Semilocal Convergence of a Newton-Secant Solver for Equations with a Decomposition of Operator

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Abstract. We provide the semilocal convergence analysis of the Newton-Secant solver with a decomposition of a nonlinear operator under classical Lipschitz conditions for the first order Fréchet derivative and divided differences. We have weakened the sufficient convergence criteria, and obtained tighter error estimates. We give numerical experiments that confirm theoretical results. The same technique without additional conditions can be used to extend the applicability of other iterative solvers using inverses of linear operators. The novelty of the paper is that the improved results are obtained using parameters which are special cases of the ones in earlier works. Therefore, no additional information is needed to establish these advantages.

Keywords: Newton-Secant solver; semilocal convergence analysis; Fréchet derivative; divided differences; decomposition of nonlinear operator

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1 Introduction

One of the important problems in Computational Mathematics including Mathematical Biology, Chemistry, Economic, Physics, Engineering and other disciplines is finding solutions of nonlinear equations and systems of nonlinear equations [1-14]. For most of these problems, to find the exact solution is difficult or impossible. Therefore, the development and research of numerical methods for solving nonlinear problems is an urgent task.

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A popular solver for dealing with nonlinear equations is Newton's [2, 3, 4]. But it is not applicable, if functions are nondifferentiable. In this case, we can apply solvers with divided differences [1, 2, 3, 7, 8, 10, 11]. If it is possible to decompose into differentiable and nondifferentiable parts, it is advisable to use combined methods [2, 3, 5, 6, 12, 13, 14].

Consider a nonlinear equation

$$F(x) + G(x) = 0, \tag{1}$$

where the operators F and G are defined on a open convex set D of a Banach space E_1 with values in a Banach space E_2 , F is a Fréchet differentiable operator, G is a continuous operator for which differentiability is not assumed. It is necessary to find an approximate solution $x_* \in D$ that satisfies equation (1).

In this paper, we consider the Newton-Secant solver

$$x_{n+1} = x_n - [F'(x_n) + G(x_{n-1}, x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, ... \tag{2}$$

This iterative process was proposed in [6] and studied in [2, 3, 13], and the convergence order $\frac{1+\sqrt{5}}{2}$ was established. It is shown that (2) converges faster than the Secant solver.

In this paper, we study solver (2) under the classical Lipschitz conditions for first-order Fréchet derivative and divided differences. Our technique allows to get the weaker convergence criteria, and tighter error estimates. This way, we extended the applicability of the results obtained in [13].

2 Convergence Analysis

Let $L(E_1, E_2)$ be a space of linear bounded operators from E_1 into E_2. Set $S(x, \tau) = \{y \in E_1 : \|y - x\| < \tau\}$ and let $\bar{S}(x, \tau)$ denote its closure.

Define quadratic polynomial φ by

$$\varphi(t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3$$

and parameters r, and r_1 by

$$r = \frac{1 - (q_0 + \bar{q}_0)a}{p_0 + q_0 + 2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0},$$

$$r_1 = \frac{1 - \bar{q}_0 a}{2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0},$$

where

$$\alpha_1 = p_0 + q_0 + 2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0,$$

$$\alpha_2 = -[1 - (q_0 + \bar{q}_0)a + (2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0)c]$$

and

$$\alpha_3 = (1 - \bar{q}_0 a)c,$$

where p_0, $\bar{p}_0$, q_0, $\bar{q}_0$, $\bar{\bar{q}}_0$, a and c are nonnegative numbers.

Suppose that $(q_0 + \bar{q}_0)a < 1$ and $\varphi\big(\frac{1}{2}r\big) \le 0$. Then, it is simple algebra to show, function φ has a unique root $\bar{r}_0 \in (0, \frac{r}{2}]$, and

$$r \le r_1,$$

$$\bar{\gamma} = \frac{p_0\bar{r}_0 + q_0(\bar{r}_0 + a)}{1 - \bar{q}_0 a - (2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0)\bar{r}_0} \in [0, 1)$$

and

$$\bar{r}_0 \ge \frac{c}{1 - \bar{\gamma}}.$$

Set $D_0 = D \cap S(x_0, r_1)$.

Definition 2.1. *We call an operator that acts from E_1 into E_2 and is denoted by $G(x, y)$ a first-order divided difference for the operator G by fixed points x and y $(x \neq y)$, if the equality*

$$G(x, y)(x - y) = G(x) - G(y)$$

is satisfied.

Theorem 2.2. *Suppose that:*

1) *F and G are nonlinear operators on an open convex set D of a Banach space E_1 into a Banach space E_2;*
2) *F is a Fréchet-differentiable operator, and let G is a continuous operator;*
3) *$G(\cdot, \cdot)$ is the first-order divided differences of the operator G defined on the set D;*
4) *the linear operator $A_0 = F'(x_0) + G(x_{-1}, x_0)$, where x_{-1}, $x_0 \in D$, is invertible;*
5) *the following conditions are satisfied for all $x, y, \in D$*

$$\|A_0^{-1}(F'(x_0) - F'(x))\| \le 2\bar{p}_0\|x_0 - x\|, \tag{3}$$

$$\|A_0^{-1}(G(x_{-1}, x_0) - G(x, x_0))\| \le \bar{q}_0\|x_{-1} - x\|, \tag{4}$$

$$\|A_0^{-1}(G(x, x_0) - G(x, y))\| \le \bar{\bar{q}}_0\|x_0 - y\|, \tag{5}$$

and for all $x, y, u \in D_0$

$$\|A_0^{-1}(F'(x) - F'(y))\| \le 2p_0\|x - y\|, \tag{6}$$

$$\|A_0^{-1}(G(x, y) - G(u, y))\| \le q_0\|x - u\|; \tag{7}$$

6) *a, c are nonnegative numbers such that*

$$\|x_0 - x_{-1}\| \le a,\ \|A_0^{-1}(F(x_0) + G(x_0))\| \le c,\ c > a, \tag{8}$$

$$(q_0 + \bar{q}_0)a < 1, \quad \varphi\big(\frac{1}{2}r\big) \le 0; \tag{9}$$

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7) $\bar{S}(x_0, \bar{r}_0) \subset D$.

Then, the solver (2) is well-defined and the sequence generated by it converges to the solution x_* *of equation (1), so that for each* $n \in \{-1, 0, 1, 2, ...\}$, *the following inequalities are satisfied*

$$\|x_n - x_{n+1}\| \leq t_n - t_{n+1}, \tag{10}$$

$$\|x_n - x_*\| \leq t_n - \bar{t}_*, \tag{11}$$

where sequence $\{t_n\}_{n \geq -1}$ *defined by the formulas*

$$t_{-1} = \bar{r}_0 + a,\ t_0 = \bar{r}_0,\ t_1 = \bar{r}_0 - c,$$

$$t_{n+1} - t_{n+2} = \bar{\gamma}_n (t_n - t_{n+1}),\ n \geq 0,$$

$$\bar{\gamma}_n = \frac{\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})},\ 0 \leq \bar{\gamma}_n < \bar{\gamma} \tag{12}$$

is decreasing, nonnegative, and converges to $\bar{t}_*$, *so that* $\bar{r}_0 - c/(1-\bar{\gamma}) \leq \bar{t}_* < t_0$, *where*

$$\tilde{p}_0 = \begin{cases} \bar{p}_0,\ n = 0 \\ p_0,\ n > 0 \end{cases},\quad \tilde{q}_0 = \begin{cases} \bar{q}_0,\ n = 0 \\ q_0,\ n > 0. \end{cases}$$

Proof. We use mathematical induction to show that, for each $k \geq 0$ the following inequalities are satisfied

$$t_{k+1} \geq t_{k+2} \geq \bar{r}_0 - \frac{1 - \bar{\gamma}^{k+2}}{1 - \bar{\gamma}} c \geq \bar{r}_0 - \frac{c}{1 - \bar{\gamma}} \geq 0, \tag{13}$$

$$t_{k+1} - t_{k+2} \leq \bar{\gamma}(t_k - t_{k+1}). \tag{14}$$

Setting $k = 0$ in (12), we get

$$t_1 - t_2 = \frac{\tilde{p}_0(t_0 - t_1) + \tilde{q}_0(t_{-1} - t_1)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_1) - \bar{\bar{q}}_0(t_0 - t_1)}(t_0 - t_1) \leq \bar{\gamma}(t_0 - t_1),$$

$$t_0 \geq t_1,\ t_1 \geq t_2 \geq t_1 - \bar{\gamma}(t_0 - t_1) \geq \bar{r}_0 - (1 + \bar{\gamma})c = \bar{r}_0 - \frac{(1 - \bar{\gamma}^2)c}{1 - \bar{\gamma}} \geq \bar{r}_0 - \frac{c}{1 - \bar{\gamma}} \geq 0.$$

Suppose that (13) and (14) are true for $k = 0, 1, ..., n-1$. Then, for $k = n$, we obtain

$$t_{n+1} - t_{n+2} = \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})}$$

$$\leq \frac{\tilde{p}_0 t_n + \tilde{q}_0 t_{n-1}}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}(t_n - t_{n+1}) \leq \bar{\gamma}(t_n - t_{n+1}),$$

$$t_{n+1} \geq t_{n+2} \geq t_{n+1} - \bar{\gamma}(t_n - t_{n+1}) \geq \bar{r}_0 - \frac{1 - \bar{\gamma}^{n+2}}{1 - \bar{\gamma}} c \geq \bar{r}_0 - \frac{c}{1 - \bar{\gamma}} \geq 0.$$

Thus, $\{t_n\}_{n\ge 0}$ is a decreasing nonnegative sequence, and converges to $\bar{t}_* \ge 0$.

Let us prove that the method (2) is well-defined, and for each $n \ge 0$ the inequality (10) is satisfied.

Since $t_{-1} - t_0 = a$, $t_0 - t_1 = c$ and conditions (8) are fulfilled then $x_1 \in S(x_0, \bar{r}_0)$ and (10) is satisfied for $n \in \{-1, 0\}$. Let conditions (8) be satisfied for $k = 0, 1, ..., n$. Let us prove that the method (2) is well-defined for $k = n+1$.

Denote $A_n = F'(x_n) + G(x_{n-1}, x_n)$. Using the Lipschitz conditions (3) – (5), we have

$$\begin{aligned} \|I - A_0^{-1}A_{n+1}\| &= \|A_0^{-1}(A_0 - A_{n+1})\| \le \|A_0^{-1}(F'(x_0) - F'(x_{n+1}))\| \\ &+\|A_0^{-1}(G(x_{-1}, x_0) - G(x_n, x_0) + G(x_n, x_0) - G(x_n, x_{n+1}))\| \\ &\le 2\bar{p}_0\|x_0 - x_{n+1}\| + \bar{q}_0(\|x_{-1} - x_0\| + \|x_0 - x_n\|) + \bar{\bar{q}}_0\|x_0 - x_{n+1}\| \\ &\le 2\bar{p}_0\|x_0 - x_{n+1}\| + \bar{q}_0 a + \bar{q}_0\|x_0 - x_n\| + \bar{\bar{q}}_0\|x_0 - x_{n+1}\| \\ &\le \bar{q}_0 a + 2\bar{p}_0(t_0 - t_{n+1}) + \bar{q}_0(t_0 - t_n) + \bar{\bar{q}}_0(t_0 - t_{n+1}) \\ &\le \bar{q}_0 a + 2\bar{p}_0\bar{r}_0 + \bar{q}_0\bar{r}_0 + \bar{\bar{q}}_0\bar{r}_0 < 1. \end{aligned}$$

According to the Banach lemma on inverse operators [2] A_{n+1} is invertible, and

$$\|A_{n+1}^{-1}A_0\| \le (1 - \bar{q}_0 a - 2\bar{p}_0\|x_0 - x_{n+1}\| - \bar{q}_0\|x_0 - x_n\| + \bar{\bar{q}}_0\|x_0 - x_{n+1}\|)^{-1}.$$

By the definition of the divided difference and conditions (6), (7), we obtain

$$\begin{aligned} &\|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\ &= \|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}) - F(x_n) - G(x_n) - A_n(x_n - x_{n+1}))\| \\ &\le \|A_0^{-1}(\textstyle\int_0^1\{F'(x_{n+1} + t(x_n - x_{n+1})) - F'(x_n)\}dt)\|\|x_n - x_{n+1}\| \\ &+\|A_0^{-1}(G(x_{n+1}, x_n) - G(x_{n-1}, x_n))\|\|x_n - x_{n+1}\| \\ &\le (\tilde{p}_0\|x_n - x_{n+1}\| + \tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|))\|x_n - x_{n+1}\|. \end{aligned}$$

In view of condition (10), we have

$$\begin{aligned} \|x_{n+1} - x_{n+2}\| &= \|A_{n+1}^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\ &\le \|A_{n+1}^{-1}A_0\|\|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\ &\le \frac{\tilde{p}_0\|x_n - x_{n+1}\| + \tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|)}{1 - \bar{q}_0 a - 2\bar{p}_0\|x_0 - x_{n+1}\| - \bar{q}_0\|x_0 - x_{n+1}\| + \bar{\bar{q}}_0\|x_0 - x_n\|}\|x_n - x_{n+1}\| \\ &\le \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} = t_{n+1} - t_{n+2}. \end{aligned}$$

Thus, the method (2) is well-defined for each $n \ge 0$. Hence it follows that

$$\|x_n - x_k\| \le t_n - t_k, \quad -1 \le n \le k. \tag{15}$$

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Therefore, the sequence $\{x_n\}_{n\geq 0}$ is fundamental, so it converges to some $x_* \in \bar{S}(x_0, \bar{r}_0)$. Inequality (11) is obtained from (15) for $k \to \infty$. Let us show that x_* solves the equation $F(x) + G(x) = 0$. Indeed, we get in turn that

$$A_0^{-1}(F(x_{n+1}) + G(x_{n+1})) \leq \big(\tilde{p}_0\|x_n - x_{n+1}\|$$
$$+\tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|)\big)\|x_n - x_{n+1}\| \to 0, \ \ n \to \infty.$$

Hence, $F(x_*) + G(x_*) = 0$. □

Remark 2.3. *The order of convergence of method (2) is equal to* $\dfrac{1+\sqrt{5}}{2}$.

Proof. In view of $t_n - t_{n+1} \leq \bar{\gamma}(t_{n-1} - t_n)$, and (12), we obtain

$$\begin{aligned}
t_{n+1} - t_{n+2} &= \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_n - t_{n+1} + t_{n-1} - t_n)\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} \\
&\leq \frac{\tilde{p}_0\bar{\gamma}(t_{n-1} - t_n) + \tilde{q}_0(1+\bar{\gamma})(t_{n-1} - t_n)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})}(t_n - t_{n+1}) \\
&= \frac{\big(\bar{p}_0\bar{\gamma} + \bar{q}_0(1+\bar{\gamma})\big)(t_n - t_{n+1})(t_{n-1} - t_n)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} \\
&\leq \frac{\tilde{p}_0\bar{\gamma} + \tilde{q}_0(1+\bar{\gamma})}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}(t_n - t_{n+1})(t_{n-1} - t_n).
\end{aligned}$$

Denote $\bar{C} = \dfrac{\bar{p}_0\bar{\gamma} + \bar{q}_0(1+\bar{\gamma})}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}$. Clearly,

$$t_{n+1} - t_{n+2} \leq \bar{C}(t_{n-1} - \bar{t}_*)(t_n - \bar{t}_*). \tag{16}$$

Since, for each $k > 2$, the estimate is satisfied

$$t_{n+k-1} - t_{n+k} \leq \bar{\gamma}^{k-2}(t_{n+1} - t_{n+2}),$$

we get

$$\begin{aligned}
t_{n+1} - t_{n+k} &= t_{n+1} - t_{n+2} + t_{n+2} - t_{n+3} + \ldots + t_{n+k-1} - t_{n+k} \\
&\leq (1 + \bar{\gamma} + \ldots + \bar{\gamma}^{k-2})(t_{n+1} - t_{n+2}) \\
&= \frac{1 - \bar{\gamma}^{k-1}}{1 - \bar{\gamma}}(t_{n+1} - t_{n+2}) \leq \frac{1}{1 - \bar{\gamma}}(t_{n+1} - t_{n+2}).
\end{aligned}$$

In view of (16), for $k \to \infty$, we have

$$t_{n+1} - \bar{t}_* \leq \frac{\bar{C}}{1 - \bar{\gamma}}(t_{n-1} - \bar{t}_*)(t_n - \bar{t}_*)$$

Hence, it follows that the order of convergence of the sequence $\{t_n\}_{n\geq 0}$ is equal to $\dfrac{1+\sqrt{5}}{2}$, and, according (11), the sequence $\{x_n\}_{n\geq 0}$ converges with the same order. □

Remark 2.4. *(a) The following conditions were used for each $x, y, u, v \in D$ in [13]*

$$\|A_0^{-1}(F'(y) - F'(x))\| \leq 2P_0\|y - x\|, \tag{17}$$

$$\|A_0^{-1}(G(x,y) - G(u,v))\| \leq Q_0(\|x-u\| + \|y-v\|), \tag{18}$$

$$\begin{aligned} r_0 &\geq \frac{c}{1-\gamma},\ Q_0a + 2P_0r_0 + 2Q_0r_0 < 1, \\ \gamma &= \frac{P_0r_0 + Q_0(r_0+a)}{1 - Q_0a - 2P_0r_0 - 2Q_0r_0},\ 0 \leq \gamma < 1. \end{aligned} \tag{19}$$

But, then we have

$$\begin{aligned} \bar{p}_0 &\leq P_0, \\ \bar{q}_0 &\leq Q_0, \\ \bar{\bar{q}}_0 &\leq Q_0, \end{aligned}$$

since $D_0 \subseteq D$, (3) and (4), (5), (7) are weaker than (17) and (18) respectively for $\bar{r}_0 \leq r_0$. Notice that sufficient convergence criteria (9) imply (19) but not necessarily vice versa, unless if $\bar{p}_0 = P_0$, $\bar{q}_0 = \bar{\bar{q}}_0 = Q_0$ and $\bar{r}_0 = r_0$.

A simple inductive argument shows that

$$\bar{\gamma}_n \leq \gamma_n, \tag{20}$$

$$t_n - t_{n+1} \leq s_n - s_{n+1}, \tag{21}$$

where

$$s_{-1} = r_0 + a,\ s_0 = r_0,\ s_1 = r_0 - c,$$

$$s_{n+1} - s_{n+2} = \gamma_n(s_n - s_{n+1}),\ n \geq 0,$$

$$\gamma_n = \frac{P_0(s_n - s_{n+1}) + Q_0(s_{n-1} - s_{n+1})}{1 - Q_0a - 2P_0(s_0 - s_{n+1}) - Q_0(2s_0 - s_n - s_{n+1})},\ 0 \leq \gamma_n \leq \gamma.$$

Notice that the corresponding quadratic polynomial φ_1 to φ is defined similarly by

$$\varphi_1(t) = b_1t^2 + b_2t + b_3$$

where

$$b_1 = 3P_0 + 3Q_0,$$

$$b_2 = -[1 - 2Q_0a + (2P_0 + 2Q_0)c]$$

and

$$b_3 = (1 - Q_0a)c.$$

We have by these definitions that

$$\alpha_1 < b_1,\ \alpha_2 < b_2,\ but\ \alpha_3 > b_3.$$

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Therefore, we cannot tell, if $r_0 < \bar{r}_0$ or $\bar{r}_0 < r_0$ or $r_0 = \bar{r}_0$. But, we have

$$\begin{aligned} \gamma \leq \bar{\gamma} &\Rightarrow r_0 \leq \bar{r}_0, \\ s_n &\leq t_n, \\ s_* \leq \bar{t}_* &= \lim_{n\to\infty} t_n \end{aligned} \tag{22}$$

and

$$\begin{aligned} \bar{\gamma} \leq \gamma \Rightarrow \bar{r}_0 &\leq r_0 \Rightarrow \bar{C} \leq C, \\ t_n &\leq s_n, \\ \bar{t}_* \leq s_* &= \lim_{n\to\infty} s_n, \end{aligned} \tag{23}$$

It is simple algebra to show that $\varphi(r) \geq 0$, and for $r_{min} = -\dfrac{\alpha_2}{2\alpha_1}$ (solving $\varphi'(t) = 0$), $r_{min} \geq \dfrac{r}{2}$, $r_{min} \leq \dfrac{r_1}{2}$. Hence, one may replace the second inequation in (9) by $\varphi(\lambda r) \leq 0$ for some $\lambda \in (0, \frac{1}{2}]$ to obtain a better information about the location of $\bar{r}_0$, if $\lambda \neq \frac{1}{2}$, especially in the case when we do not actually need to compute $\bar{r}_0$.

(b) The Lipschitz parameters $\bar{p}_0$, $\bar{q}_0$, $\bar{\bar{q}}_0$ can become even smaller, if we define the set $D_1 = D \cap S(x_1, r_1 - c)$ for $r_1 > c$ to replace D_0 in Theorem 2.2., since $D_1 \subseteq D_0$.

3 Numerical experiments

Let us define function $F + G : R \to R$, where

$$F(x) = e^{x-0.5} + x^3 - 1.3, \; G(x) = 0.2x|x^2 - 2|.$$

The exact solution of $F(x) + G(x) = 0$ is $x_* = 0.5$. Let $D = (0, 1)$. Then

$$F'(x) = e^{x-0.5} + 3x^2,$$

$$G(x, y) = \frac{0.2x(2 - x^2) - 0.2y(2 - y^2)}{x - y} = 0.2(1 - x^2 - xy - y^2).$$

$$A_0 = e^{x_0-0.5} + 3x_0^2 + 0.2(1 - x_{-1}^2 - x_{-1}x_0 - x_0^2),$$

$$|A_0^{-1}(F'(x) - F'(y))| \leq \frac{e^{0.5} + 3|x + y|}{|A_0|}|x - y|,$$

$$|A_0^{-1}(G(x, y) - G(u, v))| = \frac{0.2}{|A_0|}|(u + x + y)(u - x) + (v + y + u)(v - y)|.$$

Let $x_0 = 0.57$, $x_{-1} = 0.571$. Then, we have $a = 0.001$, $c \approx 0.0660157$, $\bar{p}_0 \approx 1.4118406$, $\bar{q}_0 \approx 0.1901483$, $\bar{\bar{q}}_0 \approx 0.2282491$, $r_1 \approx 0.3083854$,

$$D_0 = D \cap S(x_0, r_1) = (0.2616146, 0.8783854),$$

$p_0 \approx 1.5362481$, $q_0 \approx 0.2340358$, $P_0 \approx 1.6982621$, $Q_0 \approx 0.2664386$, and $r \approx 0.1994221$, $\varphi(\frac{1}{2}r) \approx -0.0051722 < 0$. So, $\bar{p}_0 < P_0$, $\bar{q}_0 < Q_0$, $\bar{\bar{q}}_0 < Q_0$.

By solving inequalities $\varphi(t) \leq 0$ and $\varphi_1(t) \leq 0$, we get

$$t \in [0.0824903,\ 0.1596319] \Rightarrow \bar{r}_0^{(1)} \approx 0.0824903,\ \bar{r}_0^{(2)} \approx 0.1596319,$$

$$t \in [0.0924062,\ 0.1211750] \Rightarrow r_0^{(1)} \approx 0.0924062,\ r_0^{(2)} \approx 0.1211750.$$

Then $\bar{r}_0 = \bar{r}_0^{(1)} \approx 0.0824903$, $r_0 = r_0^{(1)} \approx 0.0924062$, and

$$S(x_0, \bar{r}_0) = (0.4875097,\ 0.6524903),\ \bar{\gamma} \approx 0.1997151 < 1,\ \bar{C} \approx 0.8023108,$$

$$S(x_0, r_0) = (0.4775938,\ 0.6624062),\ \gamma \approx 0.2855916 < 1,\ C \approx 1.2998717.$$

In Table 1, there are results that confirm estimates (10), (11) and (21). Table 2 shows that sequences $\{t_n\}$ and $\{s_n\}$ converge to $\bar{t}_* \approx 0.0073550$ and $s_* \approx 0.0144209$, respectively, and confirms (20) and (23).

Table 1: Obtained results for $\varepsilon = 10^{-7}$

n	$\|x_{n-1} - x_n\|$	$t_{n-1} - t_n$	$s_{n-1} - s_n$	$\|x_n - x_*\|$	$t_n - \bar{t}_*$	$s_n - s_*$
1	0.0660157	0.0660157	0.0660157	0.0039843	0.0091195	0.0119695
2	0.0040123	0.0087609	0.0113203	0.0000281	0.0003586	0.0006492
3	0.0000281	0.0003573	0.0006452	1.761e-08	0.0000013	0.0000040
4	1.761e-08	0.0000040	0.0000040	7.438e-14	1.440e-10	1.033e-09

Table 2: Obtained results for $\varepsilon = 10^{-7}$

n	t_n	s_n	$\bar{\gamma}_{n-2}$	γ_{n-2}
-1	0.0834903	0.0934062		
0	0.0824903	0.0924062		
1	0.0164746	0.0263904		
2	0.0077136	0.0150701	0.1327096	0.1714793
3	0.0077136	0.0144249	0.0407873	0.0569927
4	0.0073550	0.0144209	0.0035475	0.0061771
5	0.0073550	0.0144209	0.0001136	0.0002592

4 Conclusions

We investigated the semilocal convergence of Newton-Secant solver under classical center and restricted Lipschitz conditions. This technique weakens the

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sufficient convergence criteria without adding more conditions and uses constants that are specializations of earlier ones. Moreover, tighter estimate errors are obtained. The theoretical results are confirmed by numerical experiments. Our technique can be used to extend the applicability of other iterative methods using inverses of linear operators [1-14] along the same lines.

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