Edge Domination of an Involutory Addition Cayley Graph

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ABSTRACT
Graph theory is one of the most advanced branches of discrete mathematics with variety of applications to different branches of Science and Technology. For a positive integer $n > 1$, the involutory addition Cayley graph $G^+(Z_n,I_v)$, is the graph whose vertex set is $Z_n = \{0,1,2,3,\dots,n-1\}$ and edge set $E(G^+(Z_n,I_v)) = \{xy\ /x,y \epsilon Z_n, x+y \epsilon I_v\}$, where $I_v = \{x \epsilon Z_n : x^2 \equiv 1(mod\ n)\}$ is the set of involutory elements of Z_n.By taking Involutory Addition Cayley Graphs $G^+(Z_n,I_v)$ the author's evaluated graph related Edge domination numbers and Total edge domination numbers. In this paper, Edge domination number, Total edge domination number of the Involutory addition cayley graphs $G^+(Z_n,I_v)$ were discussed.

Keywords: Involutory addition cayley graphs, Edge dominating sets, Total edge dominating sets, Edge Domination number, Total edge domination number

1. INTRODUCTION

A graph G(V, E) is a mathematical object that may be thought of as a collection of edges and a set of vertices that connect any or all of the vertices. In a graph G, two vertices are considered neighboring if an edge joins them; otherwise, the edge is considered non-adjacent. We indicate that a graph G has V (G) vertices and E(G) edges, accordingly. The cardinality of V (G) is the definition of the order of G. The cardinality of E(G) is represented by |E|, and that of V (G) by |V |. The number of edges that occur with a vertex v in a graph G is known as its degree, or deg(v). Anusha et al. [14] defined Involutory Cayley graph and some properties were discussed and some more properties were discussed by Prameela et al. [10, 11]. Involutory Addition Cayley Graph was introduced by Shanmuga Priya et al.[12, 13] and studied some properties and topological indices on this graph.

For a positive integer $n > 1$, the involutory addition Cayley graph $G^+(Z_n,I_v)$, is the graph whose vertex set is $Z_n = \{0,1,2,3,\dots,n-1\}$ and edge set $E(G_n) = \{xy\ /x,y \epsilon Z_n, x+y \epsilon I_v\}$, where $I_v = \{x \epsilon Z_n : x^2 \equiv 1(mod\ n)\}$ is the set of involutory elements of Z_n.

The theory of total domination in graphs was formalized by Cockayne et al. [4]. An extensive study of a relationship between domination and total domination of a graph G with no isolated vertices can be done by Bollobas [3].Dominating sets of edges were studied by Mitchell and Hedetniemi [6].Total edge dominating sets were introduced by Kulli and Patwari [15].

2. EDGE DOMINATION NUMBER OF A GRAPH

The theory of domination in graphs was introduced in 1958 by Claude Berge[2] in which he used the concept 'coefficient of external stability' to refer the domination number of a graph. In 1962, Oystein Ore[9] wrote another book on graph theory, in which he studied the concept of domination using the terms 'dominating set' and 'domination number' with notation d(G) for the first time. Cockayne et al. [5] discussed the review of results and applications concerning dominating sets in graphs.Dominating sets of edges were studied by Mitchell and Hedetniemi[6].

A set$F$of edges in a graph $G(V,E)$ is called an edge dominating set of G if every edge in $E-F$ is adjacent to atleast one edge in F. Equivalently, a set F of edges in G is called an edge dominating set of G if for every edge $e \epsilon E-F$, there exist an edge $e_1 \epsilon F$ such that e and e_1 have a vertex in common. The edge domination number $\gamma'(G)$ of a graph$G$ is the minimum cardinality of an edge dominating set of G

Theorem 2.1: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is even, $n > 2$ and $|I_v| = 2$, the edge domination number is

$$\gamma'(G^+(Z_n, I_v)) = \begin{cases} \frac{n}{3} & if\ n\ is\ divisible\ by\ 3 \\ \frac{n+2}{3} & if\ n+2\ is\ divisible\ by\ 3 \\ \frac{n+1}{3} & if\ n+2\ is\ divisible\ by\ 3 \end{cases}$$

Proof: Consider a graph$G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \dots, n-1\}$ where I_v denotes the set of involutory elements in Z_n.Let n be even and $n > 2$.

Case (i) : Let $n = 3s$, $|I_v| = 2$. Then the graph $G^+(Z_n, I_v)$ is a Hamilton cycle and $|E| = n$.

Consider a set $F = \{e_i / i = 3s, s \geq 1\}$.

Since 3 divides n, then every edge in $E - F$ adjacent to atleast one edge in F. Then F becomes dominating set of $G^+(Z_n, I_v)$ and it is minimum.Therefore $|F| = \frac{|E|}{3} = \frac{n}{3}$.

Hence $\gamma'(G^+(Z_n, I_v)) = \frac{n}{3}$.

Case (ii) : Let $n = 3s + 1$, $|I_v| = 2$. Then the graph $G^+(Z_n, I_v)$ is a Hamilton cycle and $|E| = n$. Consider a set $F_1 = \{e_i / i = 3s, s \geq 1\}$, $|F_1| = \frac{|E|-1}{3}$. Since 3 does not divides n, $F_2 = \{e_1\}$, $|F_2| = 1$.

Consider$F = F_1 \cup F_2$. Now every edge in $E - F$ adjacent to atleast one edge in F. Then F becomes an edge dominating set of $G^+(Z_n, I_v)$ and it is minimum.

$$|F| = |F_1| + |F_2| = \frac{(|e| - 1)}{3} + 1 = \frac{(n-1)}{3} + 1 = \frac{n-1+3}{3} = \frac{n+2}{3}.$$

Hence $\gamma'(G^+(Z_n, I_v)) = \frac{n+2}{3}$

Case (iii) : Let $n = 3s + 2$, $|I_v| = 2$. Then the graph $G^+(Z_n, I_v)$ is a Hamilton cycle and $|E| = n$. Consider a set $F_1 = \{e_i / i = 3s, s \geq 1\}$, $|F_1| = \frac{|E|-2}{3}$.

Since 3 does not divides n, $F_2 = \{e_1\}$, $|F_2| = 1$.

Consider$F = F_1 \cup F_2$. Now every edge in $E - F$ adjacent to atleast one edge in F.Then F becomes an edge dominating set of $G^+(Z_n, I_v)$ and it is minimum.

$$|F| = |F_1| + |F_2| = \frac{(|e| - 2)}{3} + 1 = \frac{(n-2)}{3} + 1 = \frac{n-2+3}{3} = \frac{n+1}{3}.$$

Hence $\gamma'(G^+(Z_n, I_v)) = \frac{n+1}{3}$

Theorem 2.2: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is even, $n > 2$ and $|I_v| = 4$, the edge domination number is

$$\gamma'(G^+(Z_n, I_v)) = \begin{cases} \frac{n}{2} & if\ n\ is\ divisible\ by\ 2 \\ \frac{n}{3} & if\ n\ is\ divisible\ by\ 3 \\ \frac{n-2}{2} & if\ n-2\ is\ divisible\ by\ 2 \end{cases}.$$

Proof: Consider a graph$G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \dots, n-1\}$ where I_v denotes the set of involutory elements in Z_n.

Let n be even, $|I_v| = 4$ and $n > 2$. These graphs contains two Hamilton cycles, each cycle contains n number of edges.

Case (i) : Let $n = 3s + 2$,$|E| = 2n$. From first Hamilton cycle, consider a set $F_1 = \{e_i / i = 3s, s \geq 1\}$, $|F_1| = \frac{|E|-2}{3} = \frac{n-2}{3}$.

Since 3does not divides n,$F_2 = \{e_1\}$, $|F_2| = 1$.
From second Hamilton cycle, consider
$F_3 = \{e_i / e_i \notin F_1 \cup F_2 \text{ and not adjacent to any edge in } F_1 \cup F_2\}, |F_3| = \frac{n-2}{6}$.
Denote $F = F_1 \cup F_2 \cup F_3$. then every edge in $E - F$ adjacent to atleast one edge in F. Then F becomes dominating set of $G^+(Z_n, I_v)$ and it is minimum.
Therefore $|F| = |F_1| + |F_2| + |F_3| = \frac{n-2}{3} + 1 + \frac{n-2}{6} = \frac{2n-4+6+n-2}{6} = \frac{3n}{6} = \frac{n}{2}$.
Hence $\gamma'(G^+(Z_n, I_v)) = \frac{n}{2}$
Case (ii) : Let $n = 3s$, From first Hamilton cycle, Since 3 divides n, consider a set
$F = \{e_i / \ i = 3s, s \geq 1\}$
then every edge in $E - F$ adjacent to atleast one edge in F. Then F becomes dominating set of $G^+(Z_n, I_v)$ and it is minimum.
Therefore $|F| = \frac{|E|}{3} = \frac{n}{3}$ and hence $\gamma'(G^+(Z_n, I_v)) = \frac{n}{3}$.
Case (iii) : Let $n = 3s + 1$, From first Hamilton cycle, consider a set $F_1 = \{e_i / \ i = 3s, s \geq 1\}$, $|F_1| = \frac{|e|-1}{3} = \frac{n-1}{3}$. Since 3 does not divides n, $F_2 = \{e_1\}$, $|F_2| = 1$.
From second Hamilton cycle, Consider
$F_3 = \{e_i / e_i \notin F_1 \cup F_2 \text{ and not adjacent to any edge in } F_1 \cup F_2\}, |F_3| = \frac{n-10}{6}$.
Denote$F = F_1 \cup F_2 \cup F_3$. Now every edge in $E - F$ adjacent to atleast one edge in F.
Then F becomes an edge dominating set of $G^+(Z_n, I_v)$ and it is minimum.

$$|F| = |F_1| + |F_2| + |F_3| = \frac{(n-1)}{3} + 1 + \frac{n-10}{6} = \frac{2n-2+6+n-10}{6} = \frac{n-2}{2}.$$

Hence $\gamma'(G^+(Z_n, I_v)) = \frac{n-2}{2}$.

Theorem 2.3: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is odd and nis either prime or prime power, $|I_v| = 2$, the edge domination number is

$$\gamma'(G^+(Z_n, I_v)) = \begin{cases} \frac{n+1}{3} & \text{if } n+1 \text{ is divisible by } 3 \\ \frac{n-1}{3} & \text{if } n-1 \text{ is divisible by } 3 \\ \frac{n}{3} & \text{if } n \text{ is divisible by } 3 \end{cases}$$

Proof: Consider a graph$G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \ldots, n-1\}$ where I_v denotes the set of involutory elements in Z_n and n is odd prime or odd prime power$|I_v| = 2$ and $|E| = n - 1$.These graphs are paths, each path contains $n - 1$ number of edges.
Case (i) : Let $n = 3s - 1$,Consider a set $F_1 = \{e_i / \ i = 3s, s \geq 1\}$, $|F_1| = \frac{n-1-1}{3} = \frac{n-2}{3}$.
Since 3 does not divides n,$F_2 = \{e_1\}$, $|F_2| = 1$.
Denote $F = F_1 \cup F_2$, then every edge in $E - F$ adjacent to atleast one edge in F. Then F becomes dominating set of $G^+(Z_n, I_v)$ and it is minimum.
Therefore $|F| = |F_1| + |F_2| = \frac{n-2}{3} + 1 = \frac{n-2+3}{3} = \frac{n+1}{3}$ and hence $\gamma'(G^+(Z_n, I_v)) = \frac{n+1}{3}$.
Case (ii) : Let $n = 3s + 1$, $|E| = 3s$. Since 3 divides $n - 1$,Consider a set
$F = \{e_i / \ i = 3s, s \geq 1\}$
then every edge in $E - F$ adjacent to atleast one edge in F. Then F becomes dominating set of $G^+(Z_n, I_v)$ and it is minimum.
Therefore $|F| = \frac{|E|}{3} = \frac{n-1}{3}$ and hence $\gamma'(G^+(Z_n, I_v)) = \frac{n-1}{3}$.
Case (iii) : Let $n = 3s$, $|E| = 3s + 2$.

Consider a set $F_1 = \{e_i / i = 3s, s \geq 1\}, |F_1| = \frac{n-1-2}{3} = \frac{n-3}{3}$.
Since 3 does not divides $n, F_2 = \{e_1\}, \ |F_2| = 1$.
Denote$F = F_1 \cup F_2$ .Now every edge in $E - F$ adjacent to atleast one edge in F.Then F becomes an edge dominating set of $G^+(Z_n, I_v)$ and it is minimum.

$$|F| = |F_1| + |F_2| = \frac{(n-3)}{3} + 1 = \frac{n}{3}.$$

Hence $\gamma'\left(G^+(Z_n, I_v)\right) = \frac{n}{3}$

Theorem 2.4: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is neither odd prime nor odd prime power, $|I_v| = 4$, the edge domination number is (first vertex of degree $I_v - 1$ should be an even vertex)$\gamma'\left(G^+(Z_n, I_v)\right) = \frac{n-3}{2} \qquad if\ n-3\ is\ divisible\ by\ 2$.

Proof: Consider a graph$G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \dots, n-1\}$ where I_v denotes the set of involutory elements in Z_n and n is neither odd prime nor odd prime power
$|I_v| = 4$ and $|E| = 2n - 2$. These graphs contains two individual paths, each path contains $n - 1$ number of edges.
Let $n = 3s, |E| = 3s + 1$, From first path, consider$F_1 = \{e_i / i = 3s, s \geq 1\}, |F_1| = \frac{n-1-2}{3} = \frac{n-3}{3}$.
Since $n - 1$ not divisible by $3, F_2 = \{e_1\}, \ |F_2| = 1$.
From second path, consider

$$F_3 = \{e_i / e_i \notin F_1 \cup F_2\ and\ not\ adjacent\ to\ any\ edge\ in\ F_1 \cup F_2\}, |F_3| = \frac{n-9}{6}.$$

Denote$F = F_1 \cup F_2 \cup F_3$. Now every edge in $E - F$ adjacent to atleast one edge in F. Then F becomes an edge dominating set of $G^+(Z_n, I_v)$ and it is minimum.

$$|F| = |F_1| + |F_2| + |F_3| = \frac{n-3}{3} + 1 + \frac{n-9}{6} = \frac{2n - 6 + 6 + n - 9}{6} = \frac{n-3}{2}.$$

Hence $\gamma'\left(G^+(Z_n, I_v)\right) = \frac{n-3}{2}$

3. TOTAL EDGE DOMINATION NUMBER OF A GRAPH

The theory of total domination in graphs was formalized by Cockayne et al. [4]. We found a detailed study of total domination in the two books Haynes et al. [7,8]. An extensive study of a relationship between domination and total domination of a graph G with no isolated vertices can be done by Bollobas [3]. An upper bound on the total domination number of a connected graph was given by Cockayne et al. [4]. A result relating the size of a graph and its order, total domination number and maximum degree can be studied by Berge[1].Total edge dominating sets were introduced by Kulli, Patwari [10].
A set F of edges in a graph $G = (V, E)$ is called total edge dominating set of G if for every edge in E is adjacent to atleast one edge in F. Equivalently a set F of edges in G is called a total edge dominating set of G if for every edge $e \epsilon E$, there exist an edge $e_1 \epsilon F$ such that e and e_1 have a vertex in common .The total edge domination number $\gamma_t'(G)$ of a graph G is the minimum cardinality of a total edge dominating set of G.

Theorem 3.1: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is even and $n = 4s$, where $s \geq 1$,the total edge domination number is

$$\gamma_t'\left(G^+(Z_n, I_v)\right) = \frac{n}{2}.$$

Proof: Consider a graph$G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \dots, n-1\}$ where I_v denotes the set of involutory elements in Z_n and n is even.
If $n = 4$ then graph contains one Hamilton cycle, $|E| = n$.
If $n \neq 4$ graph contains two Hamilton cycles and each cycle contains n number of edges.
From first Hamilton cycle, Since$n$ is divisible by 4, Consider

$$F = \{e_i, e_j / i \geq 2, i_2 = i_1 + 4, j = i + 1\}$$

Now every edge of E adjacent to atleast one edge in F. Then F becomes a total edge dominating set of $G^+(Z_n,I_v)$ and it is minimum.

$|F| = \frac{2|E|}{4} = \frac{2n}{4} = \frac{n}{2}$.

Hence $\gamma_t'(G^+(Z_n,I_v)) = \frac{n}{2}$

Theorem 3.2: For the involutory addition cayley graph $G^+(Z_n,I_v)$, if n is odd and n is not in the form of $7s(s > 1)$, where $s \geq 1$,the total edge domination number is

$$\gamma_t'(G^+(Z_n,I_v)) = \begin{cases} \frac{n+1}{2}, n \geq 3 \ and \ n = 4l-1, l\epsilon N \\ \frac{n-1}{2}, n \geq 5 \ and \ n = 4l+1, l\epsilon N \end{cases}.$$

Proof: Consider a graph$G^+(Z_n,I_v)$ with vertex set $Z_n = \{0,1,2,3,\dots,n-1\}$ where I_v denotes the set of involutory elements in Z_n and n is odd, nis not in the form of $7s(s > 1)$

Case(i): $n \geq 3, n = 4l-1, l\epsilon N$. If $I_v = 2$,For $n = 3, |E| = 2$ and for the remaining $|E| = 4s+2, s \geq 1$. These graphs are paths.

If $I_v = 4$ then $|E| = 2n-2$, These graphs contains two paths, each path contains $n-1$ number of edges.

From first path, Consider$F_1 = \{e_i, e_j / i \geq 2, i_2 = i_1 + 4, j = i+1\}$

Then $|F_1| = \frac{2(n-1-2)}{4} = \frac{n-3}{2}$.

Since 4 does not divides $n-1$, $F_2 = \{e_{n-1}, e_{n-2}\}$, $|F_1| = 2$.

Consider$F = F_1 \cup F_2$. Now every edge of E adjacent to atleast one edge in F. Then F becomes a total edge dominating set of $G^+(Z_n,I_v)$ and it is minimum.

$|F| = |F_1| + |F_2| = \frac{n-3}{2} + 2 = \frac{n-3+4}{2} = \frac{n+1}{2}$.

Hence $\gamma_t'(G^+(Z_n,I_v)) = \frac{n+1}{2}$

Case(2): $n \geq 5, n = 4l+1, l\epsilon N$. If $I_v = 2$, $|E| = 4s, s \geq 1$. These graphs are paths. If $I_v = 4$ then $|E| = 2n-2$, These graphs contains two paths, each path contains $n-1$ number of edges.From first path, Consider$F = \left\{e_i, \frac{e_j}{i} \geq 2, i_2 = i_1 + 4, j = i+1\right\}$.

Since 4 divides $n-1$,every edge of E adjacent to atleast one edge in F.Then F becomes a total edge dominating set of $G^+(Z_n,I_v)$ and it is minimum.$|F| = \frac{2(n-1)}{4} = \frac{n-1}{2}$.

Hence $\gamma_t'(G^+(Z_n,I_v)) = \frac{n-1}{2}$

Theorem 3.3: For the involutory addition cayley graph $G^+(Z_n,I_v)$, if n is even and$n = 2s$ where s is odd and $s \geq 3$, the total edge domination number is

$$\gamma_t'(G^+(Z_n,I_v)) = \frac{n}{2} + 1.$$

Proof: Consider a graph$G^+(Z_n,I_v)$ with vertex set $Z_n = \{0,1,2,3,\dots,n-1\}$ where I_v denotes the set of involutory elements in Z_n and n is even, $n = 2s, s \geq 3, s$ is odd, $|E| = n \ or \ 2n$

If $|E| = n$ graph is a Hamilton cycle.If $|E| = 2n$ graph contains two Hamilton cycles.

From first Hamilton cycle, Consider$F_1 = \{e_i, e_j / i \geq 2, i_2 = i_1 + 4, j = i+1\}$

Then $|F_1| = \frac{2(n-2)}{4} = \frac{n-2}{2}$.

Since 4 does not divides n, $F_2 = \{e_n, e_{n-1}\}$ .Then $|F_2| = 2$.

Denote$F = F_1 \cup F_2$. Now every edge of E adjacent to atleast one edge in F . Then F becomes a total edge dominating set of $G^+(Z_n,I_v)$ and it is minimum.

$|F| = |F_1| + |F_2| = \frac{n-2}{2} + 2 = \frac{n-2+4}{2} = \frac{n+2}{2} = \frac{n}{2} + 1$.

Hence $\gamma_t'(G^+(Z_n,I_v)) = \frac{n}{2} + 1$.

CONCLUSION

Using involutory addition cayley graphs, it is interesting to find the Edge domination number and total edge domination numberand the authors have also studied this aspect.

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