Weighted Lim's Geometric Mean of Positive Invertible Operators on a Hilbert Space

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Abstract

We generalize the weighted Lim's geometric mean of positive definite matrices to positive invertible operators on a Hilbert space. This mean is defined via a certain bijection map and parametrized over Hermitian unitary operators. We derive an explicit formula of the weighted Lim's geometric mean in terms of weighted metric/spectral geometric means. This kind of operator mean turns out to be a symmetric Lim-Pálfia weighted mean and satisfies the idempotency, the permutation invariance, the joint homogeneity, the self-duality, and the unitary invariance. Moreover, we obtain relations between weighted Lim geometric means and Tracy-Singh products via operator identities.

Keywords: positive invertible operator, metric geometric mean, spectral geometric mean, Lim's geometric mean, Tracy-Singh product
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1 Introduction

Recall that the Riccati equation for positive definite matrices A and B:

$$XA^{-1}X = B \tag{1}$$

has a unique positive solution

$$X = A\sharp B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}, \tag{2}$$

known as the metric geometric mean of A and B. This kind of mean was introduced by Ando [2] as the maximum (with respect to the Löwner partial

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order) of positive semidefinite matrices X satisfying

$$\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geqslant 0.$$

The above two definitions of the metric geometric mean are equivalent. See a nice discussion about the Riccati equation and the metric geometric mean of matrices in [4].

Fiedler and Pták [3] modified the notion of the metric geometric mean to the spectral geometric mean:

$$A \diamondsuit B \ = \ (A^{-1} \sharp B)^{\frac{1}{2}} A (A^{-1} \sharp B)^{\frac{1}{2}}. \tag{3}$$

One of the most important properties of the spectral geometric mean is the positive similarity between $(A \diamondsuit B)^2$ and AB. This shows that the eigenvalues of $A \diamondsuit B$ coincide with the positive square roots of the eigenvalues of AB.

Lee and Lim [5] introduced the notion of metric geometric means and spectral geometric means on symmetric cones of positive definite matrices and developed various properties of these means. Lim [6] provided a new geometric mean of positive definite matrices varying over Hermitian unitary matrices, including the metric geometric mean as a special case. The new mean has an explicit formula in terms of metric and spectral geometric means. He established basic properties of this mean including the idempotency, joint homogeneity, permutation invariance, non-expansiveness, self-duality, and a determinantal identity. He also gave this new geometric mean for the weighted case. Lim and Pálfia [7] presented a unified framework for weighted inductive means on the cone of positive definite matrices. The metric geometric mean, spectral geometric mean, and Lim geometric mean [6] are basic examples of the two-variable weighted mean.

In this paper, we extend the notion of weighted Lim's geometric mean [6] to the case of Hilbert-space operators via a certain bijection map (see Section 2). This operator mean is parametrized over Hermitian unitary operators. An explicit formula of the weighted Lim's geometric mean is given in term of weighted metric geometric means and spectral geometric means. This kind of operator mean serves the idempotency, the permutation invariance, the joint homogeneity, the self-duality, and the unitary invariance. Moreover, we establish certain operator identities involving Lim weighted geometric means and Tracy-Singh products (see Section 3). Our results include certain literature results involving weighted metric geometric means.

2 Lim's geometric mean of operators

In this section, we discuss the notion of Lim's geometric mean of positive invertible operators on any complex Hilbert space.

Throughout, let $\mathbb{H}$ be a complex Hilbert space. Denoted by $\mathfrak{B}(\mathbb{H})$ the Banach space of bounded linear operators on $\mathbb{H}$. The set of all positive invertible operators on $\mathbb{H}$ is denoted by $\mathbb{P}$.

First of all, we recall the notions of metric/spectral geometric means of operators. Recall that for any $t \in [0,1]$, the t-weighted metric geometric mean of $A, B \in \mathbb{P}$ is defined by

$$A \sharp_t B \;=\; A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}} \tag{4}$$

For briefly, we write $A \sharp B$ for $A \sharp_{1/2} B$. The spectral geometric mean of $A, B \in \mathbb{P}$ is defined by

$$A \diamondsuit B \;=\; (A^{-1} \sharp B)^{\frac{1}{2}} A (A^{-1} \sharp B)^{\frac{1}{2}}. \tag{5}$$

We list some basic properties of metric and spectral geometric means.

Lemma 1 (e.g. [1, 3, 4]). *Let $A, B \in \mathbb{P}$ and $t \in [0,1]$. Then*

(i) $A \sharp_t A = A$,

(ii) $(\alpha A) \sharp_t (\beta B) = \alpha^{1-t} \beta^t (A \sharp_t B)$,

(iii) $A \sharp_t B = B \sharp_{1-t} A$,

(iv) $(A \sharp_t B)^{-1} = A^{-1} \sharp_t B^{-1}$,

(v) *(Riccati Lemma) $A \sharp B$ is the unique positive invertible solution of $XA^{-1}X = B$,*

(vi) $(T^*AT) \sharp_t (T^*BT) = T^*(A \sharp_t B)T$ *for any invertible operator* $T \in \mathfrak{B}(\mathbb{H})$,

(vii) $(T^*AT) \diamondsuit (T^*BT) = T^*(A \diamondsuit B)T$ *for any unitary operator* $T \in \mathfrak{B}(\mathbb{H})$.

For a Hermitian unitary operator $U \in \mathfrak{B}(\mathbb{H})$, we set

$$\mathbb{P}_U^+ := \{X \in \mathbb{P} : UXU = X\}, \quad \mathbb{P}_U^- := \{X \in \mathbb{P} : UXU = X^{-1}\}$$

Lemma 2. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator. Then the map*

$$\Phi_U \;:\; \mathbb{P}_U^+ \times \mathbb{P}_U^- \;\to\; \mathbb{P}, \quad (A, B) \;\mapsto\; A^{\frac{1}{2}} B A^{\frac{1}{2}} \tag{6}$$

is bijective with the inverse map given by

$$X \;\mapsto\; (X \sharp (UXU), X \diamondsuit (UX^{-1}U)). \tag{7}$$

Proof. The proof is quite similar to [6, Theorem 2.6]. Let $A_1, A_2 \in \mathbb{P}_U^+$ and $B_1, B_2 \in \mathbb{P}_U^-$ such that $\Phi_U(A_1, B_1) = \Phi_U(A_2, B_2)$, i.e. $A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}} = A_2^{\frac{1}{2}} B_2 A_2^{\frac{1}{2}}$. Since $A_i \in \mathbb{P}_U^+$, we have

$$\begin{aligned} UA_i^{-1}U \;&=\; (UA_iU)^{-1} \;=\; A_i^{-1}, \\ UA_i^{\frac{1}{2}}U \;&=\; (UA_iU)^{\frac{1}{2}} \;=\; A_i^{\frac{1}{2}}. \end{aligned}$$

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and thus $A_i^{-1}, A_i^{\frac{1}{2}} \in \mathbb{P}_U^+$ for $i = 1, 2$. It follows that

$$\begin{aligned} B_1^{-1} &= UB_1U \\ &= U\Big(A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}B_2A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}}\Big)U \\ &= \big(UA_1^{-\frac{1}{2}}U\big)\big(UA_2^{\frac{1}{2}}U\big)\big(UB_2U\big)\big(UA_2^{\frac{1}{2}}U\big)\big(UA_1^{-\frac{1}{2}}U\big) \\ &= A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}B_2^{-1}A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}} \\ &= A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}\Big(A_2^{-\frac{1}{2}}A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}}A_2^{-\frac{1}{2}}\Big)^{-1}A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}} \\ &= \big(A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}\big)B_1^{-1}\big(A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}\big), \end{aligned}$$

i.e. $A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}$ is a solution of $XB_1^{-1}X = B_1^{-1}$. Since $B_1^{-1}\sharp B_1 = I$ is the unique solution of $XB_1^{-1}X = B_1^{-1}$ (Lemma 1 (v)), we conclude $A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}} = I$. This implies that $A_1 = A_2$ and then $B_1 = B_2$. Hence, Φ_U is injective. Next, let $X \in \mathbb{P}$. and consider $A = X\sharp(UXU)$ and $B = X\diamondsuit(UX^{-1}U) = A^{-\frac{1}{2}}XA^{-\frac{1}{2}}$. Consider

$$\begin{aligned} UAU &= U\big(X\sharp(UXU)\big)U = (UXU)\sharp(U^2XU^2) \\ &= (UXU)\sharp X = X\sharp(UXU) = A \end{aligned}$$

and

$$\begin{aligned} UBU &= U\big(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\big)U = \big(UA^{-\frac{1}{2}}U\big)(UXU)\big(UA^{-\frac{1}{2}}U\big) \\ &= A^{\frac{1}{2}}X^{-1}A^{\frac{1}{2}} = B^{-1}. \end{aligned}$$

This implies that $A \in \mathbb{P}_U^+$ and $B \in \mathbb{P}_U^-$. We have that there exist $A \in \mathbb{P}_U^+$ and $B \in \mathbb{P}_U^-$ such that $\Phi_U(A, B) = A^{\frac{1}{2}}BA^{\frac{1}{2}} = X$. Thus, Φ_U is surjective. Therefore Φ_U is bijective. □

By the bijectivity of Φ_U, we can define the t-weighted Lim geometric mean of operators as follows:

Definition 3. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0, 1]$. Let $X = \Phi_U(A_1, B_1), Y = \Phi_U(A_2, B_2) \in \mathbb{P}$. The t-weighted Lim geometric mean of X and Y is defined by*

$$\mathcal{G}_U(t; X, Y) = \Phi_U(A_1\sharp_t A_2, B_1\sharp_t B_2). \tag{8}$$

We denote $\mathcal{G}_U(X, Y) = \mathcal{G}_U(1/2; X, Y)$ the Lim geometric mean.

The next theorem gives an explicit formula of $\mathcal{G}_U(X, Y)$.

Theorem 4. *Let U be a Hermitian unitary operator and $t \in [0, 1]$. Let $X, Y \in \mathbb{P}$. We have*

$$\mathcal{G}_U(t; X, Y) = (A_1\sharp_t A_2)^{\frac{1}{2}}(B_1\sharp_t B_2)(A_1\sharp_t A_2)^{\frac{1}{2}}, \tag{9}$$

where $A_1 = X\sharp(UXU), A_2 = Y\sharp(UYU), B_1 = X\diamondsuit(UX^{-1}U)$ and $B_2 = Y\diamondsuit(UY^{-1}U)$. In particular, $\mathcal{G}_I(X, Y) = X\sharp_t Y$.

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Proof. Since f_U is surjective, there exist $A_1, A_2 \in \mathbb{P}_U^+$ and $B_1, B_2 \in \mathbb{P}_U^-$ such that $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. By using the inverse map (7), we have

$$\begin{aligned}(A_1, B_1) &= \Phi_U^{-1}(X) = (X\sharp(UXU), X\diamondsuit(UX^{-1}U))\\ (A_2, B_2) &= \Phi_U^{-1}(Y) = (Y\sharp(UYU), Y\diamondsuit(UY^{-1}U)).\end{aligned}$$

For the case $U = I$, we have $\mathbb{P}_I^+ = \mathbb{P}$ and $\mathbb{P}_I^- = \{I\}$. It follows that $B_1 = B_2 = I$. By Lemma 1, we have $A_1 = X\sharp X = X$ and $A_2 = Y\sharp Y = Y$. Hence,

$$\mathcal{G}_I(t; X, Y) = (X\sharp_t Y)^{\frac{1}{2}}(I\sharp_t I)(X\sharp_t Y)^{\frac{1}{2}} = X\sharp_t Y. \qquad \square$$

Now, we give the definition of the Lim-Pálfia weighted mean [7] in the case of operators.

Definition 5. *The (two-variable) Lim-Pálfia weighted mean of positive invertible operators is the map* $\mathbb{M} : [0, 1] \times \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ *satisfying*

(i) $\mathbb{M}(0, X, Y) = X$,

(ii) $\mathbb{M}(1, X, Y) = Y$,

(iii) (Idempotency) $\mathbb{M}(t, X, X) = X$ *for all* $t \in [0, 1]$.

We say that $\mathbb{M}$ *is symmetric if*

(iv) (Permutation invariancy) $\mathbb{M}(t, X, Y) = \mathbb{M}(1 - t, Y, X)$ *for all* $t \in [0, 1]$.

Theorem 6. *The t-weighted Lim geometric mean of operators is a symmetric Lim-Pálfia weighted mean.*

Proof. Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0, 1]$. Let $X, Y \in \mathbb{P}$. Write $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. We have by Lemma 1 that

$$\begin{aligned}\mathcal{G}_U(0; X, Y) &= \Phi_U(A_1\sharp_0 A_2, B_1\sharp_0 B_2) = \Phi_U(A_1, B_1) = X,\\ \mathcal{G}_U(1; X, Y) &= \Phi_U(A_1\sharp_1 A_2, B_1\sharp_1 B_2) = \Phi_U(A_2, B_2) = Y,\\ \mathcal{G}_U(t; X, X) &= \Phi_U(A_1\sharp_t A_1, B_1\sharp_t B_1) = \Phi_U(A_1, B_1) = X.\end{aligned}$$

This implies that $\mathcal{G}_U$ is a Lim-Pálfia weighted mean. Using Lemma 1 again, we get

$$\begin{aligned}\mathcal{G}_U(t; X, Y) &= \Phi_U(A_1\sharp_t A_2, B_1\sharp_t B_2)\\ &= \Phi_U(A_2\sharp_{1-t} A_1, B_2\sharp_{1-t} B_1)\\ &= \mathcal{G}_U(1 - t; Y, X).\end{aligned}$$

Thus, $\mathcal{G}_U$ is symmetric. $\square$

Corollary 7. *The t-weighted metric geometric mean of operators is a symmetric Lim-Pálfia weighted mean.*

Theorem 8. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0,1]$. Let $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. We have*

(i) $\mathcal{G}_U(t; X, I) = \Phi_U(A_1^{1-t}, B_1^{1-t})$ *and* $\mathcal{G}_U(t; I, Y) = \Phi_U(A_2^t, B_2^t)$*,*

(ii) (Joint Homogeneity) $\mathcal{G}_U(t; \alpha X, \beta Y) = \alpha^{1-t}\beta^t \mathcal{G}_U(t; X, Y)$ *for any* $\alpha, \beta > 0$*,*

(iii) (Self-duality) $\mathcal{G}_U(t; X, Y)^{-1} = \mathcal{G}_U(t; X^{-1}, Y^{-1})$*,*

(iv) (Unitary invariance) $\mathcal{G}_U(t; T^*XT, T^*YT) = T^*\mathcal{G}_U(t; X, Y)T$ *where* $T \in \mathfrak{B}(\mathbb{H})$ *is a unitary operator such that* $UT = TU$*,*

(v) $\mathcal{G}_U(t; UXU, UYU) = U\mathcal{G}_U(t; X, Y)U$*,*

(vi) $\mathcal{G}_U(X, X^{-1}) = I$*.*

Proof. The first assertion is immediate from Definition 3. For the joint homogeneity, note that

$$\alpha X \;=\; \alpha \Phi_U(A_1, B_1) \;=\; \alpha\big(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\big) \;=\; A_1^{\frac{1}{2}}(\alpha B_1)A_1^{\frac{1}{2}} \;=\; \Phi_U(A_1, \alpha B_1).$$

Similarly, $\beta Y = \Phi_U(A_2, \beta B_2)$. Using Lemma 1, we obtain

$$\begin{aligned} \mathcal{G}_U(t; \alpha X, \beta Y) &= \Phi_U(A_1 \sharp_t A_2, (\alpha B_1)\sharp_t(\beta B_2)) \;=\; \Phi_U(A_1\sharp_t A_2, \alpha^{1-t}\beta^t(B_1\sharp_t B_2)) \\ &= \alpha^{1-t}\beta^t \Phi_U(A_1\sharp_t A_2, B_1\sharp_t B_2) \;=\; \alpha^{1-t}\beta^t \mathcal{G}_U(t; X, Y). \end{aligned}$$

For the self-duality, consider

$$X^{-1} = \Phi_U(A_1, B_1)^{-1} = \Big(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\Big)^{-1} = A_1^{-\frac{1}{2}} B_1^{-1} A_1^{-\frac{1}{2}} = \Phi_U(A_1^{-1}, B_1^{-1}).$$

Similarly, $Y^{-1} = \Phi_U(A_2^{-1}, B_2^{-1})$. Applying Lemma 1, we get

$$\begin{aligned} \mathcal{G}_U(t; X, Y)^{-1} &= \Phi_U(A_1\sharp_t A_2, B_1\sharp_t B_2)^{-1} \;=\; \Phi_U((A_1\sharp_t A_2)^{-1}, (B_1\sharp_t B_2)^{-1}) \\ &= \Phi_U(A_1^{-1}\sharp_t A_2^{-1}, B_1^{-1}\sharp_t B_2^{-1}) \;=\; \mathcal{G}_U(t; X^{-1}, Y^{-1}) \end{aligned}$$

Now, let us prove the assertion (iv). Since T is unitary, we have by Lemma 1 that

$$\begin{aligned} (T^*XT)\sharp[U(T^*XT)U] &= (T^*XT)\sharp[T^*(UXU)T] \\ &= T^*[X\sharp(UXU)]T \\ &= T^*A_1T, \\ (T^*XT)\diamondsuit[U(T^*XT)^{-1}U] &= (T^*XT)\diamondsuit[UTX^{-1}T^*U] \\ &= (T^*XT)\diamondsuit[T^*(UX^{-1}U)T] \\ &= T^*[X\diamondsuit(UX^{-1}U)]T \\ &= T^*B_1T. \end{aligned}$$

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Similarly,

$$(T^*YT)\sharp[U(T^*YT)U] \;=\; T^*A_2T \quad \text{and} \quad (T^*YT)\diamondsuit[U(T^*YT)^{-1}U] \;=\; T^*B_2T$$

Then $T^*XT = \Phi_U(T^*A_1T, T^*B_1T)$ and $T^*YT = \Phi_U(T^*A_2T, T^*B_2T)$. Thus

$$\begin{aligned}
&\mathcal{G}_U(t; T^*XT, T^*YT) \\
&\quad= [(T^*A_1T)\sharp_t(T^*A_2T)]^{\frac{1}{2}}[(T^*B_1T)\sharp_t(T^*B_2T)][(T^*A_1T)\sharp_t(T^*A_2T)]^{\frac{1}{2}} \\
&\quad= [T^*(A_1\sharp_tA_2)T]^{\frac{1}{2}}[T^*(B_1\sharp_tB_2)T][T^*(A_1\sharp_tA_2)T]^{\frac{1}{2}} \\
&\quad= [T^*(A_1\sharp_tA_2)^{\frac{1}{2}}T][T^*(B_1\sharp_tB_2)T][T^*(A_1\sharp_tA_2)^{\frac{1}{2}}T] \\
&\quad= T^*(A_1\sharp_tA_2)^{\frac{1}{2}}(B_1\sharp_tB_2)(A_1\sharp_tA_2)^{\frac{1}{2}}T \\
&\quad= T^*\mathcal{G}_U(t; X, Y)T.
\end{aligned}$$

Setting $T = U$, we get the result in the assertion (v). For the last assertion, since $X^{-1} = \Phi_U(A_1^{-1}, B_1^{-1})$, we have

$$\mathcal{G}_U(X, X^{-1}) \;=\; \Phi_U(A_1\sharp A_1^{-1}, B_1\sharp B_1^{-1}) \;=\; \Phi_U(I, I) \;=\; I. \qquad \square$$

3 Weighted Lim geometric means and Tracy-Singh products

In this section, we investigate relations between Weighted Lim geometric means and Tracy-Singh products of operators. Let us recalling the notion of Tracy-Singh product.

3.1 Preliminaries on the Tracy-Singh product of operators

The projection theorem for Hilbert space allows us to decompose

$$\mathbb{H} \;=\; \bigoplus_{i=1}^{n} \mathbb{H}_i \tag{10}$$

where all $\mathbb{H}_i$ are Hilbert spaces. For each $i = 1, \ldots, n$, let P_i be the natural projection map from $\mathbb{H}$ onto $\mathbb{H}_i$. Each operator $A \in \mathfrak{B}(\mathbb{H})$ can be uniquely determined by an operator matrix

$$A \;=\; [A_{ij}]_{i,j=1}^{n,n}$$

where $A_{ij} : \mathbb{H}_j \to \mathbb{H}_i$ is defined by $A_{ij} = P_iAP_j^*$ for each $i, j = 1 \ldots, n$.

Recall that the tensor product of $A, B \in \mathfrak{B}(\mathbb{H})$ is the operator $A \otimes B \in \mathfrak{B}(\mathbb{H} \otimes \mathbb{H})$ such that for all $x, y \in \mathbb{H}$,

$$(A \otimes B)(x \otimes y) \;=\; (Ax) \otimes (By).$$

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Definition 9. *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ and $B = [B_{ij}]_{i,j=1}^{n,n}$ be operators in $\mathfrak{B}(\mathbb{H})$. The Tracy-Singh product of A and B is defined to be*

$$A \boxtimes B \ = \ \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij} \tag{11}$$

which is a bounded linear operator from $\bigoplus_{i,j=1}^{n,n} \mathbb{H}_i \otimes \mathbb{H}_j$ into itself.

Lemma 10 ([9, 10, 11])**.** *Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})$.*

(i) $(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.

(ii) If $A, B \in \mathbb{P}$, then $A \boxtimes B \in \mathbb{P}$.

(iii) If $A, B \in \mathbb{P}$, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any $\alpha \in \mathbb{R}$.

(iv) If A and B are Hermitian, then $A \boxtimes B$ is also.

(v) If A and B are unitary, then $A \boxtimes B$ is also.

Lemma 11 ([8])**.** *Let $A, B, C, D \in \mathbb{P}$ and $t \in [0, 1]$. Then*

$$(A \boxtimes B)\sharp_t(C \boxtimes D) \ = \ (A\sharp_t C) \boxtimes (B\sharp_t D).$$

For each $i = 1, \dots, k$, let $\mathbb{H}_i$ be a Hilbert space and decompose

$$\mathbb{H}_i = \bigoplus_{r=1}^{n_i} \mathbb{H}_{i,r}$$

where all $\mathbb{H}_{i,r}$ are Hilbert spaces. We set $\boxtimes_{i=1}^{1} A_i = A_1$. For $k \in \mathbb{N} - \{1\}$ and $A_i \in \mathfrak{B}(\mathbb{H}_i)$ $(i = 1, \dots, k)$, we use the notation

$$\mathop{\boxtimes}_{i=1}^{k} A_i \ = \ ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{k-1}) \boxtimes A_k.$$

3.2 The compatibility between weighted Lim geometric means and Tracy-Singh products

The following theorem provides an operator identity involving t-weighted Lim geometric means and Tracy-Singh products.

Theorem 12. *Let U, V be Hermitian unitary operators, $X_1, X_2, Y_1, Y_2 \in \mathbb{P}$ and $t \in [0, 1]$.*

$$\mathcal{G}_U(t; X_1, Y_1) \boxtimes \mathcal{G}_V(t; X_2, Y_2) \ = \ \mathcal{G}_{U \boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2). \tag{12}$$

Proof. Write

$$X_1 = \Phi_U(A_1, B_1), \quad Y_1 = \Phi_U(C_1, D_1), \quad X_2 = \Phi_V(A_2, B_2), \quad Y_2 = \Phi_V(C_2, D_2),$$

where $A_1, C_1 \in \mathbb{P}_U^+$, $B_1, D_1 \in \mathbb{P}_U^-$, $A_2, C_2 \in \mathbb{P}_V^+$, $B_2, D_2 \in \mathbb{P}_V^-$. Since U and V are Hermitian unitary operators, we have by Lemma 10 that $U \boxtimes V$ is also a Hermitian unitary operator. Thus $\mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2)$ is well-defined. By Lemma 10, we get

$$(U \boxtimes V)(A_1 \boxtimes A_2)(U \boxtimes V) \;=\; (UA_1U) \boxtimes (VA_2V) \;=\; A_1 \boxtimes A_2$$

and

$$\begin{aligned}(U \boxtimes V)(B_1 \boxtimes B_2)(U \boxtimes V) \;&=\; (UB_1U) \boxtimes (VB_2V) \;=\; B_1^{-1} \boxtimes B_2^{-1} \\ &=\; (B_1 \boxtimes B_2)^{-1}.\end{aligned}$$

Thus $A_1 \boxtimes A_2 \in \mathbb{P}_{U\boxtimes V}^+$ and $B_1 \boxtimes B_2 \in \mathbb{P}_{U\boxtimes V}^-$. Similarly, we have $C_1 \boxtimes C_2 \in \mathbb{P}_{U\boxtimes V}^+$ and $D_1 \boxtimes D_2 \in \mathbb{P}_{U\boxtimes V}^-$. Using Lemma 10, we get

$$\begin{aligned}X_1 \boxtimes X_2 \;&=\; \Phi_U(A_1, B_1) \boxtimes \Phi_V(A_2, B_2) \\ &=\; \big(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\big) \boxtimes \big(A_2^{\frac{1}{2}} B_2 A_2^{\frac{1}{2}}\big) \\ &=\; (A_1^{\frac{1}{2}} \boxtimes A_2^{\frac{1}{2}})(B_1 \boxtimes B-2)(A_1^{\frac{1}{2}} \boxtimes A_2^{\frac{1}{2}}) \\ &=\; (A_1 \boxtimes A_2)^{\frac{1}{2}}(B_1 \boxtimes B_2)(A_1 \boxtimes A_2)^{\frac{1}{2}} \\ &=\; \Phi_{U\boxtimes V}(A_1 \boxtimes A_2, B_1 \boxtimes B_2).\end{aligned}$$

Similarly, $Y_1 \boxtimes Y_2 = \Phi_{U\boxtimes V}(C_1 \boxtimes C_2, D_1 \boxtimes D_2)$. Then

$$\mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2) \;=\; \Phi_{U\boxtimes V}\big((A_1 \boxtimes A_2)\sharp_t(C_1 \boxtimes C_2), (B_1 \boxtimes B_2)\sharp_t(D_1 \boxtimes D_2)\big).$$

We have by applying Lemmas 10 and 11 that

$$\begin{aligned}&\mathcal{G}_U(t;X_1, Y_1) \boxtimes \mathcal{G}_V(t; X_2, Y_2) \\ &\quad=\; \Phi_U(A_1\sharp_t C_1, B_1\sharp_t D_1) \boxtimes \Phi_V(A_1\sharp_t C_2, B_2\sharp_t D_2) \\ &\quad=\; \big[(A_1\sharp_t C_1)^{\frac{1}{2}}(B_1\sharp_t D_1)(A_1\sharp_t C_1)^{\frac{1}{2}}\big] \boxtimes \big[(A_2\sharp_t C_2)^{\frac{1}{2}}(B_2\sharp_t D_2)(A_2\sharp_t C_2)^{\frac{1}{2}}\big] \\ &\quad=\; \big[(A_1\sharp_t C_1)^{\frac{1}{2}} \boxtimes (A_2\sharp_t C_2)^{\frac{1}{2}}\big]\big[(B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big]\big[(A_1\sharp_t C_1)^{\frac{1}{2}} \boxtimes (A_2\sharp_t C_2)^{\frac{1}{2}}\big] \\ &\quad=\; \big[(A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2)\big]^{\frac{1}{2}}\big[(B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big]\big[(A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2)\big]^{\frac{1}{2}} \\ &\quad=\; \Phi_{U\boxtimes V}\big((A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2), (B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big) \\ &\quad=\; \Phi_{U\boxtimes V}\big((A_1 \boxtimes A_2)\sharp_t(C_1 \boxtimes C_2), (B_1 \boxtimes B_2)\sharp_t(D_1 \boxtimes D_2)\big) \\ &\quad=\; \mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2).\end{aligned}$$

□

Corollary 13. *Let $k \in \mathbb{N}$ and $t \in [0, 1]$. For each $1 \leqslant i \leqslant k$, let $U_i \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $X_i, Y_i \in \mathbb{P}$. Then*

$$\mathop{\boxtimes}_{i=1}^{k} \mathcal{G}_{U_i}(t; X_i, Y_i) \;=\; \mathcal{G}_U\left(t; \mathop{\boxtimes}_{i=1}^{k} X_i, \mathop{\boxtimes}_{i=1}^{k} Y_i\right), \tag{13}$$

where $U = \boxtimes_{i=1}^{k} U_i$.

Proof. Since U_i is a Hermitian unitary operator for all $i = 1, \dots, k$, we have by Lemma 10 that $\boxtimes_{i=1}^{k} U_i$ is also. Using the positivity of the Tracy-Singh product, we get $\boxtimes_{i=1}^{k} X_i, \boxtimes_{i=1}^{k} Y_i \in \mathbb{P}$. Hence, the right hand side of (13) is well-defined. We reach the result by applying Thoerem 12 and induction on k. ☐

From Corollary 13, setting $U_i = I$ for all $i = 1, \dots, k$, we have

$$\boxtimes_{i=1}^{k} (X_i \sharp_t Y_i) = \left(\boxtimes_{i=1}^{k} X_i \right) \sharp_t \left(\boxtimes_{i=1}^{k} Y_i \right).$$

This equality were proved already in [8, Corollary 1].

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References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, Linear Algebra Appl. 26, 203–241 (1979), DOI: 10.1016/0024-3795(79)90179-4.

[2] T. Ando, Topics on Operator Inequalities. Hokkaido Univ., Sapporo (1978).

[3] M. Fiedler and V. Pták, A new positive definite geometric mean of two positive definite matrices, Linear Algebra Appl. 251, 1–20 (1997), DOI: 10.1016/0024-3795(95)00540-4.

[4] J.D. Lawson, and Y. Lim, The geometric mean, matrices, metrics, and more, Amer. Math. Monthly 108, 797–812 (2001), DOI: 10.2307/2695553.

[5] H. Lee and Y. Lim, Metric and spectral geometric means on symmetric cones, Kyungpook Math. J. 47(1), 133–150 (2007).

[6] Y. Lim, Factorizations and geometric means of positive definite matrices, Linear Algebra Appl. 437(9), 2159–2172 (2012), DOI: 10.1016/j.laa.2012.05.039.

[7] Y. Lim and M. Pálfia, Weighted inductive means, Linear Algebra Appl. 45, 59–83 (2014), DOI: 10.1016/j.laa.2014.04.002.

[8] A. Ploymukda and P. Chansangiam, Geometric means and Tracy-Singh products for positive operators, Communications in Mathematics and Applications 9(4), 475–488 (2018), DOI: 10.26713/cma.v9i4.547.

[9] A. Ploymukda, P. Chansangiam and W. Lewkeeratiyutkul, Algebraic and order properties of Tracy-Singh products for operator matrices, J. Comput. Anal. Appl. 24(4), 656–664 (2018).

[10] A. Ploymukda, P. Chansangiam and W. Lewkeeratiyutkul, Analytic properties of Tracy-Singh products for operator matrices, J. Comput. Anal. Appl. 24(4), 665–674 (2018).

[11] A. Ploymukda, P. Chansangiam and W. Lewkeeratiyutkul, Tracy-Singh products and classes of operators, J. Comput. Anal. Appl. 26(8), 1401–1413 (2019).