The Stability and Well-Posedness Results For Hyperbolics Non Linear Problems

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ABSTRACT

In this work, we will give a sufficient condition that guarantees the well posedness of the problem. To this end, we formulate this system in an appropriate Hilbert space setting by using the semigroups approach and we will study the stability of system and proving that the energy decays to zero exponentially. For this purpose, we shall introduce a suitable Lyapunov functional.

Keywords: Well-posedness and asymptotic stability, General energy decay estimate, Asymptotic stability.

1. INTRODUCTION

Let $]0,L[$ be a subset of R, The following system describing approximately the planar motion of a uniform prismatic beam of lenght L with memory term . The unknowns $\psi=\psi(x,t)$ and $\eta=\eta(x,t)$ represent respectively the transversal and the longitudinal displacement of the point x at time t, h and ρ are two strictly positive constants represent respectively the thickness and the mass density per unit volume of the beam, the modified von-Kármán's system is given by

$$\begin{cases}\rho h\left(I-\frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)\psi_{tt}+\psi_{xxxx}-\left[\psi_x(\eta_x+\frac{1}{2}\psi_x^2)\right]_x-\psi_{txx}-g*\psi_{xxxx}=0 & \text{in }]0,L[\times R_+^* \\ \rho h\eta_{tt}-\partial_x(\eta_x+\frac{1}{2}(\psi_x)^2)+\alpha_1\eta_t+\alpha_2\eta_t(t-\tau)=0 & \text{in }]0,L[\times R_+^*\end{cases} \quad (1)$$

with boundary conditions

$$\begin{cases}\psi(0,.)=\psi(L,.)=\psi_x(0;.)=\psi_x(L,.)=0 & \text{in } \mathrm{R}_+^* \\ \eta_x(0,.)=\eta_x(L,.)=0 & \text{in } \mathrm{R}_+^*\end{cases} \quad (2)$$

and the initial data

$$\{(\psi(.,t),\psi_t(.,0),\eta(.,0),\eta_t(.,0))=(\psi_0(.,t),\psi_1,\eta_0,\eta_1) \text{ in } \mathrm{R}_- \quad (3)$$

Here $*$ is the usual convolution product, defined by:

$$(g*\psi_{xxxx})(t)=\int_0^\infty g(s)\psi_{xxxx}(x,t-s)ds$$

and $\alpha_1$and $\alpha_2$are two positive constants.

In this paper, we will prove the well-posedness and the stability results for problem **(1) - (2),** under the assumption

$$\alpha_1>\alpha_2.$$

2. Preliminaries

In this section, we will give some assumptions and lemmas; we start by the following hypothesis:

$H_1)$ $g\ :\ R_+\mapsto R_+$ Is a non-increasing differentiable function such that

$$g(0)>0 \text{ and } \kappa=1-\int_0^\infty g(s)ds=1-g_0>0. \quad (4)$$

where $g_0=\int_0^\infty g(s)ds$

$H_2)$ There exists an increasing strictly convex function $G : R_+ \to R_+$ of class $C^1(R_+) \cap C^2(]0,+\infty[)$ satisfying:

$G(0) = G'(0) = 0$ and $\lim_{t\to\infty} G'(t) = 0$
such that

$$\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s\in R_+} \frac{g(s)}{G^{-1}(-g'(s))} < \infty \qquad \textbf{(5)}$$

Remark1: The class of functions who satisfy $(H_2)$is very large, We draw the reader to **[12].**

- If we let $g(t) = \frac{d}{(t+2)\, ln^q(t+2)}$ with $d > 0$ small enough such that (H_1) hold and $q > 1$, we can take $G(t) = e^{-t^{-p}}$ with $p > \frac{1}{q-1}$.
- for $g(t) = q_0(1+t)^{-q}$ with $q_0 > 0$ and $q \in (1,2]$ then (H_2) is satisfied with $G(t) = t^r$for all $r > \frac{q+1}{q-1}$

  For simplicity of notations, f_x (resp f_t) designs the derevative of f with respect the space variable $'x'$ (resp withe respect to the time variable $'t'$), $|f|$ represents the norme of f in $L^2(0,L)$.

3. Well-posedness

In this section, we will give a sufficient condition that guarantees the well-posedness of the problem **(1)-(2).** To this end, we formulate this system in an appropriate Hilbert space setting by using the semigroups approach.

We introduce the Hilbert space:

$H = H_0^2(0,L) \times H_0^1(0,L) \times H_\#^1(0,L) \times L^2(0,L) \times L_g^2(R_+; H_0^2(0,L)) \times L^2(0,L; L^2(0,1))$

where

$$H_\#^1(0,L) = \{v \in H^1(0,L), \int_0^L v(x)dx = 0\}$$

H is equipped with the norm

$$\|(v_1,v_2,v_3,v_4,h,z)\|_H^2 = \kappa|v_{1xx}|^2 + \frac{\rho h^3}{12}|v_{2x}|^2 + \rho h|v_2|^2 + |v_{3x}|^2 + \rho h|v_x|^2 + \|h\|_{L_g}^2 + \xi\|z\|^2$$

where :

$$L_g(R_+; H_0^2(0,L)) = \{v \; : \; R_+ \mapsto H_0^2(0,L), \int_0^L \int_0^\infty g(s)(v_{xx})^2 ds dx < +\infty\}$$

endowed with the inner product

$$(v,\tilde{v})_{L_g} = \int_0^L \int_0^\infty g(s) v_{xx}(x,s)\tilde{v}_{xx}(x,s) ds dx$$

and

$$((z,\Xi)) = \int_0^L \int_0^1 z(x,p,t)\Xi(x,p,t) dp dx$$

Also, we assume that:

$$\alpha_2\tau < \xi < (2\alpha_1 - \alpha_2)\tau. \qquad \textbf{(6)}$$

As in **[1],** let us introduce the new variable

$h(x,t,s) = \psi(x,t) - \psi(x,t-s)$
and
$z(x,p,t) = \eta_t(x,t-p\tau)$

which satisfy

$$\begin{cases} h_t(x,t,s) = -h_s(x,t,s) - \psi_t(x,t) \\ z_t(x,p,t) = -\frac{1}{\tau} z_p(x,p,t) \\ h(0,.,.) = h(L,.,.) = h_x(0,.,.) = h_x(L,.,.) = 0 \quad (s,t) \in (R_+)^2 \\ h(x,t,0) = 0 \quad (x,t) \in]0,L[\times R_+ \\ h_0(x,s) = h(x,0,s) = \psi_0(x,0) - \psi_0(x,-s) \quad (x,s) \in]0,L[\times R_+ \\ z(x,0,t) = \eta_t(x,t) \quad (x,t) \in]0,L[\times R_+ \\ z_0(x,p) = z(x,p,0) = f_0(x,-p\tau) \quad (x,p) \in]0,L[\times]0,1[\end{cases}$$

Then, the systems **(1)- (2)** becomes:

$$\begin{cases} \rho h\psi_{tt} - \frac{\rho h^3}{12}\psi_{tt} + \kappa\psi_{xxxx} - \left[\psi_x\left(\eta_x + \frac{1}{2}(\psi_x)^2\right)\right]_x - \psi_{txx} - \int_0^\infty g(s)h_{xxxx}ds = 0 \text{ in }]0,L[\times R_+^* \\ \rho h\eta_{tt} - \left[\eta_x + \frac{1}{2}(\psi_x)^2\right]_x + \alpha_1\eta_t + \alpha_2\eta_t(t-\tau) = 0 \quad \text{in }]0,L[\times R_+^* \\ h_t(s) + h_s(s) - \psi_t = 0 \quad \text{in }]0,L[\times (R_+^*)^2 \\ z_t(p) + \frac{1}{\tau}z_p(p) = 0 \quad \text{in }]0,L[\times]0,1[\times R_+^* \end{cases} \qquad \textbf{(7)}$$

With boundary conditions:

$$\begin{cases} \psi(0,.) = \psi(L,.) = \psi_x(0,.) = \psi_x(L,.) = 0 \ \text{ in } \mathrm{R}_+^* \\ \eta_x(0,.) = \eta_x(L,.) = 0 \ \text{ in } \mathrm{R}_+^* \\ h(0,.,.) = h(L,.,.) = h_x(0,.,.) = h_x(L,.,.) = 0 \ \text{ in } (R_+^*)^2 \\ z(0,.,.) = z(L,.,.) = 0 \ \text{ in }]0,1[\times R_+^* \end{cases} \qquad \textbf{(8)}$$

and initial data:

$$\begin{cases} \psi(x,t) = \psi_0(x,t) \text{ in }]0,L[\times R^- \\ (\psi_t(.,0),\eta_t(.,0),\eta_t(.,0)) = (\psi_1,\eta_0,\eta_1) \ \text{ in }]0,L[\\ h(x,0,s) = h_0(x,s) \text{ in }]0,L[\times R^+ \\ z(x,p,0) = z_0(x,p) \ \text{ in }]0,L[\times]0,1[\end{cases} \qquad \textbf{(9)}$$

The System **(7) - (9)** can be written as a semi-linear ordinary differential equation in H of the form

$$\begin{cases} BV_t = AV + F(V) \\ V(0) = V_0 \end{cases}$$

Where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho h(I - \frac{h^2}{12}\partial_x^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho h & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau \end{pmatrix}, \ V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ h \\ z \end{pmatrix}$$

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\kappa\partial_x^4 & \partial_x^2 & 0 & 0 & -\int_0^\infty g(s)\partial_x^4\,ds & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \partial_x^2 & -\alpha_1 I & 0 & -\alpha_2\gamma_{p,1} \\ 0 & I & 0 & 0 & -\partial_s & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_p \end{pmatrix}$$

$$F(V)=\begin{pmatrix} 0 \\ \left[v_{1x}\left(v_{3x}+\frac{1}{2}(v_{1x})^2\right)\right]_x \\ 0 \\ \left[\frac{1}{2}(v_{1x})^2\right]_x \\ 0 \\ 0 \end{pmatrix},\ V_0=\begin{pmatrix}\psi_0\\ \psi_1\\ \eta_0\\ \eta_1\\ h_0\\ z_0\end{pmatrix}$$

Where :

$$\gamma_{p,1}(z(x,t,p))=z(x,t,1)$$

Therefore the operator $B^{-1}A$ domain will be:

$$D(B^{-1}A)=\left\{\begin{array}{c} V\in(H_0^2\cap H^4)(0,L)\times H_0^2(0,L)\times H_\#^1(0,L)\cap H^2(0,L)\times H^1(0,L) \\ \times L_g^2(R_+;(H_0^2\cap H^4)(0,L))\times L^2(0,L;H^1(0,1)), \\ v_{3x}(0)=v_{3x}(L)=0\ ,h_s\in L_g^2(R_+;H_0^2(0,L)) \end{array}\right\}.$$

Proving that $B^{-1}A$ is dissipative on $D(B^{-1}A)$.
One has:

$$(B^{-1}AV,V)_H=\kappa(v_{2xx},v_{1xx})-\frac{h^2}{12}\kappa(\partial_x(I-\frac{h^2}{12}\partial_x^2)^{-1}v_{1xxxxx},v_{2x})+\frac{h^2}{12}(\partial_x(I-\frac{h^2}{12}\partial_x^2)^{-1}v_{2xxx},v_{2x})$$

$$-\frac{h^2}{12}(\partial_x(I-\frac{h^2}{12}\partial_x^2)^{-1}\int_0^\infty g(s)h_{xxxx}(s)ds,v_{2x})-(\kappa(I-\frac{h^2}{12}\partial_x^2)^{-1}v_{1xxxx},v_2)$$

$$+((I-\frac{h^2}{12}\partial_x^2)^{-1}v_{2xx},v_2)-((I-\frac{h^2}{12}\partial_x^2)^{-1}\int_0^\infty g(s)h_{xxxx}(s)ds,v_2)+(v_{4x},v_{3x})$$

$$-(v_{3x},v_{4x})-\alpha_1(v_4,v_4)-\alpha_2(z(1),v_4)-(h_s,h)_{L_g}+(v_2,h)_{L_g}-\frac{\xi}{\tau}((z_p,z)).$$

Integrating by part, one gets:

$$(B^{-1}AV,V)_H=\kappa(v_{2xx},v_{1xx})-\kappa((I-\frac{h^2}{12}\partial_x^2)(I-\frac{h^2}{12}\partial_x^2)^{-1}v_{1xxxx},v_2)-|v_{2x}|^2$$

$$-((I-\frac{h^2}{12}\partial_x^2)(I-\frac{h^2}{12}\partial_x^2)^{-1}\int_0^\infty g(s)h_{xxxx}(s)ds,v_2)$$

$$-(h_s,h)_{L_g}+(v_2,h)_{L_g}-\frac{\xi}{\tau}((z_p,z))$$

$$= \kappa(v_{2xx}, v_{1xx}) - \kappa(v_{1xx}, v_{2xx}) - (\int_0^\infty g(s)h_{xx}(s)ds, v_{2xx})$$
$$-|v_{2x}|^2 - \alpha_1|v_4|^2 - \alpha_2(z(1), v_4) - (h_s, h)_{L_g} + (v_2, h)_{L_g} - \frac{\xi}{\tau}((z_p, z))$$
$$= -(v_2, h)_{L_g} - |v_{2x}|^2 - \alpha_1^2|v_4|^2 - \alpha_2(z(1), v_4) - (h_s, h)_{L_g} + (v_2, h)_{L_g} - \frac{\xi}{\tau}((z_p, z))$$
$$= -|v_{2x}|^2 - \alpha_1|v_4|^2 - \alpha_2(z(1), v_4) - (h, h_s)_{L_g} - \frac{\xi}{\tau}((z_p, z)).$$

It is straightforward to see that the quantity $-(h_s, h)_{L_g}$ is negative, by definition:

$$-(\partial_s h, h)_{L_g} = -\int_0^L \int_0^\infty g(s)h_{xxs}(s)h_{xx}(s)dsdx = -\frac{1}{2}\int_0^L \int_0^\infty g(s)(h_{xx}^2(s))_s dsdx$$

integrating by part with respect to $'s'$ to get :

$$-(h_s, h)_{L_g} = \frac{1}{2}\int_0^L \int_0^\infty g'(s)h_{xx}^2(s)dsdx \leq 0,$$

since $g'(s) \leq 0$.
In the other hand, integrating by parts in p, we have,

$$((z_p, z)) = \int_0^L \int_0^1 z_p(p)z(p)dpdx = \frac{1}{2}\int_0^L \int_0^1 (z^2(p))_p dpdx = \frac{1}{2}(|z(1)|^2 - |v_4|^2)$$

after using Young's inequality, we have in total :

$$(B^{-1}AV, V)_H = -\|v_{2x}\|^2 - \alpha_1\|v_4\|^2 - \alpha_2(z(1), v_4) - \frac{\xi}{2\tau}(|z(1)|^2 - |v_4|^2) - (h_s, h)_{L_g}$$
$$\leq -\alpha_1|v_4|^2 - \alpha_2(z(1), v_4) - \frac{\xi}{2\tau}(|z(1)|^2 - |v_4|^2)$$
$$\leq (\frac{\alpha_2}{2} - \frac{\xi}{2\tau})|z(1)|^2 + (-\alpha_1 + \frac{\alpha_2}{2} + \frac{\xi}{2\tau})|v_4|^2$$

keeping in mind **(6)**, so we conclude that $(B^{-1}AV, V)_H \leq 0$. We still have to shwo that $B^{-1}A$ is maximal on $D(B^{-1}A)$.
For this purpose, we prove that for all

$$f = (f_1, f_2, f_3, f_4, f_5, f_6)^t \in H$$

there exists a unique $V \in D(B^{-1}A)$ such that:

$(I - B^{-1}A)V = f$. **(10)**
Equation **(10)** is equivalent to:

$$v_1 - v_2 = f_1 \quad (11.1)$$
$$\rho h(I - \frac{h^2}{12}\partial_x^2)v_2 + \kappa v_{1xxxx} - v_{2xx} + \int_0^\infty g(s)h_{xxxx}ds = \rho h(I - \frac{h^2}{12}\partial_x^2)f_2 \quad (11.2)$$
$$v_3 - v_4 = f_3 \quad (11.3)$$
$$\rho h v_4 - v_{3xx} + \alpha_1 v_4 + \alpha_2 z(1) = f_4 \quad (11.4)$$
$$h + h_s - v_2 = f_5 \quad (11.5)$$
$$z + \frac{1}{\tau}z_p = f_6 \quad (11.6)$$

We can easily solve (11.5) and (11.6) to get:

$$h(s)=(1-e^{-s})v_2+\int_0^s e^{y-s}f_5dy=(1-e^{-s})(v_1-f_1+)+\int_0^s e^{y-s}f_5dy \qquad \textbf{(12)}$$

$$z(p)=v_4e^{-\tau p}+\tau\int_0^p e^{\tau(y-p)}f_6dy \qquad \textbf{(13)}$$

From (11.3) we have

$$v_4=v_3-f_3 \qquad \textbf{(14)}$$

Replacing **(13)** with $p=1$ in **(2)-(5)** and use again **(14)** to obtain

$$\begin{cases}-v_{3xx}+(\rho h+\alpha_1+\alpha_2e^{-\tau})v_3=f_4+(\alpha_1+\rho h+\alpha_2e^{-\tau})f_3-\tau\alpha_2\int_0^1 e^{\tau(y-1)}f_6dy \\ v_{3x}(0)=v_{3x}(L)=0\end{cases} \qquad \textbf{(15)}$$

It is easy to show that $x\mapsto\int_0^1 e^{\tau y}f_6(y,x)dy\in L^2(0,L)$, we have:

$$\int_0^L\left(\int_0^1 e^{\tau y}f_6(y,x)dy\right)^2dx\le\int_0^L\int_0^1 e^{2\tau y}f_6^2(y,x)dydx$$

$$\le\max_{y\in[0,1]}(e^{2\tau y})\int_0^L\int_0^1 f_6^2(y,x)dydx$$

$$=\max_{y\in[0,1]}(e^{2\tau y})\|f_6\|^2<\infty$$

So the right hand side of **(15)** belong to $L^2(0,L)$, then by **Lax-Milgram's theorem**, there is unique solution of **(14)** in $H^2(0,L)$.

Reinjecting **(12)** in (11.2) with replacing v_2 by its value in (11.1) and denote $\kappa_0:=\int_0^\infty g(s)(1-e^{-s})ds$, by using **fubini's theorem** yields:

$$(\kappa_0+\kappa)v_{1xxxx}-(1+\frac{\rho h^3}{12})v_{1xx}+\rho hv_1=u \qquad \textbf{(16)}$$

where:

$$u=(\kappa_0-\int_0^\infty\int_0^s g(s)e^{y-s}dyds)f_{1xxxx}+\rho h(f_1+f_2)-(\frac{\rho h^3}{12}+1)f_{1xx}-\frac{\rho h^3}{12}f_{2xx}$$

we note that :

$$\kappa_0+\kappa=1-\int_0^\infty g(s)e^{-s}ds=1-\int_0^\infty g(s)ds+\int_0^\infty g(s)(1-e^{-s})ds>0.$$

it is easy to see that $u\in H^{-2}(0,L)$, as $f_5\in H_0^2(0,L)$, then $f_{5xxxx}\in H^{-2}(0,L)$ and the improper integral $\int_0^\infty\int_0^s g(s)e^{y-s}dyds$ converge, indeed:

$$\int_0^\infty\int_0^s g(s)e^{y-s}dyds=\int_0^\infty\int_y^\infty g(s)e^{y-s}dsdy$$

$$\le\int_0^\infty g(y)e^y\int_y^\infty e^{-s}dsdy$$

$$=\int_0^\infty g(y)dy=g_0.$$

Now, multiply **(16)** by $\vec{v}_1$ and integrat by part over $(0,L)$ we get

$$\Phi(v_1,\tilde{v}) = \phi(\tilde{v})$$

where Φ is a bilinear(respectively φ is a linear) forme defined over $H_0^2(0,L)$, it's obvious that Φ is coercive, continuous and φ is continous, then by Lax-Milgram's theorem, the equation **(15)** is solvable over $H_0^2(0,L)$, so $I-B^{-1}A$is maximal on $D(B^{-1}A)$.

To finish the proof of the nonlinear Cauchy-problem, we need to show that F is locally lipschitz continuous.

Let $V$and $\tilde{V}$ belong to $D(A)$, we have

$$\left\| B^{-1}F(V) - B^{-1}F(\tilde{V}) \right\|_{\mathcal{H}} = \frac{1}{\rho h}\left\| (I - \frac{h^2}{12}\partial_x^2)^{-1} f \right\|_{H_0^1(0,L)} + \frac{1}{\rho h}|g|$$

Where

$$f = v_{1x}(v_{3x} + \frac{1}{2}(v_{1x})^2) - \tilde{v}_{1x}(\tilde{v}_{3x} + \frac{1}{2}(\tilde{v}_{1x})^2)$$

And

$$g = \partial_x[(v_{1x})^2 - (\tilde{v}_{1x})^2].$$

To proceed further we need the following lemma:

Lemma 2 let $f \in L^2(0,L)$, so we have:

$$\left| \partial_x (I - \frac{h^2}{12}\partial_x^2)^{-1} \partial_x v \right|^2 \le C|v|^2$$

With $C > 0$.

Proof

the operators:

$$\partial_x \ : \ L^2(0,L) \to H^{-1}(0,L)$$

$$(I - \frac{h^2}{12}\partial_x^2)^{-1} \ : \ H^{-1}(0,L) \to H_0^1(0,L)$$

$$\partial_x \ : \ H_0^1(0,L) \to L^2(0,L)$$

are continuous, so their composition is continuous:

$$\partial_x (I - \frac{h^2}{12}\partial_x^2)^{-1} \partial_x \ : \ L^2(0,L) \to L^2(0,L)$$

Applying the previous lemma, there exists a positive constant c_1 such that:

$$\left\| (I - \frac{h^2}{12}\partial_x^2)^{-1} \partial_x f \right\|_{H_0^1(0,L)} \le c_1|f|$$

Adding and subtracting

$$\tilde{v}_{1x}(v_{3x} + \frac{1}{2}(v_{1x})^2)$$

Inside the norme of $|f|$ and using the fact that $H^1(0,L)$ is embidded in $L^\infty(0,L)$, we get:

$$
\begin{aligned}
|f| = & \left|\left[v_{3x}+\frac{1}{2}(v_{1x})^2+\frac{1}{2}\tilde{v}_{1x}(v_{1x}+\tilde{v}_{1x})\right](v_{1x}-\tilde{v}_{1x})+\tilde{v}_{1x}(v_{3x}-\tilde{v}_{3x})\right| \\
\leq & \left\|v_{3x}+\frac{1}{2}(v_{1x})^2+\frac{1}{2}\tilde{v}_{1x}(v_{1x}+\tilde{v}_{1x})\right\|_{L^\infty(0,L)}|v_{1x}-\tilde{v}_{1x}| \\
& +\|\tilde{v}_{1x}\|_{L^\infty(0,L)}|v_{3x}-\tilde{v}_{3x}| \\
\leq & \left[\begin{array}{c}\|v_{3x}\|_{L^\infty(0,L)}+\frac{1}{2}\|(v_{1x})^2\|_{L^\infty(0,L)} \\ +\frac{1}{2}\|\tilde{v}_{1x}\|_{L^\infty(0,L)}(\|v_{1x}\|_{L^\infty(0,L)}+\|\tilde{v}_{1x}\|_{L^\infty(0,L)})\end{array}\right]|v_{1x}-\tilde{v}_{1x}| \\
& +\|\tilde{v}_{1x}\|_{L^\infty(0,L)}|v_{3x}-\tilde{v}_{3x}| \\
\leq & C_1\left[\begin{array}{c}\|v_{3x}\|_{H^1(0,L)}+\frac{1}{2}\|v_{1x}\|^2_{H^1(0,L)} \\ +\frac{1}{2}\|\tilde{v}_{1x}\|_{H^1(0,L)}(\|v_{1x}\|_{H^1_0(0,L)}+\|\tilde{v}_{1x}\|_{H^1(0,L)})\end{array}\right]|v_{1x}-\tilde{v}_{1x}| \\
& +C_1\|\tilde{v}_{1x}\|_{H^1(0,L)}|v_{3x}-\tilde{v}_{3x}| \\
\leq & \tilde{C}_1(v_{1x},\tilde{v}_{1x},v_{3x},\tilde{v}_{3x})\left\|\mathcal{V}-\tilde{\mathcal{V}}\right\|_{\mathcal{H}}
\end{aligned}
$$

The same thing for g

$$
\begin{aligned}
|g| = & |\partial_x(v_{1x})^2-\partial_x(\tilde{v}_{1x})^2| \leq \|(v_{1x})^2-(\tilde{v}_{1x})^2\|_{H^1_0(0,L)} \\
= & \|(v_{1x}+\tilde{v}_{1x})((v_{1x}-\tilde{v}_{1x})\|_{H^1_0(0,L)} \\
\leq & \|v_{1x}+\tilde{v}_{1x}\|_{L^\infty(0,L)}\|v_{1x}-\tilde{v}_{1x}\|_{H^1_0(0,L)} \\
\leq & C_2\|v_{1x}+\tilde{v}_{1x}\|_{H^1(0,L)}\|v_{1x}-\tilde{v}_{1x}\|_{H^1_0(0,L)} \\
\leq & \tilde{C}_1(v_{1x},\tilde{v}_{1x})\left\|\mathcal{V}-\tilde{\mathcal{V}}\right\|_{\mathcal{H}}
\end{aligned}
$$

Therefore F is lipschitz continuous on $D(B^{-1}A)$, and this finishes the proof of existence and uniqueness of local solution.

To proof the existence of global solution we need to show that the energy functional associated with the system **(7) - (9)** is decreasing.

The energy functional $E(t)$ associated to the system **(7) - (9)** is defned as follows:

$$E(t)=\rho h|\psi_t|+\frac{\rho h^2}{12}|\psi_{xt}|+\kappa|\psi_{xx}|+\rho h|\eta_t|+\left|\eta_x+\frac{1}{2}(\psi_x)^2\right|^2+\|h\|^2_{L_g}+\frac{\xi}{2}\|z\|^2$$

A simple computation gives us :

$$\frac{dE(t)}{dt}=(\frac{\xi}{2\tau}-\alpha_1)|\eta_t|^2-|\psi_{xt}|^2+\int_0^\infty g'(s)\int_0^L (h_{xx})^2dsdx$$
$$-\frac{\xi}{2\tau}\|z(1)\|^2-\alpha_2(z(1),\eta_t) \quad \textbf{(17)}$$

and again we use Young's inequality, we get:

$$\frac{dE(t)}{dt}\leq(\frac{\alpha_2}{2}-\frac{\xi}{2\tau})\|z(1)\|^2+(-\alpha_1+\frac{\alpha_2}{2}+\frac{\xi}{2\tau})|\eta_t|^2+\int_0^\infty g'(s)\int_0^L (h_{xx})^2dsdx\leq 0.$$

Now we are ready to state the **well-posedness theorem.**

Theorem 3 If $V_0\in H$, then there is a unique solution of the problem (S_2) satisfies

$V \in C([0,\infty[,H)$, moreover if $V_0 \in D(B^{-1}A)$ then $V \in C([0,\infty[,D(B^{-1}A)) \cap C^1([0,\infty[;H)$.

4. General decay

In this section, we will study the stability of system (7)- (9) and proving that the energy decays to zero exponentially. For this purpose, we shall introduce a suitable **Lyapunov functional.**

One announce the principal theorem in this work.

Theorem 4 Assume that (H_1) and (H_2) are satisifed, and let $V_0 \in H$ such that

$$\exists M \geq 0 \quad : \quad \|h_{0xx}(s)\|^2 \leq M, \forall s \geq 0 \tag{18}$$

Then there is two positive constants ν, μ and ε_0 (depending continuously on $E(0)$) such that $E(t) \leq \nu G_1^{-1}(\mu t)$

Where $G_1(t) = \int_t^1 \frac{ds}{sG'(\varepsilon_0 s)}$.

To prove this theorem we will introduce a very imminent lemmas in the sequel. They will serve to establish the stability result. We start by introducing this lemma:

Lemma 5 [7], the following inequalies hold:

$$\left(\int_0^\infty g(s)h(x,t,s)ds\right)^2 \leq g_0 \int_0^\infty g(s)h^2(x,t,s)ds$$
$$\left(\int_0^\infty g'(s)h(x,t,s)ds\right)^2 \leq -g(0)\int_0^\infty g'(s)h^2(x,t,s)ds \tag{19}$$

Proof For the first one, one has:

$$\left(\int_0^\infty g(s)h(x,t,s)ds\right)^2 = \left(\int_0^\infty \sqrt{g(s)}(\sqrt{g(s)}h(x,t,s))ds\right)^2$$

Applying **Cauchy-Schwarz's inequality**, one gets:

$$\left(\int_0^\infty g(s)h(x,t,s)ds\right)^2 \leq \int_0^\infty g(s)ds \int_0^\infty g(s)h^2(x,t,s))ds$$
$$= g_0 \int_0^\infty g(s)h^2(x,t,s)ds.$$

For the second we follow the same steps by writing $g'(s) = \sqrt{-g'(s)}\sqrt{-g'(s)}$.

Introducing the auxiliary function F as:

$$F(t) = \frac{\theta}{2}I_1(t) + \theta I_2(t) + \gamma I_3(t) + \varepsilon I_4(t) \tag{20}$$

where:

$$I_1(t) = \frac{\theta}{2}(\rho h(I - \frac{h^2}{12}\partial_x^2)\psi_t, \psi)$$
$$I_2(t) = (\rho h \eta_t, \eta)$$
$$I_3(t) = -((I - \frac{h^2}{12}\partial_x^2)\rho h \psi_t, \int_0^\infty g(s)h(s)ds)$$
$$I_4(t) = \int_0^L \int_0^1 e^{-2\tau p} z^2(p)dpdx$$

Proposition 6 Let (ψ, η, h, z) be a solution of (7) - (9), then for any positive constants $(\delta_i)_{0 \leq i \leq 9}, \theta, \gamma, \varepsilon$, the functional F satisfies:

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) \leq & \left\{\theta(\frac{1}{8\delta_1}+\frac{\rho h^3}{24})+\gamma(g_0\delta_7+\frac{\rho h^3}{12}\delta_5-\frac{\rho h^3}{12}g_0)\right\}|\psi_t|^2 \\ & \left\{\frac{\theta}{2}[(\delta_0 g_0-\kappa+\delta_1 L)+(\alpha_1\delta_2+\alpha_2\delta_3)L^2 c_4\mathcal{E}(0)]+\gamma[\kappa g_0\delta_6+c_5 g_0\delta_8\delta_9\mathcal{E}(0)]\right\}|\psi_{xx}|^2 \\ & +\left\{\theta\frac{\rho h}{2}+\gamma(-\rho h g_0+\rho h\delta_4)\right\}|\psi_t|^2+\left\{\frac{\theta}{4\delta_0}+\gamma(\frac{\kappa}{4\delta_6}+\frac{L}{4\delta_7}+\frac{L}{4\delta_8}+g_0)\right\}\|h\|^2_{L_g} \\ & +\left\{\theta(-1+L(\alpha_1\delta_2+\alpha_2\delta_3)+\gamma(\frac{\delta_8\tilde{c}_5}{4\delta_9}g_0\mathcal{E}(0))\right\}\left|(\eta_x+\frac{1}{2}(\psi_x)^2)\right|^2 \\ & +\left\{\theta(\rho h+\frac{\alpha_1}{4\delta_2})+\frac{\varepsilon}{\tau}\right\}|\eta_t|^2-\gamma g(0)\left(\frac{L^2\rho h}{4\delta_4}+L\frac{\rho h^3}{48\delta_5}\right)\int_0^L\int_0^\infty g'(s)h^2_{xx}(s)\,ds\,dx \\ & +\left\{\theta\frac{\alpha_2}{4\delta_3}-\varepsilon\frac{e^{-2\tau}}{\tau}\right\}|z(1)|^2-2\varepsilon\int_0^L\int_0^1 e^{-2\tau p}z^2(p)\,dp\,dx.\end{aligned} \tag{21}$$

To prove this proposition we need some additionals lemmas.

Lemma 7 I_1 satisfies:

$$\frac{d}{dt}I_1 \leq \rho h|\psi_t|^2+(\frac{1}{4\delta_1}+\frac{\rho h^3}{12})|\psi_t|^2+(\delta_0 g_0-\kappa+\delta_1 L)|\psi_{xx}|^2$$
$$+\frac{1}{4\delta_0}\|h\|^2_{L_g}-((\psi_x(\eta_x+\frac{1}{2}(\psi_x)^2),\psi_x). \tag{22}$$

Proof

Keeping in mind the definition of $(I-\frac{h^2}{12}\partial_x^2)$, integrating par part and replace $(I-\frac{h^2}{12}\partial_x^2)\psi_{tt}$ by its value in (S_2) one gets:

$$\frac{d}{dt}I_1=\rho h|\psi_t|^2+\frac{\rho h^3}{12}|\psi_t|^2-\kappa|\psi_{xx}|^2-(\psi_t,\psi_x)-\int_0^\infty g(s)(h_{xx},\psi_{xx})ds$$
$$-((\psi_x(\eta_x+\frac{1}{2}(\psi_{xx})^2),\psi_x)$$

Applying **Young's inequality** on (h_{xx},ψ_{xx}) and **Young then Poincare's** inequalities on (ψ_t,ψ_x), one gets:

$$\frac{d}{dt}I_1 \leq \rho h|\psi_t|^2+(\frac{1}{4\delta_1}+\frac{\rho h^3}{12})|\psi_t|^2+(\delta_0 g_0-\kappa+\delta_1 L)|\psi_{xx}|^2$$
$$+\frac{1}{4\delta_0}\|h\|^2_{L_g}-((\psi_x(\eta_x+\frac{1}{2}(\psi_x)^2),\psi_x).$$

Where δ_0,δ_1 are positives constants.

Lemma 8 Let (ψ,η,h,z) be a solution of **(7) - (9)**, then I_2 satisfies:

$$\frac{d}{dt}I_2 \leq (\rho h+\frac{\alpha_1}{4\delta_2})|\eta_t|^2-(\eta_x+\frac{1}{2}(\psi_x)^2,\eta_x)+\frac{\alpha_2}{4\delta_3}\|z(1)\|^2$$
$$+(\alpha_1\delta_2+\alpha_2\delta_3)\left\{L\left|\eta_x+\frac{1}{2}(\psi_x)^2\right|^2+L^2c_4E(0)|\psi_{xx}|^2\right\} \tag{23}$$

Where δ_2,δ_3 are positive constants.

Proof

Differentiating I_2 with respect to 't', integrating by part and use again **Young's inequality**, this gives us:

$$\frac{d}{dt}I_2 \leq \rho h|\eta_t|^2-(\eta_x+\frac{1}{2}(\psi_x)^2,\eta_x)+(\alpha_1\delta_2+\alpha_2\delta_3)|\eta|^2+\frac{\alpha_1}{4\delta_2}|\eta_t|^2+\frac{\alpha_2}{4\delta_3}\|z(1)\|^2$$

We have to expressing $|\eta|^2$ in function of $E(t)$ terms. After applying **Poincare's** inequality, yields:

$$|\eta|^2 \leq L|\eta_x|^2 \leq L\left|\eta_x + \frac{1}{2}(\psi_x)^2\right|^2 + L|\psi_x{}^2|^2 \leq L\left|\eta_x + \frac{1}{2}(\psi_x)^2\right|^2 + L\|\psi_x\|_\infty^2|\psi_x|^2$$

Recall that $H^1(0,L)$ is embidded in $L^\infty(0,L)$, so:

$$\begin{aligned}|\eta|^2 &\leq L|\eta_x|^2 \leq L\left|\eta_x + \frac{1}{2}(\psi_x)^2\right|^2 + L\|\psi_x\|_\infty^2|\psi_x|^2\\ &\leq L\left|\eta_x + \frac{1}{2}(\psi_x)^2\right|^2 + L^2\tilde{c}_4|\psi_{xx}|^2|\psi_{xx}|^2\\ &\leq L\left|\eta_x + \frac{1}{2}(\psi_x)^2\right|^2 + L^2c_4E(0)|\psi_{xx}|^2\end{aligned}$$

Gathering all the terms, then **(23)** is proved.

Lemma 9 The functional I_3 satisfies along solutions of (**7)- (9)** satisfies:

$$\begin{aligned}\frac{d}{dt}I_3(t) \leq &(-\rho h g_0 + \rho h\delta_4)|\psi_t|^2 + (\kappa g_0\delta_6 + c_5 g_0\delta_8\delta_9\mathcal{E}(0))|\psi_{xx}|^2\\ &(g_0\delta_7 + \frac{\rho h^3}{12}\delta_5 - \frac{\rho h^3}{12}g_0)|\psi_t|^2 + \frac{\delta_8\tilde{c}_5}{4\delta_9}g_0\mathcal{E}(0)\left|(\eta_x + \frac{1}{2}(\psi_x)^2)\right|^2\\ &-g(0)\left(\frac{L^2\rho h}{4\delta_4} + L\frac{\rho h^3}{48\delta_5}\right)\int_0^L\int_0^\infty g'(s)h_{xx}^2(s)dsdx\\ &+(\frac{\kappa}{4\delta_6} + \frac{L}{4\delta_7} + \frac{L}{4\delta_8} + g_0)\|h\|_{L_g}^2.\end{aligned} \quad \textbf{(24)}$$

Where $\delta_4, \delta_5, \delta_6, \delta_7, \delta_8$ are positive constants.

Proof

One has:

$$\begin{aligned}\frac{d}{dt}I_3(t) &= \left(\rho h(I - \frac{h^2}{12}\partial_x^2)\psi_t, \frac{d}{dt}\int_0^\infty g(s)h(s)ds\right) - \left(\rho h(I - \frac{h^2}{12}\partial_x^2)\psi_{tt}, \int_0^\infty g(s)hds\right)\\ &= J_1(t) + J_2(t)\end{aligned}$$

Estimation of J_1

as in [4] one has:

$$\begin{aligned}\frac{d}{dt}\int_0^\infty g(s)hds &= \frac{d}{dt}\int_0^\infty g(s)(\psi(t) - \psi(t-s)ds = \frac{d}{dt}\int_0^\infty g(t-s)(\psi(t) - \psi(s)ds\\ &= \int_0^\infty g'(t-s)(\psi(t) - \psi(s)ds + \psi_t\int_0^\infty g(t-s)ds\\ &= \int_0^\infty g'(s)h(s)ds + g_0\psi_t\end{aligned}$$

So

$$\begin{aligned}J_1(t) = &-\rho h g_0|\psi_t|^2 - \frac{\rho h^3}{12}g_0|\psi_t|^2 + \rho h(\psi_t, \int_0^\infty g'(s)h(s)ds)\\ &-\frac{\rho h^3}{12}(\psi_t, \int_0^\infty g'(s)h_x(s)ds)\end{aligned}$$

one applies Young's inequality:

$$J_1(t) \leq -\rho h g_0 |\psi_t|^2 - \frac{\rho h^3}{12} g_0 |\psi_t|^2 + \rho h \delta_4 |\psi_t|^2 + \frac{\rho h}{4\delta_4} \int_0^L \left(\int_0^\infty g'(s) h(s) ds \right)^2 dx$$
$$+ \frac{\rho h^3}{12} \delta_5 |\psi_t|^2 + \frac{\rho h^3}{48\delta_5} \int_0^L \left(\int_0^\infty g'(s) h_x(s) ds \right)^2 dx \quad \textbf{(25)}$$

Making into account **(19)** and again applying **Poincare's inequality** one gets:

$$J_1(t) \leq (-\rho h g_0 + \rho h \delta_4) |\psi_t|^2 + \left(\frac{\rho h^3}{12} \delta_5 - \frac{\rho h^3}{12} g_0 \right) |\psi_t|^2$$
$$-g(0) \left(\frac{L^2 \rho h}{4\delta_4} + \frac{L\rho h^3}{48\delta_5} \right) \int_0^\infty \int_0^L g'(s) h_{xx}^2(s) ds dx$$

Estimation of $J_2(t)$

After integrating by part over $[0, L]$:

$$J_2(t) = -\left(\rho h \left(I - \frac{h^2}{12} \partial_x^2\right) \psi_{tt}, \int_0^\infty g(s) h ds\right)$$
$$= \kappa \int_0^\infty g(s) (\psi_{xx}, h_{xx}(s)) ds + \int_0^\infty g(s) (\psi_t, h_x(s)) ds$$
$$+ \left(\int_0^\infty g(s) |h_{xx}(s)|^2 ds \right)^2 + \int_0^\infty g(s) \left(\psi_x \left(\eta_x + \frac{1}{2} (\psi_x)^2\right), h_x(s)\right) ds$$

Making into account **(19)** integrating by part and using again **Young's inequality** one gets:

$$J_2(t) \leq \kappa g_0 \delta_6 |\psi_{xx}|^2 + g_0 \delta_7 |\psi_t|^2 + \left(\frac{\kappa}{4\delta_6} + \frac{L}{4\delta_7} + g_0 \right) \|h\|_{L_g}^2$$
$$+ \int_0^\infty g(s) \left(\psi_x \left(\eta_x + \frac{1}{2} (\psi_x)^2\right), h_x\right) ds \quad \textbf{(26)}$$

We will try to writing the term $\int_0^\infty g(s)(\psi_x(\eta_x + \frac{1}{2}(\psi_x)^2), h_x(s)) ds$ in function of energy's terms, after using the embedding of $H^1(0,L)$ in $L^\infty(0,L)$ and Young's inequality one gets:

$$\left(\psi_x\left(\eta_x + \frac{1}{2}(\psi_x)^2\right), h_x(s)\right) \leq \delta_8 \left| \psi_x \left(\eta_x + \frac{1}{2}(\psi_x)^2\right) \right|^2 + \frac{L}{4\delta_8} |h_{xx}(s)|^2$$
$$\leq \delta_8 \|\psi_x\|_{L^\infty(0,L)}^2 \left| \left(\eta_x + \frac{1}{2}(\psi_x)^2\right) \right|^2 + \frac{L}{4\delta_8} |h_{xx}(s)|^2$$
$$\leq \delta_8 \tilde{c}_5 |\psi_{xx}|^2 \left| \left(\eta_x + \frac{1}{2}(\psi_x)^2\right) \right|^2 + \frac{L}{4\delta_8} |h_{xx}(s)|^2$$

By applying Young's inequality again:

$$\delta_8 \tilde{c}_5 |\psi_{xx}|^2 \left| \left(\eta_x + \frac{1}{2}(\psi_x)^2\right) \right|^2 \leq \tilde{c}_5 \delta_8 \delta_9 |\psi_{xx}|^4 + \frac{\delta_8 \tilde{c}_5}{4\delta_9} \left| \left(\eta_x + \frac{1}{2}(\psi_x)^2\right) \right|^4$$
$$\leq c_5 \delta_8 \mathcal{E}(0) \left\{ \delta_9 |\psi_{xx}|^2 + \frac{1}{4\delta_9} \left| \left(\eta_x + \frac{1}{2}(\psi_x)^2\right) \right|^2 \right\} \quad \textbf{(27)}$$

Inserting (**27**) in (**26**) we get in total:

$$J_2(t) \leq \kappa g_0 \delta_6 |\psi_{xx}|^2 + g_0 \delta_7 |\psi_t|^2 + (\frac{\kappa}{4\delta_6} + \frac{L}{4\delta_7} + \frac{L}{4\delta_8} + g_0)\|h\|_{L_g}^2$$
$$+g_0 c_5 \delta_8 E(0)\left\{\delta_9 |\psi_{xx}|^2 + \frac{1}{4\delta_9}\left|(\eta_x + \frac{1}{2}(\psi_x)^2)\right|^2\right\} \quad \textbf{(28)}$$

Gathering (2**5)** and (2**8)** we get (2**4).**

Lemma 10 Let (ψ, η, h, z) be a solution of (S_2), then I_4 satisfies:

$$\frac{d}{dt} I_4(t) = -\frac{1}{\tau} e^{-2\tau} |z(1)|^2 + \frac{1}{\tau} |\eta_t|^2 - 2I_4 \quad \textbf{(29)}$$

Proof keeping in mind that $z_t(p) = -\frac{1}{\tau} z_p(p)$ yields:

$$\frac{d}{dt} I_4(t) = -\frac{2}{\tau}\int_0^L \int_0^1 e^{-2\tau p} z_p z dp dx = -\frac{1}{\tau}\int_0^L \int_0^1 e^{-2\tau p} (z^2)_p dp dx$$
$$= -\frac{1}{\tau} e^{-2\tau} |z(1)|^2 + \frac{1}{\tau} |\eta_t|^2 - 2I_4.$$

To finish the proof of the proposition above, gathring (2**2)**, (2**3)** , (2**4)** , (2**9)** We obtain the desired result.

Proposition 11 There is a positive constant C such that:

$$F(t) \leq CE(t) \quad \textbf{(30)}$$

Proof To prove this proposition, we have to analyze each term of **(17)** separately.

- **Analysis of** I_1

$$I_1 = \rho h((I - \frac{h^2}{12}\partial_x^2)\psi_t, \psi) = \rho h(\psi_t - \frac{\rho h^3}{12}\psi_{txx}, \psi)$$

Integrating by part, yields:

$$I_1 = \rho h |\psi_t|^2 + \frac{\rho h^3}{12} |\partial_x \psi_t|^2 \leq c_1 E(t)$$

- **Analysis of** I_2

After using Young's inequality, yields:

$$I_2 = (\rho h \eta_t, \eta) \leq \frac{\rho h}{2} |\eta_t|^2 + \frac{1}{2} |\eta|^2$$

Applying now **Poincare-wirtinger** inequality (see **(25))**, one gets:

$$|\eta| \leq \sqrt{L}\|\eta_x\|_{L^1(0,L)} \leq \sqrt{L}\left\|\eta_x + \frac{1}{2}(\psi_x)^2\right\|_{L^1(0,L)} + \frac{\sqrt{L}}{2}\|(\psi_x)^2\|_{L^1(0,L)}$$
$$\leq L\left|\eta_x + \frac{1}{2}(\psi_x)^2\right| + \frac{L}{2}|\psi_x|^2$$

One applies **Poincare's inequality** on the term $|\psi_x|^2$, yields:

$$|\eta| \leq \sqrt{L}\left|\eta_x + \frac{1}{2}(\psi_x)^2\right| + \frac{L^2}{4}|\psi_{xx}|^2 \leq \tilde{c}_2(\sqrt{\mathcal{E}(t)} + \mathcal{E}(t))$$
$$\leq \tilde{c}_2(\sqrt{\mathcal{E}(0)} + 1)\sqrt{\mathcal{E}(t)} \leq c_2\sqrt{\mathcal{E}(t)}$$

- **Analysis of I_3**

By applying Young's then **Poincare inequality's** one gets

$$I_3 = (\rho h\psi_t, \psi) \leq \frac{\rho h}{2}|\psi_t|^2 + \frac{1}{2}|\psi|^2 \leq \frac{\rho h}{2}|\psi_t|^2 + \frac{L^2}{2}|\psi_{xx}|^2 \leq c_3 E(t)$$

- **Analysis of I_4**

It is obvious that I_4 defines a norm on $L^2(0,L;L^2(0,1))$ equivalent with the norme induced by $L^2(0,L;L^2(0,1))$, so:

$$I_4 \leq \int_0^L \int_0^1 e^{-2\tau p} z^2 dp dx \leq \int_0^L \int_0^1 z^2 dp dx \leq E(t)$$

We deduce in total that $F(t) \leq CE(t)$ where $C = sup\{c_1, c_2, c_3, 1\}$.
Introducing now the Lyapunov functional:

$$Y(t) = \lambda E(t) + F(t)$$

Where λ is an arbitrary positive constant, by using **(30),** it is easy to see that?
$|Y(t) - \lambda E(t)| \leq CE(t)$

wich implies that

$$(\lambda - C)E(t) \leq Y(t) \leq (C + \lambda)E(t)$$

By taking $\lambda > C$, then there exist two positive constants C_1 and C_2 such that:
$C_1 E(t) \leq Y(t) \leq C_1 E(t)$ **(31)**
We deduce that $E \sim Y$.
Now differentiating the functional Y and combine (17**)-**(2**1)** one gets:

$$\begin{aligned}
\frac{d}{dt}\mathcal{Y}(t) &= \lambda \frac{d}{dt}\mathcal{E}(t) + \frac{d}{dt}\mathcal{F}(t) \\
&\leq \left\{\frac{\theta}{2}((\delta_0 g_0 - \kappa + \delta_1 L) + (\alpha_1\delta_2 + \alpha_2\delta_3)L^2 c_4 \mathcal{E}(0)) + \gamma(\kappa g_0 \delta_6 + c_5 g_0 \delta_8 \delta_9 \mathcal{E}(0))\right\}|\psi_{xx}|^2 \\
&+ \left\{\theta\frac{\rho h}{2} + \gamma(-\rho h g_0 + \rho h\delta_4) - \lambda\right\}|\psi_{tx}|^2 + \left\{\theta(\frac{1}{8\delta_1} + \frac{\rho h^3}{24}) + \gamma(g_0\delta_7 + \frac{\rho h^3}{12}\delta_5 - \frac{\rho h^3}{12}g_0)\right\}|\psi_t|^2 \\
&+ \left\{\frac{\theta}{4\delta_0} + \gamma(\frac{\kappa}{4\delta_6} + \frac{L}{4\delta_7} + \frac{L}{4\delta_8} + g_0)\right\}\|h\|_{L_g}^2 \\
&+ \left\{\theta(-1 + L(\alpha_1\delta_2 + \alpha_2\delta_3) + \gamma(\frac{\delta_8 \tilde{c}_5}{4\delta_9} g_0 \mathcal{E}(0))\right\}\left|(\eta_x + \frac{1}{2}(\psi_x)^2)\right|^2 \\
&+ \left\{\theta(\rho h + \frac{\alpha_1}{4\delta_2}) + \frac{\varepsilon}{\tau} + \lambda(-\alpha_1 + \frac{\alpha_2}{2} + \frac{\xi}{2\tau})\right\}|\eta_t|^2 + \left\{\theta\frac{\alpha_2}{4\delta_3} - \varepsilon\frac{e^{-2\tau}}{\tau} + \lambda(\frac{\alpha_2}{2} - \frac{\xi}{2\tau})\right\}\|z(1)\|^2 \\
&+ \left\{\lambda - \gamma g(0)(\frac{L^2\rho h}{4\delta_4} + \frac{\rho h^3}{48\delta_5})\right\}\int_0^\infty g'(s)\int_0^L h_{xx}^2(s)dxds - 2\varepsilon\int_0^L\int_0^1 e^{-2\tau p}z^2 dpdx
\end{aligned}$$

(32)

In this case, we must choose $(\delta_i)_{0\leq i\leq 9}, \theta, \gamma, \varepsilon$ carefully.
from **(31)** on has :

$$\lambda > C$$

so we have to choose λ large enough to make the **L.H.S** of the quantities $|z(1)|^2$, $|\eta_t|^2$, $|\psi_{tx}|^2$ negative and $\int_0^L \int_0^\infty g'(s)h_{xx}^2(s)dxds$ positive
One invokes (H_2) then

$$\left\{\lambda - \gamma g(0)\left(\frac{L^2\rho h}{4\delta_4} + \frac{\rho h^3}{48\delta_5}\right)\right\}\int_0^L \int_0^\infty g'(s) h_{xx}^2 dx ds \leq -\beta\zeta \|h\|_{L_g}^2.$$

In order to make all the terms of **(32)** negatives, we must solve the following system:

$$\begin{cases} \frac{\theta}{2}((\delta_0 g_0 + \delta_1 L) + (\alpha_1\delta_2 + \alpha_2\delta_3)L^2 c_4 \mathcal{E}(0)) + \gamma(\kappa g_0 \delta_6 + c_5 g_0 \delta_8 \delta_9 \mathcal{E}(0)) < \frac{\theta}{2}\kappa & (32.1) \\ \theta\left(\frac{1}{8\delta_1} + \frac{\rho h^3}{24}\right) + \gamma\left(g_0\delta_7 + \frac{\rho h^3}{12}\delta_5\right) < \frac{\rho h^3}{12} g_0 & (32.2) \\ \theta(L(\alpha_1\delta_2 + \alpha_2\delta_3) + \gamma\left(\frac{\delta_8\tilde{\delta}_5}{4\delta_9} g_0 \mathcal{E}(0)\right) < \theta & (32.3) \end{cases}$$

Firstly, we choose $\delta_0, \delta_1, \delta_2, \delta_3, \delta_6, \delta_8, \delta_9$ small enough to make (32.1) hold, for (32.1), we pick θ and δ_5, δ_7 small enough and for (32.3) we pick δ_2, δ_3 and γ small enough to make to hold therefore, we deduce that there is a positive constant C_1 and C_2 such that:

$$\frac{d}{dt} Y(t) \leq -C_1 E(t) + C_2 \|h\|_{L_g}^2 \qquad \textbf{(33)}$$

Lemma 12 Let $\varepsilon_0 > 0$ the the following inequality hold

$$\|h\|_{L_q}^2 G'(\varepsilon_0 E(t)) \leq -cE'(t) + c\varepsilon_0 E(t) G'(\varepsilon_0 E(t))$$

Where $c > 0$.

Proof Since E is non-increasing, then we have

$$|h_{xx}(s)|^2 = \int_0^L (\psi_{xx}(t) - \psi_{xx}(t-s))^2 dx \leq 4\sup_{s\in R} \int_0^L \psi_{xx}^2(s) dx$$

$$\leq c\sup_{s>0} \int_0^L \psi_{xx}^2(s) dx + cE(0)$$

$$\leq c\sup_{s>0} \int_0^L h_{0xx}^2(s) dx + cE(0)$$

Keeping in mind (1**8**) then there is a positive constant $m = c(M^2 + E(0))$ such that

$$|h_{xx}(s)|^2 \leq m, \forall t, s \in R_+$$

let $\varepsilon_0, \tau_1, \tau_2$ be a strictly positive constants, and denote $K(s) = \frac{s}{G^{-1}(s)}$, then K is non-increasing function and keeping in mind that G^{-1} is concave and $G^{-1}(0) = 0$, indeed, for any $t_1 > t_2 \geq 0$

$$K(t_1) = \frac{t_1}{G^{-1}(t_1)} = \frac{t_1}{G^{-1}\left(\frac{t_1}{t_2} t_2 + \left(1 - \frac{t_1}{t_2}\right) 0\right)}$$

$$\leq \frac{t_1}{\frac{t_1}{t_2} G^{-1}(t_2) + \left(1 - \frac{t_1}{t_2}\right) G^{-1}(0)}$$

$$= \frac{t_1}{G^{-1}(t_1)} = K(t_2)$$

and we have

$$K\left(-\tau_2 g'(s) \int_0^L (\psi_{xx}(t) - \psi_{xx}(t-s))^2 dx\right) \leq K(-m\tau_1 g'(s)) \; \textbf{(34)}$$

After using **(34)** we arrive to

$$\|h\|_{L_g}^2 = \int_0^L g(s)\int_0^L (\psi_{xx}(t)-\psi_{xx}(t-s))^2 dxds$$
$$= \int_0^\infty \frac{1}{\tau_1 G'(\varepsilon_0 E(t))} G^{-1}\left(-\tau_2 g'(s)\int_0^L (\psi_{xx}(t)-\psi_{xx}(t-s))^2\right)$$
$$\times \frac{\tau_1 G'(\varepsilon_0 E(t))g(s)}{-\tau_2 g'(s)} K\left(-\tau_2 g'(s)\int_0^L (\psi_{xx}(t)-\psi_{xx}(t-s))^2\right)ds$$
$$\le \int_0^\infty \frac{1}{\tau_1 G'(\varepsilon_0 E(t))} G^{-1}\left(-\tau_2 g'(s)\int_0^L (\psi_{xx}(t)-\psi_{xx}(t-s))^2 dx\right)$$
$$\times \frac{\tau_1 G'(\varepsilon_0 E(t))g(s)}{-\tau_2 g'(s)} K(-m\tau_1 g'(s))ds$$
$$\le \int_0^\infty \frac{1}{\tau_1 G'(\varepsilon_0 E(t))} G^{-1}\left(-\tau_2 g'(s)\int_0^L (\psi_{xx}(t)-\psi_{xx}(t-s))^2 dx\right)$$
$$\times \frac{m\tau_1 G'(\varepsilon_0 E(t))g(s)}{G^{-1}(-\tau_2 m g'(s))} ds$$

We denote by G^* the convex conjugate of G defined by

$$G^*(t) = tG'^{-1}(t) - G(G'^{-1}(t)) = \sup_{s\in R_+}[ts - G's)]$$

Recall the Young inequality of convex function

$$t_1 t_2 \le G(t_1) + G^*(t_1)$$

if we let

$$t_1 = G^{-1}\left(-\tau_2 g'(s)\int_0^L (\psi_{xx}(t)-\psi_{xx}(t-s))^2\right)$$

and

$$t_2 = \frac{m\tau_1 G'(\varepsilon_0 E(t))g(s)}{G^{-1}(-\tau_2 m g'(s))}$$

then we get

$$\|h\|_{L_g}^2 \le \frac{-\tau_2}{\tau_1 G'(\varepsilon_0 E(t))}\int_0^\infty g'(s)\int_0^L h_{xx}^2(s)dxds + \frac{-\tau_2}{\tau_1 G'(\varepsilon_0 E(t))}\int_0^\infty G^*\left(\frac{m\tau_1 G'(\varepsilon_0 E(t))g(s)}{G^{-1}(-\tau_2 m g'(s))}\right)ds$$

bearing in mind that

$$E'(t) \le \int_0^\infty g'(s)\int_0^L h_{xx}^2(s)dxds$$

And

$$G^*(t) \le tG'^{-1}(t)$$

yields

$$\|h\|_{L_g}^2 \le \frac{-\tau_2}{\tau_1 G'(\varepsilon_0 E(t))} E'(t) + m\int_0^\infty \frac{g(s)}{G^{-1}(-\tau_2 m g'(s))} G'^{-1}\left(\frac{m\tau_1 G'(\varepsilon_0 E(t))g(s)}{G^{-1}(-\tau_2 m g'(s))}\right)ds$$

Thanks to (5), denote $\sup_{s\in R_+} \frac{g(s)}{G^{-1}(-g'(s))} = m''$ and using the fact that G'^{-1} is non-decreasing, and for more of simplicity, we choose $\tau_2 = \frac{1}{m}$ we get

$$\|h\|_{L_g}^2 \leq \frac{-2}{m\tau_1 G'(\varepsilon_0 E(t))} E'(t) + mG'^{-1}(\tau_1 mm' G'(\varepsilon_0 E(t))) \int_0^\infty G'^{-1}\left(\frac{g(s)}{G^{-1}(-g'(s))}\right) ds$$

We denote $\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))} = m''$, choosing $\tau_1 = \frac{1}{mm'}$ finally we get

$$\|h\|_{L_g}^2 \leq \frac{-2}{m\tau_1 G'(\varepsilon_0 E(t))} E'(t) + mm'' \varepsilon_0 E(t). \qquad (35)$$

Let us continue our proof of **(theorem 3)**
Multiply by $G'(\varepsilon_0 E(t))$ **(33)** and use **(lemma 12)** we get

$$G'(\varepsilon_0 E(t))Y'(t) \leq (c\varepsilon_0 - C_1)G'(\varepsilon_0 E(t))E(t) - cE'(t)$$

We pick $\varepsilon_0 < \frac{C_1}{c}$ then we obtain

$$G'(\varepsilon_0 E(t))Y'(t) + cE'(t) \leq -c'G'(\varepsilon_0 E(t))E(t) \qquad (36)$$

Where $c' > 0$.
Now let Z be the functional

$$Z(t) = \gamma\left[G'(\varepsilon_0 E(t))Y(t) + cE(t)\right]$$

Recall that G' is non-increasing function, therefore, $Z \sim E$, we use (36) then Z satisfies

$$Z'(t) = \gamma\left[G''(\varepsilon_0 E)Y + G'(\varepsilon_0 E)Y' + cE'\right] \leq -\gamma c'' G'(\varepsilon_0 E)Z \qquad (37)$$

We choose γ small enough such that $Z \leq E$ and $Z(0) \leq 1$, therefore we get

$$Z' = -\tilde{c}G'(\epsilon_0 Z)Z \qquad (38)$$

And from **(38)** we have

$$\frac{-Z'}{G'(\epsilon_0 Z)Z} \geq \tilde{c}$$

Integrating over $(0,t)$ we get

$$G_1(Z(t)) \geq \tilde{c}t + G_1(Z(0)) \geq \tilde{c}t$$

Where $G_1(t) = \int_t^1 \frac{ds}{sG'(\varepsilon_0 s)}$. (G_1 is decreasing function)
And finally,

$$Z(t) \leq G_1^{-1}(\tilde{c}t)$$

And this finishes the proof.

REFERENCES

[1] Dafermos, C.M. Asymptotic stability in viscoelasticity. Arch. Rational Mech. Anal. 37, 297-308 (1970).
[2] F. D. Araruna, P. Braz E Ssilva. E. Zuazua. Asymptotic limits and stabilization for the 1-D Nonlinear Mindlin-Timoshenko system.J Syst Sci Complex (2010) 23: 1-17.
[3] Aissa Guesmia, Mokhtar Kirane.Uniform and weak stability of Bresse system with two infinite

memories.Z. Angew.
[4] Math. Phys. (2016) 67:124.
[5] Mohamed Ferhat, Ali Hakem. Well-posedness and asymptotic stability of solutions to a bresse system with time varying delay terms and infinite memories. Ser. Math. Inform. Vol. 31, No 1 (2016), 97-124.
[6] G. Perla Menzala, Enrique Zuazua. On a one-dimensional version of the dynamical Marguerre-Vlasov system. Bol. Soc. Bras. Mat., Vol.32, No. 3, 303-319.
[7] Laid Djilali, Abbes Benaissa, Abdelkader Benaissa. Global existence and energy decay of solutions to a viscoelastic Timoshenko beam system with a nonlinear delay term.Applicable analysis,2015.
[8] Aissa Guesmia, Salim A. Messaoudi, Abdelaziz Soufyane. Stabilization of a linear Timoshenkosystem with infinite history and applications to the Timoshenko-heat systems. Electronic Journal of Differential Equations, Texas State University, Department of Mathematics, 2012. <hal-01281868>.
[9] G. PerlaMenzala, Erique Zuazua. One a one-dimentional versionof the dynamical
[10] Marguerre-Vlasov system. AMS Classification: 35B40, 73C50, 73K15.
[11] Jum-Ran Kang. Energy decay rates for von Karman system with memory
[12] and boundary feedback. Applied Mathematics and Computation 218 (2012) 9085-9094.
[13] Serge Nicaise, Cristina pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks.SIAM J.Control optim,Vol. 45, No. 5, pp. 1561-1585.
[14] G.Perla Menzala, Enrique Zuazua,Timoshenko's beam equation as a limit of a nonlinea one-dimentional Von-karman system, Supported by Grants PB 96-0663 and the DGES (Spain), ERB FMRX CT 960033 of the European Union and partially by PRONEX (MCT, Brasil).
[15] A. Guesmia; Asymptotic stability of abstract dissipative systems with infinite memory, J. Math. Anal. Appl., 382 (2011), 748-760.
[16] A. Guesmia, S. A. Messaoudi; On the control of solutions of a viscoelastic equation, Appl. Math. Computations, 206 (2008), 589-597.
[17] A. Guesmia, S. A. Messaoudi; General energy decay estimates of Timoshenko systems with frictional versus viscoelastic damping, Math. Meth. Appl. Sci., 32 (2009), 2101-2122.
[18] F. Ammar-Khodja, A Benabdallah, J. E. Munoz Rivera, R. Racke; Energy decay for Timoshenko systems of memory type, J. Diff. Equa., 194 (2003), 82-115.
[19] H. Brezis, Analyse Fonctionnelle: Theorie et Applications, Dunod, Paris, 1999.