5$. Hence C_q is projection closed.

Theorem 4.6. C_s is projection closed.

Proof: Consider $\mu \in C_s$. Therefore $\mu(x^3) = \mu(x)\ \forall\ x \in S$. Theorem follows similar to Theorem 4.1.

5. Fuzzy Ideals of a ternary Semigroup

Theorem 5.1. The following assertions hold in a ternary semigroup S:

(1) Any fuzzy left (right, lateral) ideal of S is a fuzzy bi-ideal of S.

(2) Any fuzzy quasi-ideal of S is also a fuzzy bi-ideal of S.

Proof: We establish (2). Since Both C_{q} and C_{b} are projection closed, therefore, $C_q \subseteq C_b \Leftrightarrow C_q \subseteq C_b$ by Proposition 2.13. By Theorem 3.2 of [2] any quasi-ideal of a ternary semigroup *S* is also a bi-ideal of S and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_q \subseteq C_b$. Hence $C_q \subseteq C_b$.

The remaining results of this theorem extends. Proposition 3.6 of [2] to fuzzy context and their proofs can be derived analogously.

The subsequent results extend Propositions 3.2, 3.3 and 3.5 of [11] to the fuzzy context and their proofs can be derived analogously.

Theorem 5.2 The following assertions hold in a ternary semigroup S:

(1) Any fuzzy left ideal (right ideal, lateral ideal, ideal) of S is a fuzzy quasi-ideal of S.

(2) Any fuzzy quasi-ideal of S is also a fuzzy ternary subsemigroup of S.

Theorem 5.3. The following assertions hold in a ternary semigruop S.

(1) Let $\alpha \in C_q$ and $\gamma \in C_i$, then $\alpha \cap \gamma \in C_q$.

(2) Let $\alpha \in C_q$ and $\gamma \in C_{ss}$, then $\alpha \cap \gamma \in C_q$.

(3) Let $\alpha \in C_i$ and $\gamma \in C_{ss}$, then $\alpha \cap \gamma \in C_q$.

Proof (1). Define the classes $C_{q,\,i} = \{\alpha_1 \cap \alpha_2 : \alpha_1 \in C_q,\ \alpha_2 \in C_i\}$ and $C_{q,\,i} = \{\alpha \cap \gamma : \alpha \in C_q,\ \gamma \in C_i\}$ in S. Claim that $C_{q,\,i}$ is projection closed. Let $\alpha \cap \gamma \in C_{q,\,i}$. Now, for all $\alpha \in C_q$ and $\gamma \in C_i$, $\mathrm{Rep}(\alpha \cap \gamma)(r) = \mathrm{Rep}(\alpha)(r) \cap \mathrm{Rep}(\gamma)(r)\ \forall r \in J$ by Proposition 2.8. $\mathrm{Rep}(\alpha)(r) \in C_q$ and $\mathrm{Rep}(\gamma)(r) \in C_i$ since both C_q and C_i are projection closed. Thus $\mathrm{Rep}(\alpha \cap \gamma)(r) \in C_{q,\,\mathrm{i}}\ \ \forall\ r \in J$. Hence, the classes $C_{q,\,i}$ is projection closed. Since C_q is projection closed, therefore, $C_{q,\,i} \subseteq C_q \Leftrightarrow C_{q,\,\mathrm{i}} \subseteq C_q$ by Proposition 2.13. Since the intersection of a quasi-ideal and an ideal of a ternary semigroup is a quasi-ideal by Proposition 3.3 of [2] and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{q,i} \subseteq C_q$. Hence, $C_{q,\,i} \subseteq C_q$.

Next to establish (2), we define $C_A = \{\alpha_1 \cap \chi_A : \alpha_1 \in C_q$ and A is a subsemigroup of S} and $\mathcal{C}_A = \{\alpha \cap \mu : \alpha \in \mathcal{C}_q \text{ and } \mu \in \mathcal{C}_{ss}\}$. Claim : $\mathcal{C}_A$ is projection closed. Now $\text{Rep}\{\alpha \cap \mu)(r)$ $= \text{Rep}\{\alpha)(r) \cap \text{Rep}\{\mu)(r) \forall r \in J$ by Proposition 2.8. Since $\mathcal{C}_q$ and $\mathcal{C}_{ss}$ are projection closed, therefore $\text{Rep}\{\alpha)(r) \in \mathcal{C}_q$ and $\text{Rep}\{\mu)(r) \in \mathcal{C}_{ss}$. Thus $\text{Rep}\{\alpha \cap \mu)(r) \in \mathcal{C}_A \forall r \in J$ and hence $\mathcal{C}_A$ is closed under projection. Therefore $\mathcal{C}_A \subseteq \mathcal{C}_q \Leftrightarrow C_A \subseteq C_q$ by Proposition 2.13.As the intersection of a quasi-idealand a subsemigroup of S is a quasi-ideal by Preposition 3.7of [11] and $P(S) \cong C(S)$ underthe isomorphism *Chi*. we get, $C_A \subseteq C_q$ .Hence $\mathcal{C}_A \subseteq \mathcal{C}_q$.

(3) is anextension of Proposition 3.2 of [2]to fuzzy context and its proof can be derived analogously.

Theorem 5.4 Let $h \in \mathcal{C}_{ss}$. Then $h \in \mathcal{C}_b$ if there exists $h \in \mathcal{C}_i, \mu \in \mathcal{C}_b$ and $\delta \in \mathcal{C}_r$ such that $a \text{o} \mu \text{o} \delta \subseteq h \subseteq \alpha \cap \mu \cap \delta$

Proof. Define the classes D = $\{\chi_H \in C(S)$ where H is a subsemigroup of $S : A_1 B_1 C_1 \subseteq H \subseteq A_l \cap B_1 \cap C_1$ for some leftideal A$_1$, lateral idealB$_1$ and right ideal C$_1$ of S} and $\mathcal{D}$ = $\{h \in C_{ss} : \alpha \text{o} \mu \text{o} \delta \subseteq h \subseteq \alpha \cap \mu \cap \delta$ for some $\alpha \in \mathcal{C}_1$, $\mu \in \mathcal{C}_b$ and $\delta \in \mathcal{C}_r\}$, where $C_r (C_b,\ C_1)$ arecrisps classes of right (lateral, left) ideals of S. Firstly claim: $\mathcal{D}$is projection closed. Let $h \in \mathcal{D}$. Therefore, $h \in \mathcal{C}_{ss}$ such that $\alpha \text{o} \mu \text{o} \delta \subseteq h \subseteq \alpha \cap \mu \cap \delta$ for some $\alpha \in \mathcal{C}_r$, $\mu \in \mathcal{C}_b$ and $\mu \in \mathcal{C}_l$. $\text{Rep}\{\alpha \text{o} \mu \text{o} \delta)(r) \leq \text{Rep}(h)(r) \leq \text{Rep}(a \cap \mu \cap \delta)(r)\ \forall\ r \in J$ byProposition 2.9.Now for $r \in J$, $\text{Rep}(\alpha)(r) \text{o} \text{Rep}(\mu)(r) \text{o} \text{Rep}(\delta)(r) \leq \text{Rep}(h)(r) \leq \text{Rep}(\alpha)(r) \cap$ $\text{Rep}(\mu)(r) \cap \text{Rep}(\delta)(r)$ by Proposition2.8 and 2.11.Since $\mathcal{C}_l$, $\mathcal{C}_{lt}$ and $\mathcal{C}_r$ are projection closed, we have, $\text{Rep}(\alpha)(r) \in \mathcal{C}_l$, $\text{Rep}(\mu)(r) \in \mathcal{C}_{lt}$ and $\text{Rep}(\delta)(r) \in \mathcal{C}_r$.Therefore, $\text{Rep}(h)(r) \in \mathcal{D}$ $\forall\ r \in J$. Hence $\mathcal{D}$ is Projection closed. Also $\mathcal{C}_b$ is projection closed. Therefore, by Proposition 2.13, $\mathcal{D} \subseteq \mathcal{C}_b \Leftrightarrow D \subseteq C_b$.

The later proposition follows as a ternary semigroup H of a ternary semigroup S is a bi-ideal if $A_l B_l C_1 \subseteq H \subseteq A_1 \cap B_1 \cap C_1$ forsomeleft idealA$_1$, lateralideal B$_1$ and right idealC$_1$ of S byProposition 4.8 of [7] and $P(S) \cong C(S)$ under the isomorphism *Chi*.

Theorem 5.5. Let μ be a fuzzy set of a ternary semigroup S. Then following assertions hold in S:

(1) $S \text{o} S \text{o} \mu \in \mathcal{C}_l$

(2) $\{S \text{o} \mu \text{o} S\} \cup (S \text{o} S \text{o} \mu \text{o} S \text{o} S) \in \mathcal{C}_{lt}$

(3) $\mu \text{o} S \text{o} S \in \mathcal{C}_r$

Proof. To establish [2], define the classes $C = \{\chi_A \in C(S)$, where $A \in P(S) : SAS \cup SSSASS\}$ and $\mathcal{C} = \{\mu \in F(S) : (S \text{o} \mu \text{o} S) \cup (S \text{o} S \text{o} \mu \text{o} S \text{o} S)\}$. Firstly claim that $\mathcal{C}$ is projection closed. Let $\mu \in \mathcal{C}$ $r \in J$, we have $\text{Rep}\{(S \text{o} \mu \text{o} S) \cup (S \text{o} S \text{o} \mu \text{o} S \text{o} S)\}(r)$

$= \text{Rep}\{S \cup \mu \text{o} S)(r) \cup \text{Rep}(S \text{o} S \text{o} \mu \text{o} S \text{o} S)(r)$ by Proposition 2.8.

$= \text{Rep}(S)(r) \,o\, \text{Rep}(\mu)(r) \,o\, \text{Rep}(S)(r) \cup \text{Rep}(S)(r) \cup \text{Rep}(S)(r) \,o\, \text{Rep}(\mu)(r) \,o\, \text{Rep}(S)(r) \,o\, \text{Rep}(S)(r)\}$ by Proposition 2.11.

$= (S \,o\, \text{Rep}(\mu)(r) \,o\, S) \cup (S \,o\, S \,o\, \text{Rep}(\mu)(r) \,o\, S \,o\, S)$

It can be seen easily that $\text{Rep}(\mu)(r) \in F(S)$. Thus $\text{Rep}(\mu)(r) \in \mathcal{C}\ \forall r \in J$

and hence C is projection closed. Also by Theorem 4.2, the class $\mathcal{C}_{lt}$ of fuzzy lateral ideals is projection closed. Therefore, $\mathcal{C} \subseteq \mathcal{C}_{lt} \Leftrightarrow C \subseteq C_{lt}$ by Proposition 2.13.

By Proposition 1.1. of [1] $(SA_1S \cup SSA_1SS)$ is a lateralideal of S for a non-empty subset A_1 of a ternary semigroup and $P(S) \cong C(S)$ under the isomorphisms*Chi*, we get, $C \subseteq C_{lt}$. Hence $\mathcal{C} \subseteq \mathcal{C}_{lt}$.

The proof of following theorem is an alternative proof of the Theorem 3.20 of [7].

Theorem 5.6. Let $\mu \in \mathcal{C}_r$, $\gamma \in \mathcal{C}_{lt}$ and $\delta \in \mathcal{C}_l$. Then $\mu \cap \gamma \cap \delta \in \mathcal{C}_q$.

Proof. We define the classes $C_{r,lt,i} = \{\mu_1 \cap \mu_2 \cap \mu_3 : \mu \in C_r,\ \mu_2 \in C_{lt}$ and $\mu_3 \in C_l\}$ and $\mathcal{C}_{r,lt,l} = \{\mu \cap \gamma \cap \delta : \mu \in \mathcal{C}_r,\ \gamma \in \mathcal{C}_{lt}$ and $\delta \in \mathcal{C}_l\}$. Firstly claim: $\mathcal{C}_{r,lt,l}$ is projection closed. Let $\mu \cap \gamma \cap \delta \in \mathcal{C}_{r,lt,l}$. We have for all $\mu \in \mathcal{C}_r,\ \gamma \in \mathcal{C}_{lt}$ and $\delta \in \mathcal{C}_l$ $\text{Rep}(\mu \cap \gamma \cap \partial)(r) = \text{Rep}(\mu)(r) \cap \text{Rep}(\gamma)(r) \cap \text{Rep}(\delta)(r) \forall r \in J$ by Proposition 2.8. Since $\mathcal{C}_r$, $\mathcal{C}_{lt}$ and $\mathcal{C}_l$ are projection closed by Theorem 4.2 and 4.3, we have, $\text{Rep}(\mu)(r) \in \mathcal{C}_r$, $\text{Rep}(\gamma)(r) \in \mathcal{C}_{lt}$ and $\text{Rep}(\delta)(r) \in \mathcal{C}_l$. Hence $\text{Rep}(\mu \cap \gamma \cap \delta)(r) \in \mathcal{C}_{r,lt,l}\ \forall r \in J$. Thus $\mathcal{C}_{r,lt,l}$ is projection closed. Therefore, by Proposition 2.13, $\mathcal{C}_{r,lt,l} \subseteq \mathcal{C}_q \Leftrightarrow C_{r,lt,l} \subseteq C_q$. By Proposition 3.6 of [11], the intersection of a right ideal, a lateral ideal and a leftideal of a ternary semigroup S is a quasi-ideal of S and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{r,lt,l} \subseteq C_q$. Hence, $\mathcal{C}_{r,lt,l} \subseteq \mathcal{C}_q$.

Theorem 5.7. .Let $\mu \in F(S)$. Then $\mu \in \mathcal{C}_q$ if μ is an intersection of a fuzzy right ideal $\mu \cup \mu \,o\, S \,o\, S$, a fuzzy lateral ideal $\mu \cup (S \cup \mu \,o\, S \cup S \,o\, S \,o\, \mu \,o\, S \,o\, S)$ and a fuzzy left ideal $\mu \cup S \,o\, S \,o\, \mu$.

Proof. Define the classes:

$C = \{\chi_A \in C(S)$,where $A \in P(S) : A = (A \cup SSA) \cap (A \cup SAS \cup SSASSA) \cap A \cup ASS\}$
and
$\mathcal{C} = \{\mu \in F(S) : \mu = (\mu \cup S \,o\, S \,o\, \mu) \cap \mu \cup (S \,o\, \mu \,o\, S \cup S \,o\, S \,o\, \mu \,o\, S \,o\, S \,o\, \mu) \cap (\mu \cup \mu \,o\, S \,o\, S)\}$

Claim: $\mathcal{C}$ is projection closed. Let $\mu \in \mathcal{C}$. Therefore $\mu \in F(S)$ such that $\mu = (\mu \cup S \,o\, S \,o\, \mu) \cap \mu \cup (S \,o\, \mu \,o\, S \cup S \,o\, S \,o\, \mu \,o\, S \,o\, S) \cap (\mu \cup S \,o\, S \,o\, \mu)$. Now $\forall r \in J$, $\text{Rep}(\mu)(r) = \text{Rep}\{(\mu \cup S \,o\, S \,o\, \mu) \cap \mu \cup (S \,o\, \mu \,o\, S \cup S \,o\, S \,o\, \mu \,o\, S \,o\, S\} \cap (\mu \cup \mu \,o\, S \,o\, S))$ (r) by Proposition 2.9.Also $\text{Rep}(\mu)(r) = \text{Rep}\{(\mu \cup S \,o\, S \,o\, \mu)(r) \cap (\mu \cup S \,o\, \mu \,o\, S \cup S \,o\, S \,o\, \mu \,o\, S \,o\, S)(r) \cap \text{Rep}(\mu \cup \mu \,o\, S \,o\, S)(r)\ \forall r \in J$ by Proposition 2.8.

$= (\text{Rep}(\mu)(\text{r}) \cup \text{Rep}(S)(\text{r})\, \text{o}\text{Rep}(S)(\text{r})\, \text{o}\text{Rep}(\mu)(\text{r})) \cap \text{Rep}(\mu)(\text{r}) \cup\ (\text{Rep(S)(r)}\ \text{o}\text{Rep}(\mu)(\text{r}) \cup \text{Rep}(S)(\text{r})\ \cup \text{Rep}(S)(\text{r})\ \text{o}\text{Rep(S)(r)}\, \text{o}\text{Rep}(\mu)(\text{r})\ \text{Rep}(S)(\text{r}) \cap \text{Rep(S)(r)}) \cap \text{Rep}(\mu)(\text{r})\ \text{o}\text{Rep}(\mu)(\text{r})\ \text{o}\text{Rep}(S)(\text{r})\ \text{o}\text{Rep}(S)(\text{r})$ by Proposition 2.8 and 2.11 . That is for $r \in J$, $\text{Rep}\{\mu)(r) = (\text{Rep}\{\mu)(r)\ \cup S\ \text{o}S\ \text{o}\text{Rep}\{\mu)(r) \cap \text{Rep}\{\mu)(r) \cup\ (S\ \text{o}\text{Rep}\{\mu)(r)\ \text{o}S \cup S\ \text{o}\ \text{Rep}\{\mu)(r)\ \text{o}\ S\ \text{o}S) \cap (\text{Rep}\{\mu)(r) \cup \text{Rep}\{\mu)(r)\ \text{o}S\ \text{o}S)$. Hence $\text{Rep}(\mu)(r) \in C\ \forall\ r \in J$. Therefore C is projection closed.Also by Theorem 4.5, the class C_q of fuzzy quasi-ideals is projection closed. Therefore, $C \subseteq C_q \Leftrightarrow C \subseteq C_q$ by Proposition 2.13.

By Theorem 3.10 of [11],. a non subset A_1 of a ternary semigroup S is a quasi-ideal of S if $A_1 = \{A_1 \cup SSA_1) \cap (A_i \cup SA_1S \cup SSA_1SS) \cap (A_1 \cup A_1SS)$ and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C \subseteq C_q$. Hence, $C \subseteq C_q$.

Similarly it can be prove that:

Theorem 5.8. If λ and μ be two non-empty subset of a ternary semigroup S. Then $\lambda\, \text{o}S\, \text{o}\mu$ is a fuzzy bi-ideal of S.

The proof of following theorem is an alternative proof of Lemma 2.1 of [6].

Theorem 5.9. Let μ be a fuzzy set in a ternary semigroup S. Then, following assertions hold in S:

(1) $\mu \in C_{ss} \Leftrightarrow \mu\, \text{o}\mu\, \text{o}\mu \subseteq \mu$

(2) $\mu \in C_l \Leftrightarrow S\, \text{o}S\, \text{o}\mu \subseteq \mu$

(3) $\mu \in C_{lt} \Leftrightarrow S\, \text{o}\mu\, \text{o}S \subseteq \mu$

(4) $\mu \in C_r \Leftrightarrow \mu\, \text{o}S\, \text{o}S \subseteq \mu$

Proof. To establish (1), define $C = \{\chi_A \in C(S)$ where $A \in P(S) : A^3 \subseteq A\}$ and $C = \{\mu \in C_{SS} : \mu\, \text{o}\mu\, \text{o}\mu \subseteq \mu\}$. To show C is projection closed. Let $\mu \in C$. Then $\mu \in C_{ss}$ such that $\mu\, \text{o}\mu\, \text{o}\mu \subseteq \mu$. By Proposition 2.9, $\text{Rep}(\mu)(r) \geq \text{Rep}(\mu\, \text{o}\mu\, \text{o}\mu)(r)\ \forall r \in J$. Since Rep commutes with the operations 'o'(Proposition 2.11), $\text{Rep}(\mu)(r) \geq \text{Rep}(\mu)(r)\, \text{o}\text{Rep}(\mu)(r)\, \text{o}\text{Rep}(\mu(r)\ \forall r \in J$. Since C_{ss} is closed under projection, $\text{Rep}(\mu)(r) \in C_{ss}$. Thus $\text{Rep}(\mu)(r) \in C\ \forall r \in J$ and hence C is projection closed. By Theorem 4.1, C_{SS} is projection closed. Therefore $C = C_{ss}$ if and only if $C = C_{SS}$ by Proposition 2.13. Since a non-empty set A of a ternary semigroup S is a fuzzy subsemigroup of $S \Leftrightarrow A^3 \subseteq A$ by Lemma 2.1 and *Chi* is a isomorphism from $P(S)$ to $C(S)$, we have $C = C_{SS}$. Hence, $C = C_{ss}$.

The remaining parts are extensionof Lemma 2.1 to fuzzy context and their proofs can be derived analogously.

6. Fuzzy Ideals of a Regular and Intra-Regular Ternary Semigroups.

Theorem 6.1. Let *S* be ternary semigroup. Then,

(1) Any fuzzy bi-ideal of a regular ternary semigroup S is also a fuzzy quasi-deal.

(2) Let μ be a fuzzy subsemigroup of an intra-regular ternary semigroup *S*. Then μ is a fuzzy lateral ideal of *S* if and only if it is a fuzzy ideal of *S*.

Proof. (1) Both the classes $\mathcal{C}_b$ and $\mathcal{C}_q$ are projection closed by Theorem4.3 and 4.4. Therefore, $\mathcal{C}_b \subseteq \mathcal{C}_q$ if and only if $C_b \subseteq C_q$ by Proposition 2.13. By Proposition 3.13 of [2] any bi-ideal of a regular ternary semigroup is a fuzzy quasi-ideal and $P(S) \cong C(S)$ under the isomorphism *Chi*, $C_b \subseteq C_q$. Hence $\mathcal{C}_b \subseteq \mathcal{C}_q$

The proof is an alternative proof of the Proposition4.19 of [7].

(2) his result extends Proposition 3.20 from [3] to fuzzy context and its proofs canbe derived analogously.

The proof of following theorem is an alternative proof of the Proposition 3.15 and 3.17 of [7].

Theorem 6.2. Let *S* be a regular ternary semigroup and $\mu \in F(S)$. Then $\mu \in \mathcal{C}_q$ if and only if $(\mu \circ S \circ \mu \circ S \circ \mu) \cap (\mu \circ S \circ S \circ \mu \circ S \circ S \circ \mu) \subseteq \mu$.

Proof. Define Classes: $C = \{\chi_A \in C(S),$ where $A \in P(S) : ASASA \cap ASSASSA \subseteq A\}$ and $\mathcal{C} = \{\mu \in F(S) : \mu \circ S \circ \mu \circ S \cap \mu \circ S \circ S \circ \mu \subseteq \mu\}$. To show $\mathcal{C}$ is projection closed, let $\mu \in C$. Therefore, $\mu \in F(S)$ such that $\mu \circ S \circ \mu \circ S \circ \mu \cap \mu \circ S \circ S \circ \mu \circ S \circ S \circ \mu \subseteq \mu$ for $\forall\ r \in J$, $\mathrm{Rep}\{\mu \circ S \circ \mu \circ S \circ \mu \cap \mu \circ S \circ S \circ \mu(r) \le \mathrm{Rep}(\mu)(r)$ by Proposition 2.9. Then $\forall\ r \in J$, $\mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(\mu)(r) \cap \mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(S)(r)$ $\circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(\mu)(r) \le \mathrm{Rep}(\mu)(r)$ by Proposition 2.8 and 2.11. That is $\mathrm{Rep}(\mu)(r) \circ S \circ \mathrm{Rep}(\mu)(r) \circ S \circ \mathrm{Rep}(\mu)(r) \cap$ $\mathrm{Rep}(\mu)(r) \circ S \circ S \circ \mathrm{Rep}(\mu)(r) \circ S \circ S \circ \mathrm{Rep}(\mu)(r) \le \mathrm{Rep}(\mu)(r)\ \forall r \in J$. It can be seen easily that $\mathrm{Rep}(\mu)(r) \in F(S)$. Hence $\mathrm{Rep}(\mu)(r) \in \mathcal{C}\ \forall r \in J$ and thus $\mathcal{C}$ is projection closed. Also $\mathcal{C}_q$ is projection closed. Therefore, $\mathcal{C} = \mathcal{C}_q \Leftrightarrow C = C_q$ by Proposition 2.13.

By Theorem3.18of [11], in a regular ternary semigroup S, a subset Q_1 is a quasi-ideal of $S \Leftrightarrow Q_1SQ_1SQ_1 \cap Q_1SSQ_1SSQ_1 \subseteq Q_1$ and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C = C_q$. Hence $\mathcal{C} = \mathcal{C}_q$.

Theorem 6.3. Let Se be a regular ternary semigroup. If $\lambda \in C_b$ and $h_1, h_2 \in C_{ss}$, then $\lambda \circ h_1 \circ h_2,\ h_1 \circ \lambda \circ h_2,\ h_1 \circ h_2 \circ \lambda \in \mathcal{C}_b$.

Proof. Let $C_A = \{T_1 \chi_{K_1} \chi_{K2} : T_1 \in C_b,\ K_1, K_2 \text{ are subsemigroups of } S\}$ and $\mathcal{C}_A = \{\lambda \circ g \circ h : \lambda \in \mathcal{C}_b, g, h \in C_{ss}\}$ be crisp and fuzzy classes defined on *S*. By proceeding as in Theorem 5.3, it can be established easily that $\mathcal{C}_A$ is closed under

projection. Also by Theorem 4.4, $\mathcal{C}_b$ is closed under projection. Therefore, by Proposition 2.13, $\mathcal{C}_A \subseteq \mathcal{C}_b \Leftrightarrow C_A \subseteq C_b$. The later proposition follows in view of Proposition 4 of [17] and $P(S) \cong C(S)$ under the isomorphism *Chi*.

$h_1\lambda h_2$ and $h_1h_2 \text{o} \lambda$ are fuzzy bi-ideals of *S* can be proved similarly.

Theorem 6.4. Let *S* is a regular ternary semigroup. If $\mu_1, \mu_2, \mu_3 \in C_q$, then $\mu_1 \text{o} \mu_2 \text{o} \mu_3 \in C_q$.

Proof. We define the classes $C_{q,q,q} = (Q_1Q_2Q_3,\ Q_2,\ Q_2,\ Q_3 \in C_q)$ and $C_{q,q,q} = (\mu_1 \text{o} \mu_2 \text{o} \mu_3 : \mu_1, \mu_2,\ \mu_3 \in C_q\}$ Proceeding as in Theorem 5.3. It was he shown easily that $C_{q,q,q}$ is projection closed. Also C_q is projection closed. Therefore, $C_{q,q,q} \subseteq C_q \Leftrightarrow C_{q,q,q} \subseteq C_q$ by Proposition 2.13. By Theorem3.19 of [11], the product of three quasi-ideals of regular ternary semigroup is itself a quasi-ideal and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{q,q,q} \subseteq C_q$ .Hence $C_{q,q,q} \subseteq C_q$.

Theorem 6.5. Following assertions held on ternary semigroup *S*:

(1) S is regular.
(2) $\alpha \cap \lambda \subseteq \alpha \text{o} S \text{o} \delta \cap \delta \text{o} S \text{o} \alpha \ \ \forall \alpha, \delta \in C_b$.
(3) $\alpha \cap \delta \subseteq \alpha \text{o} S \text{o} \delta \cap \delta \text{o} S \text{o} \alpha \ \forall a \in C_b$ and $\forall \delta \in C_q(C_l)$
(4) $\alpha \cap \delta \subseteq \alpha \text{o} S \text{o} \delta \cap \delta \text{o} S \text{o} \alpha \ \forall \in C_l(C_r)$ *and* $\forall \delta \in C_q$
(5) $\alpha \cap \delta \subseteq \alpha \text{o} S \text{o} \delta \cap \delta \text{o} S \text{o} \alpha \forall \alpha \in C_r$ and $\delta \in C_l$
(6) $\alpha \cap \delta \subseteq \alpha \text{o} S \text{o} \delta \cap \delta \text{o} S \text{o} \alpha \ \forall \alpha \in C_b$ and $\delta \in C_r$

Proof. We establish (1) $\Leftrightarrow$ (6). Let (S, o) be a ternary semigroup. Consider the algebra- $(F(S), \inf, \sup, \text{o})$. let $L(a,c) = \alpha \cap c$ and $M(a,c) = aSc \cap cSa$ be expression over the variable set v and operations set $\{\inf, \sup, \text{o})$. Since C_b is projection closed, therefore, $L(\alpha,\ \delta) = M(\alpha,\ \delta)\ \forall\ \alpha \in C_b$ and $\delta \in C_r$ if and only if $L(\alpha, \delta) = \text{M}(\alpha, \delta)\ \forall\ \alpha \in C_b$ *and* $\forall\ \alpha \in C_r$ by metatheorem.

Now, a ternary semigroup *S* is regular

$\Leftrightarrow B_1 \cap R_1 = B_1SR_1 \cap R_1SB_1$ for any bi-ideal B_1 and right ideal R_1 of *S* by Theorem 8 of [17].

$\Leftrightarrow \chi_{B_1 \cap R_1} = \chi_{B_1SR_1 \cap R_1SB_1}$ for any bi-ideal B_1 and right ideal R_1 of *S* since $P(S) \cong C(S)$ under the isomorphism *Chi*.

$\Leftrightarrow \chi_{B_1} \cap \chi_{R_1} = \chi_{B_1SR_1} \cap \chi_{R_1SB_1}$ for any bi-ideal B_1 and right ideal R_1 of *S* by Proposition 2.8.

$\Leftrightarrow \chi_{B_1} \cap \chi_{R_1} = \chi_{B_1}\chi_S\chi_{R_1} \cap \chi_{R_1}\chi_S\chi_{B_1}$ for any bi-ideal B_1 and right ideal R_1 of *S* by Proposition 2.6.

$\Leftrightarrow \chi_{B_1} \cap \chi_{R_1} = \chi_{B_1}S\chi_{R_1} \cap \chi_{R_1}S\chi_{B_1}$ for any bi-ideal B_1 and right ideal R_1 of *S*.

$\Leftrightarrow L(\alpha, \delta) = M(\alpha, \delta)\ \forall \alpha \in C_b$ and $\forall \delta \in C_r$.

$\Leftrightarrow L(\alpha, \delta) = M(\alpha, \delta)\ \forall \alpha \in C_b$ and $\forall \delta \in C_r$.

Theorem 6.6. The following assertions are equivalent in a ternary semiring*S*.

(1) S is regular.
(2) $\mu = \mu \, \mathrm{o} S \, \mathrm{o} \mu \ \ \forall \mu \in C_b(C_q)$.

Proof. Similar to Theorem 6.5, defined expressions $L(a) = a \ S \ a$ and $M(a) = a$. Since C_b is projection closed, therefore, by metatheorem, $L(\delta) = M(\delta) \ \forall \delta \in C_b$ if and only if $L(\delta) = M(\delta) \ \forall \delta \in C_b$.

Now, a ternary semigroup *S* is regular

$\Leftrightarrow B_1 = B_1 S B_1$ for any bi-ideal B_1 of *S* by Theorem 4 of [17].

$\Leftrightarrow \chi_{B_i} = \chi_{B_i S \beta_1}$ for any bi-ideal B_1 of *S* since $P(S) \cong C(S)$ under the isomorphism *Chi*.

$\Leftrightarrow \chi_{B_1} = \chi_{B_1} \chi_S \chi_{B_1}$ for any bi-ideal B_1 of *S* by Proposition 2.6.

$\Leftrightarrow \chi_{B_1} = \chi_{B_1} S \chi_{B_1}$ for any bi-ideal B_1 of *S*.

$\Leftrightarrow L(\delta) = M(\delta) \ \forall \delta \in C_b$

$\Leftrightarrow L(\delta) = M(\delta) \ \forall \delta \in C_b$

The subsequent theorem can be derived in a similar manner.

Theorem 6.7.A ternary semigroup S is regular if and only if $\delta \, \mathrm{o} S \, \mathrm{o} \delta \, \mathrm{o} S \, \mathrm{o} \delta = \delta \ \ \forall \ \delta \in C_b(C_q)$.

Theorem 6.8. A ternary semigroup *S* is regular if $\alpha \cap \delta = \alpha \, \mathrm{o} S \, \mathrm{o} \delta \ \forall \alpha \in C_r$ and $\forall \ \delta \in C_l$.

Proof. As in Theorem 6.5 define expressions $L(a_1, a_2) = a_1 \cap a_2$ and $M(a_1, a_2) = a_1 S a_2$. Since the classes C_r and C_l are projection closed by Theorem 4.3, therefore, by metatheorem $L(\alpha, \delta) = M(\mu, \delta) \forall \ \mu \in C_r$ and $\forall \delta \in C_l \Leftrightarrow L(\mu, \delta) = M(\mu, \delta) \ \forall \mu \in C_r$ and $\forall \delta \in C_l$.

Now, a ternary semigroup *S* is regular

$\Leftrightarrow R_1 \cap L_1 = R_1 S L_1$ for any right ideal R_1 and any left ideal L_1 of *S* by Corollary 3.8 of [3].

$\Leftrightarrow \chi_{R_1 \cap L_1} = \chi_{R_1 S L_1}$ for any right ideal R_1 and any left ideal L_1 of *S* since $P(S) \cong C(S)$ under the isomorphism *Chi*.

$\Leftrightarrow \chi_{R_1 \cap L_1} = \chi_{R_1 S L_1}$ for any right ideal R_1 and any left ideal L_1 of *S* by Proposition 2.6.

$\Leftrightarrow \chi_{R_1} \cap \chi_{L_1} = \chi_{R_1} S \chi_{L_1}$ for any right ideal R_1 and any left ideal L_1 of *S*.

$\Leftrightarrow L(\mu, \delta) = \mathrm{M}(\mu, \delta) \ \forall \mu \in C_r$ and $\forall \delta \in C_l$

$\Leftrightarrow L(\mu,\delta) = \mathrm{M}(\mu,\delta)\ \forall\, \mu \in C_r$ and $\forall\, \delta \in C_l$

The subsequent theorem extends Theorem 7 of [17] to the fuzzy context and its roofs can be derived analogously.

Theorem 6.9. Following assertions hold in a ternary semigroup *S*:

(1) $\alpha \cap \delta \subseteq \alpha\, \mathrm{o} S\, \mathrm{o} \delta\ \forall\ \alpha \in C_b(C_q)$ and $\forall\, \delta \in C_l$.

(2) $\alpha \cap \delta \subseteq \alpha\, \mathrm{o} S\, \mathrm{o} \delta\ \forall\ \alpha \in C_b(C_q)$ and $\forall\, \delta \in C_r$.

The subsequent results extends Theorem 6 of [17] to the fuzzy context and its proofs can be derived analogously.

Theorem 6.10. A ternary semigroup *S* is regular if and only if $\alpha \cap \delta = \delta\, \mathrm{o} \alpha\, \mathrm{o} \delta\ \forall\, \alpha \in C_{lt}$ and $\forall\ \delta \in C_l(C_q)$.

The following theorem is an alternative proof of the theorem 3.14 of [7].

Theorem 6.11. A ternary semigroup *S* is regular if and only if $\alpha \cap S \cap \mu = \alpha\, \mathrm{o} S\, \mathrm{o} \mu$ $\forall \alpha \in C_r,\ \forall \delta \in C_{lt}$ and $\forall\ \mu \in C_l$.

Proof. As in Theorem 6.5, defined expression $L(a,b,c) = a \cap b \cap c$ and $M(a,b,c) = abc$. Since the classes C_r, C_b and C_l are projection closed, therefore, by metatheorem, $L(\alpha,\delta,\mu) = \mathrm{M}(\alpha,\delta,\mu)\ \forall \alpha \in C_r,\ \forall \delta \in C_{lt}$ and $\forall\, \mu \in C_l \Leftrightarrow L(\alpha,\delta,\mu)$ $L(\alpha,\delta,\mu) = \mathrm{M}(\alpha,\delta,\mu)\ \forall \alpha \in C_r,\ \forall \delta \in C_{lt}$ and $\forall\ \mu \in C_l$.

The later proposition follows as a ternary semigroup *S* is regular $\Leftrightarrow R \cap A \cap L = RAL$ for any right R, lateral ideal A and left ideals L of *S* by Theorem 3.7 of [3], $P(S) \cong C(S)$ under the isomorphism *Chi* and Proposition 2.6.

Following theorem is an extension of Theorem 3.19 of [3] to the fuzzy context and its proofs can be derived analogously.

Theorem 6.12. A ternary semigroup *S* is intra-regular if $\alpha \cap \mu \cap \delta \subseteq \alpha\, \mathrm{o} \mu\, \mathrm{o} \delta$ $\forall \alpha \in C_l,\ \forall \mu \in C_{lt}$ and $\forall\, \delta \in C_r$.

Theorem 6.13. A ternary semigroup *S* is regular if and only if $\forall\ \mu \in C_i,\ \mu\, \mathrm{o} \mu\, \mathrm{o} \mu = \mu$

Proof. Similar to Theorem 6.5, defined expressions $L(c) = c^3$ and $M(c) = c$. Since C_i is projection closed, therefore, by metatheorem, $L(\mu) = M(\mu)\ \forall\, \mu \in C_i$ if and only if $L(\mu) = M(\mu)\ \forall\, \mu \in C_i$.

The later oppositionfollows as a ternary semigroup *S* is a regular $\Leftrightarrow A^3 = A$ for any ideal A of *S* by Therefore 3.10 of [3], $P(S) \cong C(S)$ under the isomorphism *Chi* and Proposition 2.8.

Similarly we can prove.

Theorem 6.14. Let *S* be a regular ternary semiring. Then $\delta \circ \delta \circ \delta = \delta \circ \delta \circ \delta \circ \delta \circ \delta$ $\forall \delta \in C_b(C_q)$.

Theorem 6.15. A commutative ternary semigroup *S* is completely regular if and only if any fuzzy ideal of Sis fuzzy semiprime.

Proof. By Theorem 3.11 of [3], a commutative ternary semigroup S is completely regular $\Leftrightarrow$ any ideal of *S* is semiprime and $P(S) \cong C(S)$ under the isomorphism *Chi,* we get, $C_i \subseteq C_s$. Since both C_i and C_s are projection closed therefore by Proposition 2.13, $C_i \subseteq C_s$ if and only if $C_i \subseteq C_s$. Hence, $C_i \subseteq C_s$.

Following theorem is an extension of Theorem 3.10 and 3.17 of [3] to the fuzzy context an its p roofs can be derived analogously.

Theorem 6.16. A ternary semigroup *S* is completely regular if and only if $\mu \circ \mu \circ \mu = \mu \quad \forall \in C_i(C_b)$.

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