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# On the Localization of Factored Fourier Series

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**Abstract**

In the present paper, a theorem concerning local property of $|A, p_n|_k$ summability of factored Fourier series, which generalizes a result dealing with $|\bar{N}, p_n|_k$ summability of factored Fourier series, has been obtained. Also, some results have been given.

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## 1 Introduction

Let $\sum a_n$ be an infinite series with its partial sums $(s_n)$ and $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

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The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [21])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

If we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability (see [2]). If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of $n$ (resp. $a_{nv} = \frac{p_v}{P_n}$ and $k = 1$), $|A, p_n|_k$ summability reduces to $|C, 1|_k$ summability (see [11]) (resp. $|\bar{N}, p_n|$) summability. Also, if we take $p_n = 1$ for all values of $n$, then $|A, p_n|_k$ summability reduces to $|A|_k$ summability (see [22]). Furthermore, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A|_k$ summability reduces to $|R, p_n|_k$ summability (see [4]).

A sequence $(\lambda_n)$ is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer $n$, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ (see [24]).

Let $f(t)$ be a periodic function with period $2\pi$, and integrable $(L)$ over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) \quad = \quad \sum_{n=1}^{\infty} C_n(t),$$

where $(a_n)$ and $(b_n)$ denote the Fourier coefficients. It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function $f$ (i.e. it depends only on the behaviour of $f$ in an arbitrarily small neighbourhood of $x$), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function $f$ (see [23]).

## 2 Known Results

There are many different applications of Fourier series. Some of them can be find in [1], [5]-[10], [12]-[20]. Furthermore, Bor [3] has proved the following theorem.

**Theorem 1** *Let $k \geq 1$ and $(p_n)$ be a sequence such that*

$$P_n = O(np_n), \tag{1}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{2}$$

*Then the summability $|\bar{N}, p_n|_k$ of the series $\sum \frac{C_n(t)\lambda_n P_n}{np_n}$ at a point can be ensured by local property, where $(\lambda_n)$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.*

## 3 Main Result

The purpose of this paper is to generalize Theorem 1 by using the definition of $|A, p_n|_k$ summability. Now, let us introduce some further notations. Let $A = (a_{nv})$ be a normal matrix, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{3}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{4}$$

and it is well known that

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{5}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{6}$$

Now, we will prove the following theorem.

**Theorem 2** *Let $k \geq 1$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \ n = 0, 1, ..., \tag{7}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1, \tag{8}$$

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$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{9}$$

$$|\hat{a}_{n,v+1}| = O\left(v\,|\Delta_v \hat{a}_{nv}|\right), \tag{10}$$

*where* $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}$. *Let the sequence* $(p_n)$ *be such that the conditions (1) and (2) of Theorem 1 are satisfied. Then the summability* $|A, p_n|_k$ *of the series* $\sum \frac{C_n(t)\lambda_n P_n}{np_n}$ *at a point can be ensured by local property, where* $(\lambda_n)$ *is as in Theorem 1.*

Here, if we take $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1.

We should give the following lemmas for the proof of Theorem 2.

**Lemma 3** ([13]) *If the sequence* $(p_n)$ *is such that the conditions (1) and (2) of Theorem 1 are satisfied, then*

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{11}$$

**Lemma 4** ([10]) *If* $(\lambda_n)$ *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, then* $(\lambda_n)$ *is non-negative and decreasing, and* $n\Delta\lambda_n \to 0 \quad as \quad n \to \infty$.

**Lemma 5** *Let* $k \geq 1$ *and let the sequence* $(p_n)$ *be such that the conditions (1) and (2) of Theorem 1 are satisfied. If* $(s_n)$ *is bounded and the conditions (7)-(10) are satisfied, then the series*

$$\sum_{n=1}^{\infty} \frac{a_n \lambda_n P_n}{np_n} \tag{12}$$

*is summable* $|A, p_n|_k$, *where* $(\lambda_n)$ *is as in Theorem 1.*

**Remark 6** *Since* $(\lambda_n)$ *is a convex sequence, therefore* $(\lambda_n)^k$ *is also convex sequence and*

$$\sum \frac{1}{n}(\lambda_n)^k < \infty. \tag{13}$$

## 4 Proof of Lemma 5

Let $(M_n)$ denotes the $A$-transform of the series $\sum \frac{a_n\lambda_n P_n}{np_n}$. Then, we have

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v \lambda_v P_v}{v p_v}$$

by (5) and (6).

Now, we get

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv}\lambda_v P_v}{v p_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv}\lambda_v P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= \frac{a_{nn} P_n \lambda_n}{n p_n} s_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v(\hat{a}_{nv})}{v p_v} s_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \Delta\lambda_v P_v}{v p_v} s_v \\
&+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left( \frac{P_v}{v p_v} \right) s_v \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}
\end{aligned}$$

by applying Abel's transformation. For the proof of Lemma 5, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |M_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

First, we have

$$\begin{aligned}
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{k-1} |M_{n,1}|^k &= \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{k-1} \left| \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \right|^k \\
&= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{n^k} \left( \frac{P_n}{p_n} \right)^k (\lambda_n)^k |s_n|^k \\
&= O(1) \sum_{n=1}^{m} \frac{1}{n} (\lambda_n)^k = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by (9), (1) and (13).

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From Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v\Delta_v(\hat{a}_{nv})}{vp_v}s_v\right|^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)|\Delta_v(\hat{a}_{nv})|\,(\lambda_v)|s_v|\right\}^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k|s_v|^k\right\}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.\end{aligned}$$

By (4) and (3), we have that

$$\begin{aligned}\Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv}-\hat{a}_{n,v+1}\\ &= \bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}\\ &= a_{nv}-a_{n-1,v}. \qquad (14)\end{aligned}$$

Thus using (8), (3) and (7)

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv}) \leq a_{nn}. \qquad (15)$$

Hence, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k\right\}\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|.\end{aligned}$$

Here, from (14) and (8), we obtain

$$\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| = \sum_{n=v+1}^{m+1}(a_{n-1,v}-a_{nv}) \leq a_{vv}.$$

Then,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k = O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k a_{vv}$$

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$$\begin{aligned} &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}(\lambda_v)^k \\ &= O(1)\sum_{v=1}^{m}v^{k-1}\frac{1}{v^k}(\lambda_v)^k \\ &= O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_v)^k = O(1) \quad as \quad m\to\infty, \end{aligned}$$

by (9), (1) and (13).

Now, by (1) and Hölder's inequality, we have

$$\begin{aligned} \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_v P_v}{vp_v}s_v\right|^k \\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v|s_v|\right\}^k \\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v|s_v|^k\right\}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}^{k-1}. \end{aligned}$$

Now, (4), (3), (7) and (8) imply that

$$\begin{aligned} \hat{a}_{n,v+1} = \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1} &= \sum_{i=v+1}^{n}a_{ni} - \sum_{i=v+1}^{n-1}a_{n-1,i} \\ &= \sum_{i=0}^{n}a_{ni} - \sum_{i=0}^{v}a_{ni} - \sum_{i=0}^{n-1}a_{n-1,i} + \sum_{i=0}^{v}a_{n-1,i} \\ &= 1 - \sum_{i=0}^{v}a_{ni} - 1 + \sum_{i=0}^{v}a_{n-1,i} \\ &= \sum_{i=0}^{v}(a_{n-1,i} - a_{ni}) \geq 0 \qquad (16) \end{aligned}$$

and from this, using (4), (3) and (8), we have

$$\begin{aligned} |\hat{a}_{n,v+1}| &= \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1} \\ &= \sum_{i=v+1}^{n}a_{ni} - \sum_{i=v+1}^{n-1}a_{n-1,i} \\ &= a_{nn} + \sum_{i=v+1}^{n-1}(a_{ni} - a_{n-1,i}) \\ &\leq a_{nn}. \end{aligned}$$

Hence, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\left\{\sum_{v=1}^{n-1}\Delta\lambda_v\right\}^{k-1}\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}\\ &= O(1)\sum_{v=1}^{m}\Delta\lambda_v\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.\end{aligned}$$

Now, by (16), (3) and (7), we find

$$\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|\leq 1. \tag{17}$$

Thus,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k = O(1)\sum_{v=1}^{m}\Delta\lambda_v=O(1)\quad as\quad m\to\infty,$$

by Lemma 4.

Since $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$ by Lemma 3 and also by using (10), we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\Delta\left(\frac{P_v}{vp_v}\right)s_v\right|^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})|s_v|\right\}^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k|s_v|^k\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.\end{aligned}$$

From (15) and (9),

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k\\ &= O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_{v+1})^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.\end{aligned}$$

Again using (17),

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k = O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_{v+1})^k=O(1)\quad as\quad m\to\infty,$$

by (13). Hence the proof of Lemma 5 is completed.

## 5 Proof of Theorem 2

The convergence of the Fourier series at $t = x$ is a local property of $f$ (i.e., it depends only on the behaviour of $f$ in an arbitrarily small neighbourhood of $x$), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of $f$. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of $x$ depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 2 is a consequence of Lemma 5.

## 6 Conclusions

For $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of $n$, then we get a result concerning $|C,1|_k$ summability factors of Fourier series. If we take $a_{nv} = \frac{p_v}{P_n}$ and $k = 1$, then we get a result concerning $|\bar{N},p_n|$ summability factors of Fourier series (see [13]).

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