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A unified convergence analysis for single step-type methods for non-smooth operators

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Abstract

This paper is devoted to the approximation of solutions for nonlinear equations by using iterative methods. We present a unified convergence analysis for some Newton-type methods. We consider both semilocal and local analysis. In the first one, the hypotheses are imposed on the initial guess and in the second on the solution. The results can be applied for smooth and non-smooth operators. In the numerical section we study two applications, first one, it is devoted to a nonlinear integral equation of Hammerstein type and in second one, we approximate the solution of a nonlinear PDE related to image denoising.

1 Introduction

There are several situations in which the modeling of a problem leads us to calculate a solution of an equation

$$F(x) = 0. \tag{1}$$

This equation can represent a differential equation, ordinary or partial, an integral equation, an integro-differential equation or a simple system of equations. In general, mathematical methods that obtain exact solutions of (1) are not known, so that iterative methods are usually used to solve (1) [9, 10, 1, 2, 3, 4, 5, 7, 12]. For a greater generality,

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in this study, we consider $F : D \subset X \to Y$, where X, Y are Banach spaces and D is a nonempty, open and convex set. And we pay attention to F is continuous and Fréchet non-differentiable. In this case, to approximate a solution of (1), iterative methods using divided differences are usually applied instead of using derivatives [12]-[11]. It is common to approximate derivatives by divided differences for obtaining derivative free iterative schemes. So, given an operator $G : D \subset X \to Y$, let us denote by $\mathfrak{L}(X, Y)$ the space of bounded linear operators from X into Y, an operator $[x, y; G] \in \mathfrak{L}(X, Y)$ is called a first order divided difference for the operator G on the points x and y $(x \neq y)$ in D if

$$[x, y; G](x - y) = G(x) - G(y). \tag{2}$$

Steffensen's method [13] is the most used iterative method using divided differences in the algorithm, which is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [x_n, x_n + F(x_n); F]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{3}$$

As we can see in [14], Steffensen's method has a problem of accessibility that can be solved by using a procedure of decomposition ([15]) for operator F, the Fréchet differentiable part and the non-differentiable part. So, we consider

$$F(x) = F_1(x) + F_2(x) \tag{4}$$

where $F_1, F_2 : D \subset X \to Y$, F_1 is Fréchet differentiable and F_2 is continuous and Fréchet non-differentiable. Thus, in [14], we consider the method of Newton-Steffensen, given by the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - \left(F_1'(x_n) + [x_n, x_n + F(x_n); F_2]\right)^{-1} \left(F_1(x_n) + F_2(x_n)\right), \quad n \geq 0, \end{cases} \tag{5}$$

with $X = Y$, which improves significantly the accessibility of method (3) and has quadratic convergence.

By using this procedure of decomposition for operator F, we see that we can also consider the application of iterative methods that use derivatives when F is non-differentiable. So, for example, we can consider the well-known Newton's method, which algorithm is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F'(x_n)]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{6}$$

Obviously, Newton's method is not applicable, under form (6), when F is not Fréchet differentiable. However, if we consider decomposition of F given in (4), we can use the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F_1'(x_n)]^{-1} (F_1(x_n) + F_2(x_n)), \quad n \geq 0, \end{cases} \tag{7}$$

which is known as method of Zincenko [17].

The main aim of this paper consists of defining one-point iterative methods of Newton-type, as we can see previously, to obtain a general study for the convergence, local and semilocal, for these type of iterative methods. Moreover, in view of the last two considerations, with these one point iterative methods we can to improve the accessibility of one-point iterative methods that use divided differences and, in addition, to extend the application of iterative methods that use derivatives when F is Fréchet non-differentiable. For this aim, we consider the one-point iterative methods of Newton-type given by the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - L_n^{-1}(F_1(x_n) + F_2(x_n)), \quad n \geq 0, \end{cases} \tag{8}$$

where $L_n := L(x_n)$ with $L(.) : D \to \mathfrak{L}(X, Y)$. Clearly, method (8) can be used to solve equations containing a nondifferentiable term.

There are a lot of iterative methods that can be written as algorithm (8), in addition to modifications of Steffensen and Newton given in (5) and (7), where $L(x) = F_1'(x) + [x, x + F_2(x); F_2]$ and $L(x) = F_1'(x)$, respectively. At the same time, we can also consider two interesting cases. Firstly, the generalized Steffensen methods [6], that are very used in the approximation of solutions of non-differentiable operators equations and the algorithm is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [x_n - aF(x_n), x_n + bF(x_n); F]^{-1}F(x_n), \quad n \geq 0. \end{cases}$$

Then, it is clear that we can define the generalized Newton-Steffensen method from 8) with $L(x) = F_1'(x) + [x - aF_2(x), x + bF_2(x); F_2]$, so we have the final iterative function given as:

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - (F_1'(x_n) + [x_n - aF_2(x_n), x_n + bF_2(x_n); F_2])^{-1}F(x_n), \quad n \geq 0. \end{cases} \tag{9}$$

where $a, b \in \mathbb{R}$.

In the same way as Newton's method, from Stirling method [16],

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F_1'(x_n - F(x_n))]^{-1}F(x_n), \quad n \geq 0, \end{cases} \tag{10}$$

we can define a modification of Newton-type, that can be applied to Fréchet non-differentiable operators. For this, just consider (8) with $L(x) = F_1'(x - F(x))$. In both cases, we choose $X = Y$. Obviously, we can include a lot of iterative methods in (8) if F is Fréchet differentiable.

So, in this paper, we study the convergence of algorithm (8). We analyze the semilocal and local convergences, so that we have a study of convergence of a lot of iterative methods that are usually used and can be written by algorithm (8).

Section 2 is devoted to the theoretical analysis about local and semilocal convergence for a very general single step Newton-like methods. In Section 3 we make a comparison for the behavior of some of these methods by solving a non-differentiable problem. In Section 4, we consider an application related to image denoising. Finally, in Section 5 we give some conclusions.

2 Convergence Analysis for single step Newton-like methods

In this section, we present both semilocal and local convergence analysis. In the first one, the hypotheses are imposed on the initial guess and in the second on the solution. The results can be applied for smooth and non-smooth operators.

2.1 Local Convergence Analysis

In this section, we first present the local followed by the semilocal convergence of method (8). Let $v_0 : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function with $v_0(0) = 0$. Suppose that the equation

$$v_0(t) = 1 \tag{11}$$

has at least one positive root r_0. Let also $v : [0, r_0) \to [0, +\infty)$ be a nondecreasing continuous function. Define function $\bar{v}$ on the interval $[0, r_0)$ by $\bar{v}(t) = \frac{v(t)}{1-v_0(t)} - 1$.

Suppose equation

$$\bar{v}(t) = 0 \tag{12}$$

has at least one positive root. Denote by r the smallest such root. It follows that for each $t \in [0, r)$

$$0 \leq v_0(t) < 1 \tag{13}$$

and

$$0 \leq \bar{v}(t) < 1. \tag{14}$$

The local convergence analysis of method (8) uses the conditions (A):

- (a_1) There exist a solution $x^* \in D$ of equation (4), and $B \in \mathfrak{L}(X, Y)$ such that $B^{-1} \in \mathfrak{L}(Y, X)$.

- (a_2) Condition (11) holds and for each $x \in D$

  $$\|B^{-1}(L(x) - B)\| \leq v_0(\|x - x^*\|),$$

  where v_0 is defined previously and r_0 is given in (11).

  Set $D_0 = D \cap \bar{U}(x*, r_0)$.

- (a_3) For $L : D_0 \to \mathfrak{L}(X, Y)$, any solution y of equation (4) and each $x \in D_0$

$$\|B^{-1}(F_1(x) + F_2(x) - L(x)(x - y))\| \leq v(\|x - y\|)\|x - y\|,$$

  where v is defined previously.

- (a_4) $\bar{U}(x^*, r) \subset D$, where r is given in (12).

- (a_5) There exist $r^* \geq r$ such that

$$\xi := \frac{v(r^*)}{1 - v_0(r)} \in [0, 1).$$

  Set $D_1 = D \cap \bar{U}(x*, r^*)$.

Remark 1

- *Condition (a_3) can be replaced by the stronger: for each $x, y, z \in D_0$*

$$\|B^{-1}(F_1(x) + F_2(x) - L(x)(x - y))\| \leq v_1(\|x - y\|)\|x - y\|,$$

  where function v_1 is as v. But for each $t \geq 0$

$$v(t) \leq v_1(t).$$

- *Linear operator B does not necessarily depend on the solution x^*. It is used to determine the invertibility of linear operator $L(\cdot)$ appearing in the method. The invertibility of B can be assured by an additional condition of the form $||I - B|| < 1$ or some other way. A possible choice for B is $B = B(x^*)$ or $B = F_1'(x^*)$.*

- *It follows from the definition of r_0 and r that $r_0 \geq r$.*

We can present the local convergence analysis of method (8) based on the aforementioned conditions (A).

Theorem 2 *Suppose that the conditions (A) hold. Then, sequence x_k generated by method (8) for $x_0 \in U(x^*, r) - x^*$ is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ and converges to x^*. Moreover, the following estimates hold.*

$$\|x_{k+1} - x^*\| \leq \frac{v(\|x_k - x^*\|)}{1 - v_0(\|x_k - x^*\|)}\|x_k - x^*\| \leq \|x_k - x^*\| < r. \tag{15}$$

The vector x^* is the only solution of equation (4) in D_1, where D_1 is given in $(a5)$.

Proof We base the proof on k and mathematical induction. Let $x \in U(x^*, r)$. Using (8), $(a1)$ and $(a2)$, we have in turn that

$$\|B^{-1}(L(x) - B)\| \leq v_0(\|x - x^*\|) \leq v_0(r) < 1. \tag{16}$$

It follows by (16) and the Banach lemma on invertible operators [] that $L(x)^{-1} \in \mathfrak{L}(Y, X)$ and

$$\|L(x)^{-1}B\| \leq \frac{1}{1 - v_0(\|x - x^*\|)}. \tag{17}$$

In particular, estimate (17) holds for $x = x_0$, so x_1 is well defined by method (8) for $k = 0$. We also get by method (8) (for $k = 0$), $(a1)$, $(a3)$, (14) and (17) (for $k = 0$) that

$$\begin{aligned}
\|x_1 - x^*\| &= \|x_0 - x^* - L(x_0)^{-1}(F_1(x_0) + F_2(x_0))\| \\
&= \|[-L(x_0)^{-1}B][B^{-1}(F_1(x_0) + F_2(x_0) - L(x_0)(x_0 - x^*))]\| \\
&\leq \|L(x_0)^{-1}B\|\|B^{-1}(F_1(x_0) + F_2(x_0) - L(x_0)(x_0 - x^*))\| \\
&\leq \frac{v(\|x_0 - x^*\|)}{1 - v_0(\|x_0 - x^*\|)}\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (18)
\end{aligned}$$

which shows estimate (15) for $k = 0$, and $x_1 \in U(x^*, r)$.

Simply, replace x_0, x_1 by x_i, x_{i+1} in the preceding estimates to complete the induction for estimate (15). Then, in view of the estimate

$$\|x_{i+1} - x^*\| \leq \xi\|x_i - x^*\| < r, \tag{19}$$

where

$$\xi = \frac{v(\|x_0 - x^*\|)}{1 - v_0(\|x_0 - x^*\|)} \in [0, 1),$$

we deduce that $\lim_{i\to+\infty} x_i = x^*$ and $x_{i+1} \in U(x^*, r)$. Moreover, to show the uniqueness part, let $y^* \in D_1$ with $F_1(y^*) + F_2(y^*) = 0$. Using $(a3)$, $(a5)$ and estimate (18), we obtain in turn that

$$\begin{aligned}
\|x_{i+1} - y^*\| &\leq \|L(x_i)^{-1}B\|\|B^{-1}(F_1(x_i) + F_2(x_i) - L(x_i)(x_i - y^*))\| \\
&\leq \frac{v(\|x_i - y^*\|)}{1 - v_0(\|x_i - x^*\|)}\|x_i - y^*\| \\
&\leq \xi\|x_i - y^*\| < \xi^{i+1}\|x_0 - y^*\|, \qquad (20)
\end{aligned}$$

which shows $\lim_{i\to+\infty} x_i = y^*$. But, we showed $\lim_{i\to+\infty} x_i = x^*$. Hence, we conclude that $x^* = y^*$.

□

2.2 Semilocal Convergence Analysis

As in the local case it is convenient to define some functions and parameters for the semilocal analysis. Let $w_0 : [0, +\infty) \to [0, +\infty)$ be a continuous and nondecreasing function.

Suppose that equation

$$w_0(t) = 1. \tag{21}$$

has at least one positive root. Denote by ρ_0 the smallest such root. Let also $w : [0, \rho_0) \times [0, \rho_0) \to [0, +\infty)$ be a nondecreasing continuous function. Moreover, for $\eta \geq 0$, define parameters C_1 and C_2 by

$$\begin{aligned} C_1 &= \frac{w(\eta, 0)}{1 - w_0(\eta)}, \\ C_2 &= \frac{w(\frac{\eta}{1-C_1}, \eta)}{1 - w_0(\frac{\eta}{1-C_1})} \end{aligned}$$

and function $C : [0, \rho_0) \to [0, +\infty)$ by $C(t) = \frac{w(t,t)}{1-w_0(t)}$. Suppose that equation

$$(\frac{C_1 C_2}{1 - C(t)} + C_1 + 1)\eta - t = 0 \tag{22}$$

has as least one positive root. Denote by ρ the smallest such root.

Next, we show the semilocal convergence analysis of method (8) in an analogous way, under the conditions (H):

- $(h1)$ There exists $x_0 \in D$ and $B \in \mathfrak{L}(X, Y)$ such that $B^{-1} \in \mathfrak{L}(Y, X)$.
- $(h2)$ Condition (21) holds and for each $x \in D$

  $$\|B^{-1}(L(x) - B)\| \leq w_0(\|x - x_0\|),$$

  where w_0 is as defined previously, and ρ_0 is given in (21).

  Set $D_2 = D \bigcap \bar{U}(x_0, \rho_0)$.
- $(h3)$ For $L(\cdot) : D_2 \to \mathfrak{L}(X, Y)$, and each $x, y \in D_2$

  $$\begin{aligned} &\|B^{-1}(F_1(y) - F_1(x) + F_2(y) - F_2(x) - L(x)(y - x))\| \\ &\qquad \leq w(\|y - x_0\|, \|x - x_0\|)\|y - x\|, \end{aligned}$$

  where w is as defined previously.
- $(h4)$ $\bar{U}(x_0, \rho) \subseteq D$ and condition (22) holds for ρ, where $\|x_1 - x_0\| \leq \eta$.
- $(h5)$ There exists $\rho^* \geq \rho$ such that

  $$\xi_0 := \frac{w(\rho, \rho^*)}{1 - w_0(\rho)} \in [0, 1).$$

  Set $D_2 = D \bigcap \bar{U}(x^*, \rho^*)$.

Then, as in the local case but using the (H) instead of the (A) conditions, we have in turn the estimates:

$$\begin{aligned}
\|x_2 - x_1\| &\leq \frac{w(\|x_1 - x_0\|, \|x_0 - x_0\|)}{1 - w_0(\|x_1 - x_0\|)} = C_1\|x_1 - x_0\|, \\
\|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (1 + C_1)\|x_1 - x_0\| \\
&= \frac{1 - C_1^2}{1 - C_1}\|x_1 - x_0\| \\
&< \frac{\|x_1 - x_0\|}{1 - C_1}\eta < \rho, \\
\|x_3 - x_2\| &\leq \frac{w(\|x_2 - x_0\|, \|x_1 - x_0\|)}{1 - w_0(\|x_2 - x_0\|)}\|x_2 - x_1\| \\
&\leq \frac{w(\frac{\eta}{1-C_1}, \eta)}{1 - w_0(\frac{\eta}{1-C_1})}\|x_2 - x_1\| = C_2\|x_2 - x_1\| \\
\|x_3 - x_0\| &\leq \|x_3 - x_2\| + \|x_2 - x_1\| + \|x_1 - x_0\| \\
&\leq C_2\|x_2 - x_1\| + C_1\|x_1 - x_0\| + \|x_1 - x_0\| \\
&\leq (C_2C_1 + C_1 + 1)\|x_1 - x_0\|, \\
\|x_4 - x_3\| &\leq \frac{w(\|x_3 - x_0\|, \|x_2 - x_0\|)}{1 - w_0(\|x_3 - x_0\|)}\|x_3 - x_2\| \\
&\leq C(\rho)\|x_3 - x_2\| \leq C(\rho)C_2\|x_2 - x_1\| \\
&\leq C(\rho)C_2C_1\|x_1 - x_0\|, \\
&\quad \ldots \\
\|x_{i+1} - x_i\| &\leq C(\rho)\|x_i - x_{i-1}\| \leq C(\rho)^{i-2}\|x_3 - x_2\| \\
\|x_{i+1} - x_0\| &\leq \|x_{i+1} - x_i\| + \ldots + \|x_4 - x_3\| + \|x_3 - x_0\| \\
&\leq C(\rho)\|x_i - x_{i-1}\| + \ldots + C(\rho)\|x_3 - x_2\| \\
&\quad + (C_2C_1 + C_1 + 1)\|x_1 - x_0\| \\
&\leq C(\rho)^{i-2}\|x_3 - x_2\| + \ldots + C(\rho)\|x_3 - x_2\| \\
&\quad + (C_2C_1 + C_1 + 1)\|x_1 - x_0\| \\
&\leq (\frac{1 - C(\rho)^{i-1}}{1 - C(\rho)}C_2C_1 + C_1 + 1)\|x_1 - x_0\| \\
&< (\frac{C_1C_2}{1 - C(\rho)} + C_1 + 1)\eta \leq \rho, \qquad (23)
\end{aligned}$$

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$$\begin{aligned}
\|x_{i+j}-x_i\| &\leq \|x_{i+j}-x_{i+j-1}\|+\|x_{i+j-1}-x_{i+j-2}\|+...+\|x_{i+1}-x_i\| \\
&\leq (C(\rho)^{i+j-3}+...+C(\rho)^{i-2})\|x_3-x_2\| \\
&\leq C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}\|x_3-x_2\| \\
&\leq C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}C_2C_1\|x_1-x_0\| \\
&\leq C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}C_2C_1\eta. \qquad (24)
\end{aligned}$$

It follows from (23) that $x_i \in U(x_0,\rho)$ and from (24) that sequence x_i is complete in X and as such it converges to some $x^* \in \bar{U}(x_0,\rho)$. By letting $i \to +\infty$ in the estimate

$$\|B^{-1}(F_1(x_i)+F_2(x_i))\| = \|B^{-1}(F_1(x_i)+F_2(x_i)-F_1(x_{i-1})-F_2(x_{i-1})-B_{i-1}(x_i-x_{i-1}))\|$$

$$\leq \frac{w(\|x_i-x_0\|,\|x_{i-1}-x_0\|)\|\|x_i-x_{i-1}\|}{1-w_0(\|x_i-x_0\|)} \leq \frac{w(\rho,\rho)}{1-w_0(\rho)}\|x_i-x_{i-1}\|,$$

we obtain $F_1(x^*)+F_2(x^*)=0$. The uniqueness part is omitted as identical to the one in the local convergence case.

Hence, we arrived at the semilocal convergence result for method (8).

Theorem 3 *Suppose that the conditions (H) hold. Then, sequence x_k generated by method (8) for $x_0 \in D$ is well defined in $U(x_0,\rho)$ remains in $U(x_0,\rho)$ and converges $x^* \in \bar{U}(x_0,\rho)$ to a solution of equation (4). Moreover, the vector x^* is the only solution of equation (4) in D_3, where D_3 is defined previously.*

The same comments introduced in the previous remark are valid.

We emphasize the theoretical importance of this theorem because it presents a unified studied of the local and semilocal convergence of a big variety of Newton-Type methods and Steffensen type methods, so the study is applicable to differentiable an non differentiable equations.

3 Numerical Experiments

In this section, we consider a nonlinear integral equation of Hammerstein type, which can be used to describe applied problems in the fields of electro-magnetics, fluid dynamics, in the kinetic theory of gases and, in general, in the reformulation of boundary value problems. These equations are of the form:

$$x(s) = f(s) - \int_a^b K(s,t)\Phi(x(t))dt, \quad a \leq s \leq b, \qquad (25)$$

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where $x(s), f(s) \in C[a,b]$, with $-\infty < a < b < \infty$, and Φ is a polynomial function. One of the most used techniques to solve this kind of equations consists of expressing them as a nonlinear operator in a Banach space and solving the following operator equation:

$$F(x)(s) = x(s) - f(s) + \int_a^b K(s,t)\Phi(x(t))dt = 0, \tag{26}$$

where $F : D \subseteq C[a,b] \to C[a,b]$ with D a non-empty open convex subset of $C[a,b]$ with the max-norm $\|\nu\| = \max_{s\in[a,b]} |\nu(s)|$.

We consider (25), where K is the Green function in $[a,b] \times [a,b]$, and then use a discretization process to transform equation (26) into a finite dimensional problem by approximating the integral by an adequate quadrature formula

$$\int_a^b q(t)\,dt \simeq \sum_{i=1}^{p} w_i q(t_i),$$

where the nodes t_i and the weights w_i are known.

If we denote the approximations of $x(t_i)$ and $f(t_i)$ by x_i and f_i, respectively, with $i = 1, 2, \dots, p$, then equation (26) is equivalent to the following system of nonlinear equations:

$$x_i = f_i + \sum_{j=1}^{p} a_{ij}\,\Phi(x_j), \quad j = 1, 2, \dots, p, \tag{27}$$

where

$$a_{ij} = w_j K(t_i, t_j) = \begin{cases} w_j \frac{(b-t_i)(t_j-a)}{b-a}, & j \le i, \\ w_j \frac{(b-t_j)(t_i-a)}{b-a}, & j > i. \end{cases}$$

Now, system (27) can be written as

$$\mathbb{F}(\mathbf{x}) \equiv \mathbf{x} - \mathbf{f} - A\,\mathbf{z} = 0, \qquad \mathbb{F} : \Delta \subseteq \mathbb{R}^p \longrightarrow \mathbb{R}^p, \tag{28}$$

where

$$\mathbf{x} = (x_1, x_2, \dots, x_p)^T, \qquad \mathbf{f} = (f_1, f_2, \dots, f_p)^T, \qquad A = (a_{ij})_{i,j=1}^{p},$$
$$\mathbf{z} = (\Phi(x_1), \Phi(x_2), \dots, \Phi(x_p))^T.$$

After that, we choose $a = 0$, $b = 1$, $K(s,t)$ as the Green function in $[0,1] \times [0,1]$ and $\Phi(x(t)) = x(t)^3 + |x(t)|$ in (25). Then, the system of nonlinear equations given in (28) is of the form

$$\mathbb{F}(\mathbf{x}) = \mathbf{x} - \mathbf{f} - A\,(\mathbf{v_x} + \mathbf{w_x}) = 0, \qquad \mathbb{F} : \mathbb{R}^p \longrightarrow \mathbb{R}^p, \tag{29}$$

where

$$\mathbf{v_x} = (x_1^3, x_2^3, \dots, x_p^3)^T, \qquad \mathbf{w_x} = (|x_1|, |x_2|, \dots, |x_p|)^T.$$

It is obvious that the function $\mathbb{F}$ defined in (29) is nonlinear and non-differentiable. So, we consider $\mathbb{F}(\mathbf{x}) = \mathbb{F}_1(\mathbf{x}) + \mathbb{F}_2(\mathbf{x})$ where:

$$\mathbb{F}_1(\mathbf{x}) = \mathbf{x} - \mathbf{f} - A\mathbf{v_x} \quad \text{and} \quad \mathbb{F}_2(\mathbf{x}) = -\mathrm{A}\mathbf{w_x}.$$

As in $\mathbb{R}^p$ we can consider divided difference of first order that do not need that the function $\mathbb{F}$ is differentiable (see [16]), we use the divided difference of first order given by $[\mathbf{u},\mathbf{v};\mathbb{G}] = ([\mathbf{u},\mathbf{v};\mathbb{G}]_{ij})_{i,j=1}^{p} \in \mathcal{L}(\mathbb{R}^p,\mathbb{R}^p)$, where

$$[\mathbf{u},\mathbf{v};\mathbb{G}]_{ij} = \frac{1}{u_j - v_j}\left(\mathbb{G}_i(u_1,\ldots,u_j,v_{j+1},\ldots,v_p) - \mathbb{G}_i(u_1,\ldots,u_{j-1},v_j,\ldots,v_p)\right), \tag{30}$$

if $u_j \neq v_j$, in other case $[\mathbf{u},\mathbf{v};\mathbb{G}]_{ij} = 0$, for $\mathbf{u} = (u_1,u_2,\ldots,u_p)^T$ and $\mathbf{v} = (v_1,v_2,\ldots,v_p)^T$.

Now, to compare the behavior of different methods we consider the case $\mathbf{f} = \mathbf{0}$ in (29). Obviously, for this problem, $\mathbf{x}^* = \mathbf{0}$ is a solution of $\mathbb{F}(\mathbf{x}) = \mathbf{0}$. Then, the system of nonlinear equations given in (29) is of the form

$$\mathbb{F}(\mathbf{x}) = \mathbf{x} - A\,\mathbf{z}, \qquad z_j = x_j^3 + |x_j|,\ j = 1,\ldots,p. \tag{31}$$

The numerical results are obtained by using MATLAB 2018 and working with variable precision arithmetic with 100 digits. In Table 1 we can see the results obtained by using the methods mentioned in our study. First of all we take nodes and weights of Trapezoidal rule with $n = 10$ subintervals for approximatting the integral and starting guess $x_0(t) = 1/2\ \forall t \in [0,1]$. We compare the distance between consecutive iterates of the first 7 iterations of each method. In the case of the Newton-Steffensen General method (9), the parameters involved are $a = 0.5$ and $b = 1.5$.

	Stirling (10)	Zincenko (7)	Steffensen (3)	New-Steff. (5)	New-Steff. Gen.(9)
1	1.5887	1.1637	7.4375	2.9044	2.9044
2	$6.0578e-01$	$3.0210e-01$	$2.7350e-01$	1.3867	1.3867
3	$4.7941e-01$	$1.2065e-01$	$1.8235e-02$	$3.2041e-01$	$1.2942e-01$
4	$4.1942e-01$	$4.9511e-02$	$5.5411e-05$	$2.8725e-04$	$2.8725e-04$
5	$3.5456e-01$	$2.0403e-02$	$2.8134e-09$	$1.3552e-12$	$1.3552e-12$
6	$1.9024e-01$	$8.4133e-03$	$3.0173e-18$	$3.1538e-37$	$3.3246e-37$
7	$2.9676e-02$	$3.4697e-03$	$3.9490e-36$	$1.7796e-111$	$2.1782e-111$

Table 1: Results with different methods in the first iterations.

In Table 2 we work with same conditions, we obtain the iterations that each method needs to satisfy the stopping criterion $||x_{k+1} - x_k|| \leq 10^{-40}$. It should be noted that the first two methods never meet the required tolerance because they are not convergent and, therefore, the methods end when the required iterations are completed (in this case 15 iterations at most). Second, we observe a good approximation to the order of convergence of each method p in case the method converges. In the last two rows of Table 2 we compare

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	Stirling (10)	Zincenko (7)	Steffensen (3)	New-Steff. (5)	New-Steff. Gen.(9)
k	15	15	8	7	7
p	1.0000	1.0000	1.9994	3.0142	3.0148
$\|\|x_{k-1}-x_k\|\|$	$2.3258e-04$	$2.9041e-06$	$6.9382e-72$	$1.7796e-111$	$2.1782e-111$
$\|\|F(x_k)\|\|$	$9.5985e-05$	$1.1977e-06$	$1.2745e-107$	$7.8863e-219$	$6.8587e-219$

Table 2: Numerical results for comparing the proposed methods.

the difference between the last iterates of each method and we also see the norm of the function evaluted in the last iteration.

Now, we also want to use the Gauss-Legendre quadrature to approximate the integral of equation (25). Moreover, by using the Newton-Steffensen method we compare two different possibilities for implementing the divided differences given in (30), that is, in Tables 1 and 2 we obtain the divided difference like $[x_n, x_n + F_1(x_n) + F_2(x_n), F_2]$ but we want to compare with $[x_n, x_n + F_2(x_n), F_2]$. The results in Table 3 show that the use of first form used for obtaining the divided differences gives better residual errors, which was expected because $F_1(x_n) + F_2(x_n)$ tends to zero quicker than $F_2(x_n)$. Even in some different example the value $F_2(x_n)$ could not tend to zero, in this case only first form of obtaining the divided differences considered would work. In Table 3 we have also included the computational time, as can be observed in the last row, notice that the use of Gauss-Legedre quadrature needs much more time than the trapezoidal rule although in some cases reaches better accuracy.

	$\|\|x_n - x_{n-1}\|\|$			
$Iterations$	$Trapezoidal\quad rule$		$Gauss-Legendre$	
n	$[x, x+F_1+F_2, F_2]$	$[x, x+F_2, F_2]$	$[x, x+F_1+F_2, F_2]$	$[x, x+F_2, F_2]$
1	2.9044	2.9044	2.7204	2.7204
2	1.3867	1.3867	1.1355	1.1355
3	$3.2041e-01$	$1.2942e-01$	$6.6978e-02$	$6.6978e-02$
4	$2.8725e-04$	$2.8725e-04$	$3.4608e-05$	$3.4608e-05$
5	$1.3552e-12$	$1.3552e-12$	$2.1448e-15$	$2.1448e-15$
6	$3.1538e-37$	$3.3489e-28$	$1.124e-45$	$1.124e-45$
7	$1.7796e-111$	$1.3651e-43$	$8.0773e-137$	$7.8571e-137$

Table 3: Results with Trapezoidal rule and Gauss-Legendre method by using different form of obtaining the divided differences.

	$Trapezoidal\quad rule$		$Gauss-Legendre$	
n	$[x, x+F_1+F_2, F_2]$	$[x, x+F_2, F_2]$	$[x, x+F_1+F_2, F_2]$	$[x, x+F_2, F_2]$
k	7	8	7	7
p	3.0142	$unstable$	3.0099	3.0103
$\|\|x_{k-1}-x_k\|\|$	$1.7796e-111$	$4.3463e-59$	$8.0772e-137$	$7.8571e-137$
$\|\|F(x_k)\|\|$	$7.8863e-219$	$1.0160e-74$	$1.3057e-243$	$1.5367e-138$
$time$	17.796129	20.6134	282.5403	309.3090

Table 4: Numerical results and computational time for comparing the proposed methods.

4 Approximating the solution of a nonlinear PDE related to image denoising

In some steps of the manipulation of an image, some random noise is usually introduced. This noise makes the later steps of processing the image difficult and inaccurate.

In many applications like astrophysics, astronomy or meteorology we have to manipulate images contaminated by noise. The image processing becomes difficult and inaccurate. For these reasons, usually some image denoising strategies are developed. In this paper, we center our attention in the PDE framework.

Let $f\colon \Omega \to \mathbb{R}$ be a signal or image which we would like to denoise.

The usual PDE frameworks start with constrained optimization problems like

$$\begin{aligned}&\text{Minimize in } u\colon R(u)\\ &\text{subject to } \|u-f\|^2_{L^2(\Omega)} = |\Omega|\sigma^2.\end{aligned}$$

where $n = u - f$ denotes the noise. If there is no good estimate of the variance of the noise, then we may consider the unconstrained optimization problem.

Different linear regularization functionals $R(u)$ can be consider, the most used is $\|\nabla u\|_{L^2}$. This type of functionals introduce diffusion near the edges of the images, this is their main limitation.

The TV norm does not penalize discontinuities in u, and thus allows us to improve the approximation near the edges.

$$\int_\Omega |\nabla u(x)|dx.$$

For the linear model its Euler–Lagrange equation, with Neumann's boundary conditions for u, is

$$-\triangle u + \lambda(u-f) = 0, \tag{32}$$

which comes from the corresponding unconstrained problem with the norm $\|\nabla u\|^2_{L^2(\Omega)}$ and where the positive parameter λ determines the relative importance of the smoothness of u and the quality of the approximation to the given signal f..

For the TV- model we have

$$-\nabla \cdot (\frac{\nabla u}{|\nabla u|}) + \lambda(u - f) = 0. \tag{33}$$

In practice, the term $|\nabla u|$ is replaced by $\sqrt{|\nabla u|^2 + \epsilon}$, but even after this regularization, Newton's method does not work satisfactorily in the sense that its domain of convergence is very small. This is especially true if the regularizing parameter ϵ is small.

On the other hand, while the singularity and nondifferentiability of the term $w = \nabla u/|\nabla u|$ is the source of numerical problems, w itself is usually smooth because it is in fact the unit vector normal to the level sets of u. The numerical difficulties arise only because we linearize it the wrong way.

Thus we should introduce a new variable w; namely

$$w = \frac{\nabla u}{\sqrt{|\nabla u|^2}},$$

and replace (33) by the equivalent system of nonlinear PDEs:

$$\begin{aligned} -\nabla \cdot w + \lambda(u - f) &= 0, \\ w\sqrt{|\nabla u|^2} - \nabla u &= 0. \end{aligned}$$

Without the inclusion of the above regularization parameter ϵ, this system is nonlinear and nondifferentiable .

4.1 Discretization and numerical implementation

We present a comparison between the nonlinear model and the linear model using a simple finite difference discretization procedure.

For a regular mesh of size $h = 1/m$, $m \in \mathbb{N}$ $(x_i = i \cdot h,\ i = 0, \ldots, m)$, if in each iteration k we approximate the divergence and the gradient operators (these operators are the same in 1D) by

$$\nabla \cdot v(x_i) = \nabla v(x_i) \approx \frac{v_i - v_{i-1}}{h},$$

we obtain a nonlinear system for the unknowns w_i and u_i.

That is,

$$\begin{aligned} -\frac{w_i - w_{i-1}}{h} - \lambda(u_i - f_i) &= 0, \quad w_1 = w_m = 0, \\ w_i \cdot \sqrt{(\frac{u_i - u_{i-1}}{h})^2} - \frac{u_i - u_{i-1}}{h} &= 0, \quad u_0 = f_0, u_m = f_m, \end{aligned}$$

for $i = 1, \ldots, m-1$.

We then consider the nonlinear and nondifferentiable operator

$$\begin{aligned} F_{2i-1}(u, w, \lambda_h) &= w_i - w_{i-1} + \lambda_h(u_i - f_i) \;=\; 0, \\ F_{2i}(u, w, \lambda_h) &= w_i\sqrt{(u_i - u_{i-1})^2 +} - (u_i - u_{i-1}) \;=\; 0, \quad 1 \le i \le m-1, \end{aligned}$$

with $\lambda_h = h\,\lambda$, $w_0 = w_m = 0$, $u_0 = f_0$ and $u_m = f_m$.

For the discretization of the linear model we can consider the system

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \lambda(u_i - f_i) \;=\; 0, \quad u_0 = f_0, u_m = f_m,$$

for $i = 1, \ldots, m-1$.

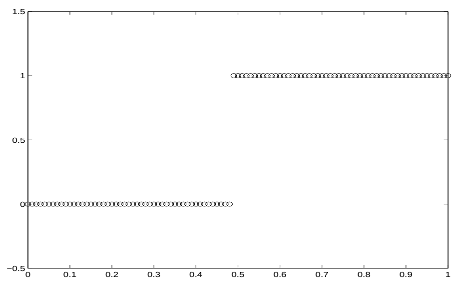

Figure 1: Original signal with a jump singularity.

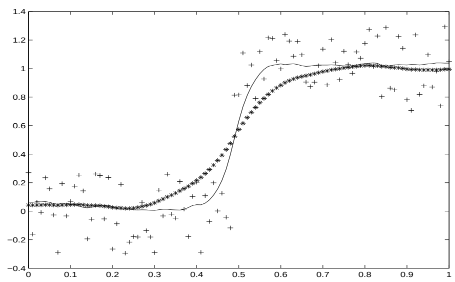

Figure 2: Solid lines = nonlinear model, starred lines = linear model and + lines = signal with noise. Noise level = 0.3, $\lambda = 10$.

In Figure 2, the solid lines are the function reconstructed by the nonlinear model approximated by the linearization based on a dual variable, solving the nonlinear system of equations by Steffensen's method 3 and the starred lines are given by the standard linear model, solving the associated linear system of equations by Gauss's method. The line with '+' is the noisy signal. The linear model introduces too much diffusion, giving a continuous function.

5 Conclusions

We have to point out the generalization of this study in which we have analyzed the local and semilocal convergence for Newton type methods and Steffensen like methods, so we can consider Newton-Steffensen's methods. The main idea it is to apply these kind of study to non-differentiable equations by taking in to account the advantages of consider the decomposition of the nonlinear equation into a sum of the differentiable part and the one non-differentiable.

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