The stability of the two-dimensional incompressible mmiscible displacement in porous media using the Petrov Galerkin finite element method

Wafaa S. Mohammed

Basrah Education Directorate, Ministry of Education, Basrah, Iraq, Email: wafaasamer21@gmail.com

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ABSTRACT

A two-dimensional displacement fluid in a porous material that is incompressible and immiscible is studied via the lens of a nonlinear system of two pair partial differential equations models. A petrov Galerkin mixed finite element approach is used to estimate the mixture's pressure and Darcy velocity, while a petrov Galerkin Finite Element Method (PGFEM) is stability prove for the pressure, velocity and Saturation.

Keywords: petrov Galerkin finite element method, petrov mixed finite element method, incompressible, immiscible displacement, porous media, stability.

1. INTRODUCTION

The capillary pressure fields' discontinuities at interfaces complicates Simulating two-phase immiscible incompressible flows numerically over heterogeneous porous media. separating subdomains with different rock characteristics. In the event that one phase is missing, the pressure within capillaries may also be discontinuous because of the varying pressures of entry of the various rocks. However, when the interface's two sides exhibit both phases, to ensure capillary pressure continuity, the saturation must be discontinuous at the interface. The global pressure and saturation in these applications may show prominent discontinuities because of large (up to several orders) permeability changes and fast variations in capillary forces [1]. The creation of numerical techniques that are compatible with non-linearity for two-phase movements in heterogeneous porous medium.

Over the past few decades, a great deal of research has been done on numerical simulations and analyses for the systems (3.1, 3.2, and 3.3). The Darcy velocity and pressure mixed finite element technique, as well as the saturation equation for immiscible displacement of a single incompressible fluid in porous media using the upwind Galerkin finite element method, have been applied [2]. developed discontinuous Galerkin techniques to compute incompressible two-phase flow numerically in porous media [3]. Modeled the three-dimensional Two-phase incompressible and irreversible flow in porous medium using The updated explicit saturation method with implicit pressure in conjunction with the finite volume method [4]. In order to Generally Dual-phase incompressible and immiscible flows in porous heterogeneous media with varying capillary pressures, the sequential discontinuous Galerkin finite element method was designed and investigated [5]. introduced a method for simulating finite element level sets [6]. Boltzmann technique for immiscible fluid displacement numerical modeling in a two-dimensional porous medium [7]. The linearized discontinuous error analysis for incompressible, immiscible displacement in porous media using the Galerkin Finite Element Method [8]. The linearized discontinuous error analysis of the GFEM for incompressible, immiscible displacement in porous media [9]. Impermeable Immiscible Migration in Permeable Materials: HP-Discontinuous GFEM [10]. Petrov - Galerkin in (1978) D.F. Griffith and J. Lorenz introduced the one-dimensional PGFE method, they also characterized the trial and test (weight) spaces, stated that the trial and test functions belong to different spaces, provided the trial and test functions' shapes, explained the asymmetric matrices' shapes, and examined the error [11]. Tezduyer T.E. and Akin J. ED, stabilizing parameter δ utilized in the PG formulations was computed in three distinct ways: using linear, triangular, and quadrilateral elements. They also compared how the obtained δ values varied depending on the definitions used in the computational domains [12]. Volker and Julia proved that standard energy arguments yield estimates for stabilization parameters that are dependent on the time step length and those that are independent of the time step length when paired with the backward Euler scheme. They also researched other methods for obtaining inaccuracy for the SUPGFE method to utilized to CDR equations, explored conditions on the stabilization parameters [13]. D. Broersen and R. Stevenson investigated a PG discretization of the CDR equation's ultra-weak version in a mixed form [14]. N. Ahmed and G. Mattie they offered an approximation for solving time-dependent linear CDR issues. Time discretization's using CPGFE and discontinuous GFE techniques. To assess the correctness of the temporal and spatial discretization schemes, they ran a number of numerical tests. Depending on the stabilizing parameters for the spatial estimations, the results are explained [15]. Regarding fractional boundary value problems in one dimension, variational formulations of the PG type were devised by Zhou, B. Jin, and R. Lazarov, Either the The diffusion term is Riemann-Liouville, or the caputo fractional derivative of order (3/2,2). a novel FE technique that uses partial powers for the trial space and continuous piecewise linear FEs for the test space [16]. L. Wang, S. Hou, L. Shi, and P. Zhang: For elliptic problems, a bilinear PGFE method is proposed to solve the variable matrix coefficient. In contrast to previous research focused solely on triangular parts, this new work addresses the intricate techniques used to cut interfaces as rectangular elements. i.e., half as many elements as triangular elements with the same mesh size [17]. D. A. Keshaish and H.A. Kashkool explain and evaluate the Petrov Galekin Finite Element (PGFE) approach for resolving linear problems including diffusion, reaction, and convection [18].

The structure of the paper is as follows:

In Sect. 2, We present some key definitions and lemmas. In Sect. 3, It explains the Petrov Galerkin finite element methodology. It also introduces some of the fundamental concepts of the method, including element matrices. For Darcy velocity and pressure, the Petrov Galerkin mixed finite element approach is utilized. while the Petrov GFEM is used for the saturation. In Sect. 4, we have demonstrated to us that there is stability for pressure, saturation, and velocity, respectively. Lastly, the conclusions in Sect 5.

2. Definitions in advance and Lemmas

A general framework for the numerical solution of partial differential equations is the FEM. The fundamental resources for understanding functional analysis theory and finite element methods are covered in this part. The following list includes certain symbols and definitions that will be used frequently.

Definition 2.1[19]: **(Lebesgue space)**

Given ; $n \geq 1$, let $ב$ be an open set of $\mathbb{R}^n$ The set of real-valued Lebesgue measurable functions $\aleph$ defined on $ב$ such that $|\aleph|^a$is integrable on $ב$ with regard to the Lebesgue measure in $\mathbb{R}^n$; is denoted by $L_a(ב),\ 1 \leq a < \infty$.

The Lebesgue space $L^a(ב)$ is defined by:

$$L^a(ב) = \left\{ \aleph : \int_{ב} |\aleph|^a dx < \infty \right\} \qquad , a \in [1, \infty)$$

equipped with the norm,

$$\|\aleph\|_{L^a(ב)} = \left(\int_{ב} |\aleph(x)|^a \ dx \right)^{\frac{1}{a}} \qquad (2.1)$$

For $a = 2$, the space $L^2(ב)$ of " functions that are square integrable " will be seen as of particular importance.

$$L^2(ב) = \{ \ \aleph: \int_{\Omega} |\aleph(x)|^2 dx < \ \infty \}$$

prepared with the norm,

$$\|\aleph\|_{L^2(ב)} = (\int_{\Omega} |\aleph|^2 \, dx)^{1/2}$$

for real-valued functions $\aleph, v \ \in L^2(ב)$, we defined the L^2–inner product by

$$(\aleph, v) = \int_{ב} \aleph(x) v(x) dx. \qquad (2.2)$$

For $a = \infty$, $L^\infty(ב)$ shows the space of all functions, all of which have a bound for nearly all $x \in ב$;

$$L^\infty(ב) = \{\aleph : |\aleph(x)| < \infty \ \ for\ almost\ all\ \ x \in ב\}$$

equipped with the norm,

$$\|\aleph\|_{L^\infty(ב)} = \max_{x \in ב} |\aleph(x)| \qquad (2.3)$$

Definition 2.2 [20]: Hilbert Space $H^a(ב)$

Letב be an open set ,contained in $\mathbb{R}^n$; $n \geq 1$ .We denote the boundary of ב by $\partial ב$ .A Hilbert space is complete inner product space. Hilbert space of order a such that $a \geq 1$ denoted by $H^a(ב)$ of function $\aleph(x)$ on Ω is defined as

$$H^a(ב) = \{\aleph : \aleph \in L^2(ב)\ and\ D^\rho \aleph \in L^2(ב)\ ;\ \forall\ |\rho| \leq a\} \qquad (2.4)$$

is endowed for $u, w \in H^a(ב)$ with the inner product

$$(ב, w)_{H^a(ב)} = \sum_{|\rho| \leq a} \int_{ב} D^\rho ב, D^\rho w \, d ב, \qquad (2.5)$$

equipped with the norm,

$$\|\aleph\|_{H^a(ב)} = \sum_{|\rho| \leq a} (D^\rho \aleph, D^\rho \aleph)_{L^2(ב)} = (\aleph, \aleph)_{H^a(ב)} < \infty \qquad (2.6)$$

or

$$\|\aleph\|^2_{H^a(ב)} = \sum_{|\rho| \leq a} \|D^\rho \aleph\|^2_{L^2(ב)} \qquad (2.7)$$

and the semi-norm:

$$|\aleph|^2_{H^a(ב)} = \sum_{|\rho| = a} (D^\rho \aleph, D^\rho \aleph)_{L^2(ב)} \ . \qquad (2.8)$$

Remark 2. 1 ***(Certain types of Hilbert spaces)*** **· [21]**

*.In the space $H^1(ב)$, is defined by

$$H^1(ב) = \{\aleph : \aleph \in L^2(ב), \nabla \aleph \in L^2(ב)\}, \qquad (2.9)$$

With the norm

$$\|u\|_{H^1(ב)} = \left(\int_{ב} (|\aleph|^2 + |\nabla \aleph|^2)\, d\, x\right)^{1/2}, \qquad (2.10)$$

*.In the space $H_0^1(ב)$, is defined by

$$H_0^1(ב) = \{\aleph : \aleph \in H^1(ב), \aleph = 0\ on\ \partial ב\}, \qquad (2.11)$$

With semi-norm is

$$|\aleph|_{H_0^1(ב)} = \left(\int_{\Omega} |\nabla \aleph|^2 \, d ב\right)^{1/2}, \qquad (2.12)$$

*.In the H^2, is defined by

$$H^2(ב) = \{u : u \in L^2(ב), \nabla \aleph, \nabla^2 \aleph \in L^2(ב)\}, \qquad (2.13)$$

With the norm

$$\|\aleph\|_{H^2(\beth)} = \left(\int_\beth (|\aleph|^2 + |\nabla\aleph|^2 + |\nabla^2\aleph|^2)\, d\beth\right)^{1/2}, \qquad (2.14)$$

and the semi-norm

$$|\aleph|_{H^2(\beth)} = \left(\int_\beth |\nabla^2\aleph|^2\, d\beth\right)^{\frac{1}{2}}, \qquad (2.15)$$

Lemma 2.1 [22] [21]: (Young's inequality)

Let $\chi, \alpha \geq 0$ real numbers and $\rho > 0$, the Young's-inequality is

$$\chi\alpha \leq \frac{1}{2\rho}\chi^2 + \frac{\rho}{2}\alpha^2$$

or

$$\chi\alpha \leq \frac{1}{4\rho}\chi^2 + \rho\alpha^2 \qquad (2.16)$$

Lemma 2.2 [22]: (Cauchy-Schwartz inequality)

If V is inner product space then, Cauchy-Schwartz inequality is

$$|(\aleph, v)| \leq \sqrt{(\aleph,\aleph)(v,v)} \leq \|\aleph\|\|v\| \;, \;\forall\, \aleph, v \in V$$
(2.17)

and the equality holds $\Leftrightarrow$ $\aleph$ and v are linearly dependent.

Definition 2.3 [23]:

The space $H(div;\beth)$is defined as the collection of vector-valued functions on $\beth$ that are square integrable along with their divergence; ie,

$$H(div;\beth) = \{\aleph : \aleph \in [L^2(\beth)]^2, \nabla\cdot\aleph \in L^2(\beth)\}$$

The norm in $H(div;\beth)$ is defined by

$$\|\aleph\|_{H(div(\beth)} = (\|\aleph\|^2 + \|\nabla\cdot\aleph\|^2)^{1/2} \qquad (2.18)$$

3.The petrov Galerkin Finite Element Methods

An immiscible displacement system can be used to characterize incompressible immiscible flow in porous media in various engineering applications. Let $\beth$ be a bounded domain (oil field) in the plan $R^2$with a smooth boundary Γ and T>0. The following are the classical systems in two dimensions [29].

$$\nabla. v = q(x,t), (x,t) \in \beth \times (0,T] \qquad (3.1)$$

$$v = -\sigma(s).\nabla p, (x,t) \in \beth \times (0,T] \qquad (3.2)$$

$$\emptyset s_t + \nabla.(f(s)v - \nabla.(a(s)\nabla s) = g(x,t,s), (x,t) \in \beth \times (0,T] \qquad (3.3)$$

Where

$$g(x,t,s) = \begin{cases} q_+(x,t)\tilde{s}(x,t) \;, & at\ injection\ wells \\ -q_-(x,t)s(x,t)\,, & at\ production\ wells \\ 0 \;, & otherwise \end{cases}$$

where ϕ is the porosity of the rock's medium (ϕ∈(0,1) in the domain $\beth$),s,v and p are the saturation, Darcy's velocity and pressure, respectively, $f(s)$ is the fractional flow function, $g(x,t,s)$ is the gravitational acceleration, σ (s)=$k\mu$ is a smooth function, k=$k(x)$ is the porous rock's permeability, μ=$\mu(s)$ is the fluid's viscosity, and q is the source and sink terms.

If s=$soil$, then s =0 and we may denote $g(x,t,s)$= $-q_-(x,t)s(x,t)$ where $-q_-(x,t) \geq 0$. As a result, the previous equation (2.3)becomes

$$\emptyset s_t + \nabla.(f(s)v - \nabla.(a(s)\nabla s) + q_-(x,t)s(x,t) = 0, (x,t) \in \beth \times (0,T] \qquad (3.4)$$

The starting condition is

$$s(x,0) = s^0(x), x \in \beth, \qquad (3.5)$$

and the boundary conditions are

$$\frac{\partial p}{\partial t} = 0, s_t = 0, (x,t) \in \Gamma \times (0,T] \qquad (3.6)$$

The aforementioned system's variational formulation. For the unknown, the finite element approximation space is formed by the pair $H_0(div,ב) \times L_0^2(ב) \times H^1(ב)$. Find $v \in H_0(div,ב), p \in L_0^2(ב)$ and $s \in H^1(ב)$ such that

$$(\nabla. v, w) = (q, w);\ \forall w \in L_0^2(ב), \qquad (3.7)$$

$$(v, א) = (\sigma(s)p, \nabla. א;\ \forall\, א \in H_0(div,ב) \qquad (3.8)$$

$$(\emptyset s_t, \varphi) + (a(s)\nabla.\ s, \nabla.\varphi) + \nabla.(f(s)v, \varphi) + (q\_s, \varphi) = 0;\quad \forall \varphi \in H_0^1(\Omega), \qquad (3.9)$$

We now define the Galerkin spaces of Petrov. Let W, U, ψ be three trail space and ϑ, ξ, η be three a test space which are define

$$W = \{w: w \in L_0^2(ב)\}, \qquad (3.10)$$

$$U = \{א: א \in H_0(div,ב)\}, \qquad (3.11)$$

$$\psi = H_0^1(ב) = \{\varphi: \varphi \in H^1(ב);\ \varphi|_\Gamma = 0\}, \qquad (3.12)$$

and,

$$\vartheta = \{r : r = w + \delta\,\beta \cdot \nabla w\ ;\ w \in W\} \qquad (3.13)$$

$$\xi = \{\zeta : \zeta = א + \delta\,\beta \cdot \nabla א\ ;\ א \in U\} \qquad (3.14)$$

$$\eta = \{\rho : \rho = \varphi + \delta\,\beta \cdot \nabla \varphi\ ;\ \varphi \in \psi\} \qquad (3.15)$$

here δ shows a constant stability parameter. It will be chosen as [24];

$$\delta = \begin{cases} \eta h & if \quad \epsilon < h \\ 0 & if \quad \epsilon \geq h \end{cases} \quad ;\ 0 < \eta < \frac{1}{4}\ , (\text{small constant}) \cdot$$

and $dim\, W, U, \psi = dim\, \vartheta, \xi, \eta$.

The petrov Galerkin mixed finite element mothed for Darcy velocity and pressure. Find $v_h \in W$ and $p_h \in U$ such that

$$(\nabla. v_h, w + \delta\,\beta \cdot \nabla w) = (q, w + \delta\,\beta \cdot \nabla w);\quad \forall w \in \vartheta \qquad (3.16)$$

$$(v_h, א + \delta\,\beta \cdot \nabla א) = (\sigma(s)p_h, \nabla. א) + (\sigma(s)p_h, \delta\,\beta \cdot \nabla א);\quad \forall\, א \in \xi \qquad (3.17)$$

ie

$$(\nabla. v_h, w) + (\nabla. v_h, \delta\,\beta \cdot \nabla w) = (q, w) + (q, \delta\,\beta \cdot \nabla w);\ \forall w \in \vartheta \qquad (3.18)$$

$$(v_h, א) + (א, \delta\,\beta \cdot \nabla א) = (\sigma(s)p_h, \nabla. א) + (\sigma(s)p_h, \delta\,\beta \cdot \nabla א);\quad \forall א \in \xi \qquad (3.19)$$

The petrov GFEM for the saturation. Find $s_h \in \psi_{h,r} \subset \eta$ such that

$$\big(\emptyset s_{h,t}, \varphi + \delta\,\beta \cdot \nabla \varphi\big) + (a(s)\nabla. s, \nabla.\varphi) + \nabla.(f(s)v_h, \varphi + \delta\,\beta \cdot \nabla \varphi),$$
$$+(q\_s, \varphi + \delta\,\beta \cdot \nabla\varphi) = 0;\ \ s(x,0) = s^0(x),\ \forall \varphi \in \eta \qquad (3.20)$$

rewrite equation

$$\big(\emptyset s_{h,t}, \varphi\big) + (\emptyset s_{h,t}, \delta\,\beta \cdot \nabla\varphi\big) + (a(s)\nabla.\ \varphi, \nabla.\varphi) + (\nabla.(f(s)v_h, \varphi) +$$
$$(\nabla.(f(s)v_h, \delta\,\beta \cdot \nabla\varphi) + (q\_s, \varphi) + (q\_s, \delta\,\beta \cdot \nabla \varphi) = 0;\ \ s(x,0) = s^0(x)\ \ \forall \varphi \in \eta$$

(3.21)

4. Petrov Galerkin Finite Element Methods' Stability

We demonstrated the stability of the PGFE technique, as stated in equation (3.18) and (3.19), which means that the problem data can be used to determine an appropriate norm for a solution.

Lemma 4.1: (the stability for the pressure)

Let a solution $p_h \in W$ and C is a constant and independent h then

$$\|p_h\|^2 \leq C\, \|q\|^2 \qquad (4.1)$$

Proof : From equation (3.18) , (3.19) and put $w = p_h$ we get

$$(\sigma(s_h)p_h, \nabla. א) + (\sigma(s_h)p_h, \delta\,\beta \cdot \nabla א) = (q, w) + (q, \delta\,\beta \cdot \nabla w) \qquad (4.2)$$

by using Cauchy – Schwartz and Young's-inequality, we obtain;

$$|(\sigma(s_h)p_h, \nabla. א)| \leq \|\sigma(s_h)\|\ \|\nabla. p_h\|^2 \qquad (4.3)$$

$$\leq c_1 \|\sigma(s_h)\| \, \|\nabla . p_h\|^2 \qquad (4.4)$$

and, term

$$|(\sigma(s_h)p_h, \delta\, \beta \cdot \nabla\, \aleph)| \leq \|\sigma(s)\nabla . p_h\| \, \|\delta\, \beta . \nabla p_h\|^2$$

$$\leq \frac{z}{2}\|\sigma(s_h)\| \;\; \|\nabla . p_h\|^2 + \frac{1}{2z}\, \delta \,\|\, \beta . \nabla p_h\|^2 \qquad (4.5)$$

Then

$$|(\sigma(s_h)p_h, \nabla . \aleph) + (\sigma(s_h)p_h, \delta\, \beta \cdot \nabla\, \aleph)| \leq c_2 \|\sigma(s_h)\| \, \|p_h\|^2 + \lambda \,\|\, \beta . \nabla p_h\|^2, \qquad (4.6)$$

Where $c_2 = (c_1 + \frac{z}{2})$, $\lambda = \frac{1}{2z}\, \delta$

and, the term by using Cauchy – Schwartz and Young's-inequality, we get;

$$|q, w) + (q, \delta\, \beta \cdot \nabla\, w\,| \leq \frac{1}{4} \;\; \|q\|^2 + \frac{z}{2}\|q\|^2 + \frac{1}{2z}\, \delta \,\|\, \beta . \nabla p_h\|^2 \qquad (4.7)$$

$$\leq \mu \;\; \|q\|^2 + \|p_h\|^2 + \lambda \,\|\, \beta . \nabla p_h\|^2 \qquad (4.8)$$

Where $\mu = (\frac{1}{4} + \frac{z}{2})$, $\lambda = \frac{1}{2z}\, \delta$

From (4.6) the stability for pressure, saturation, and velocity, in that order
and (4.8), we have ;

$$c_2 \|\sigma(s_h)\| \, \|p_h\|^2 + \lambda \,\|\, \beta . \nabla p_h\|^2 \leq \mu \;\; \|q\|^2 + \|p_h\|^2 + \lambda \,\|\, \beta . \nabla p_h\|^2 \qquad (4.9)$$

Then

$$c_2 \|\sigma(s_h s)\| \, \|p_h\|^2 \leq \mu \;\; \|q\|^2 + \|p_h\|^2 \qquad (4.10)$$

$$\|p_h\|^2 \leq C \;\; \|q\|^2 \qquad (4.11)$$

Where $C = \frac{1}{\min\{c_2\|\sigma(s)\|, -\mu \;\; \|p_h\|^2\}}$

Lemma 4.2: (the stability for the velocity)

Let a solution $v_h \in U$ and L is a constant and independent h then

$$\|v_h\|^2 \;\; \leq L\, \{\|p_h\|^2\} \qquad (4.12)$$

Proof: put $\aleph = v_h$ in equation (3.19) we get;

$$(v_h v_h) + (v_h, \delta\, \beta \cdot \nabla\, v_h) = (\sigma(s_h)p_h, \nabla . v_h) + (\sigma(s_h)p_h, \delta\, \beta \cdot \nabla\, v_h) \qquad (4.13)$$

Then

$$|(v_h v_h)| \;\; \leq \alpha_1 \|v_h\|^2 \qquad (4.15)$$

through the use of Cauchy – Schwartz and Young's-inequality, we obtain;

$$|(v_h, \delta\, \beta \cdot \nabla\, v_h)| \leq \|v_h\| \, \|\delta\, \beta \cdot \nabla\, v_h\| \qquad (4.16)$$

$$\leq \frac{z}{2} \;\; \|v_h\|^2 + \frac{1}{2z}\, \delta \,\|\, \beta . \nabla v_h\|^2 \qquad (4.17)$$

Let $z = 2$, we have ;

$$|(v_h, v_h) + (v_h, \delta\, \beta \cdot \nabla\, v_h)| \;\; \leq \;\; \|v_h\|^2 + \frac{1}{4}\, \delta \,\|\, \beta . \nabla v_h\|^2 \qquad (4.18)$$

$$\leq \alpha_2 \;\; \|v_h\|^2 + \;\; \mu \,\|\, \beta . \nabla v_h\|^2 \qquad (4.19)$$

Where $\alpha_2 = (\alpha_1 + 1)$, and $\mu = \frac{1}{4}\, \delta$.

and,

$$|(\sigma(s_h)p_h, \nabla . v_h)| \leq 2\|\, \sigma(s_h)\|^2 \|p_h\|^2 + \frac{r}{2}\|v_h\|^2 \qquad (4.20)$$

through the use of Cauchy – Schwartz and Young's-inequality, we obtain;

$$\left|(\sigma(s_h)p_h, \delta\, \beta \cdot \nabla\, v_{h)}\right| \;\; \leq \|\sigma(s_h)\| \, \|\delta\, \beta \cdot \nabla\, v_h\| \qquad (4.21)$$

$$\leq 2 \;\; \|\sigma(s_h)\|^2 \|p_h\|^2 + \frac{1}{4}\, \delta \,\|\, \beta . \nabla v_h\|^2 \qquad (4.22)$$

$$\leq 2 \;\; \|\sigma(s_h)\|^2 \|p_h\|^2 + \;\; \mu \;\; \|\, \beta . \nabla v_h\|^2 \qquad (4.23)$$

Then

$$|(\sigma(s_h)p_h, \nabla . v_h) + (\sigma(s_h)p_h, \delta\, \beta \cdot \nabla\, v_h)| \leq 4 \;\; \|\sigma(s_h)\|^2 \|p_h\|^2 + \;\; \mu \;\; \|\, \beta . \nabla v_h\|^2, \qquad (4.24)$$

$$\alpha_2 \;\; \|v_h\|^2 + \;\; \mu \,\|\, \beta . \nabla v_h\|^2 \leq 4 \;\; \|\sigma(s_h)\|^2 \|p_h\|^2 + \;\; \mu \;\; \|\, \beta . \nabla v_h\|^2 \qquad (4.25)$$

We obtain;

$$\alpha_2 \;\; \|v_h\|^2 \;\; \leq 4 \;\; \|\sigma(s_h)\|^2 \|p_h\|^2 \qquad (4.26)$$

$$\|v_h\|^2 \le L\{\|p_h\|^2\} \qquad (4.27)$$

Where

$$L = \max\{\frac{4\|\sigma(s_h)\|^2}{\alpha_2}\}$$

Theorem 4.1: (the stability for the velocity and the pressure)

Let the dual solution $v_h \in U$ and $p_h \in W$ and $\mathcal{B}$ a constant and independent h ,then

$$\|p_h\|^2 + \|v_h\|^2 \le \mathcal{B}\{\|q\|^2 + \|v_h\|^2\} \qquad (4.28)$$

Proof : From equation (3.18) and (3.19) , replacement test function with $\aleph \in U$ and $v_h \in U$, we have ;

$$(\sigma(s_h)p_h, \nabla.\aleph_h) + (\sigma(s_h)p_h, \delta\,\beta \cdot \nabla\,\aleph_h) = (q, \aleph) + (q, \delta\,\beta \cdot \nabla\,\aleph), \forall \aleph \in \mathrm{U} \qquad (4.29)$$

put $\aleph = v_h$ in above equation, we have;

$$(\sigma(s_h)\nabla p_h, \nabla.v_h) + (\sigma(s_h)p_h, \delta\,\beta \cdot \nabla v_h) = (q, v_h) + (q, \delta\,\beta \cdot \nabla\, v_h), \qquad (4.30)$$

Then

$$|(\sigma(s_h)\nabla p_h, \nabla.v_h)| \le \frac{\chi_{2\|\sigma(s_h)\|^2}}{2}\|\nabla p_h\|^2 + 2\chi_2\|\nabla.v_h\|^2, \qquad (4.31)$$

$$\le L\|p_h\|^2 + L\|v_h\|^2, \qquad (4.32)$$

$$\left|(\sigma(s_h)p_h, \delta\,\beta \cdot \nabla\, v_{h)}\right| \le 2\ \|\sigma(s_h)\|^2\|p_h\|^2 + \ \mu\ \|\beta.\nabla v_h\|^2, \qquad (4.33)$$

$$\le \xi\ \|p_h\|^2 + \ \mu\ \|\beta.\nabla v_h\|^2, \qquad (4.34)$$

$$\le n\,\|p_h\|^2 + L\|v_h\|^2 + \ \mu\ \|\beta.\nabla v_h\|^2, \qquad (4.35)$$

Where $\xi = \min\{2\|\sigma(s_h)\|^2\}$,$n = (L + \xi)$,

$$|(q, v_h)| \le \frac{1}{4}\ \|q\|^2 + \|v_h\|^2 \qquad (4.36)$$

and,

$$|(q, \delta\,\beta \cdot \nabla v_h| \le \ \|q\|^2 + \mu\,\|\beta.\nabla v_h\|^2 \qquad (4.37)$$

Then,

$$|(q, v_h) + (q, \delta\,\beta \cdot \nabla v_h| \le \tau\,\|q\|^2 + \|v_h\|^2 + \mu\,\|\beta.\nabla v_h\|^2, \qquad (4.38)$$

$$\left|(\sigma(s_h)\nabla p_h, \nabla.v_h) + (\sigma(s_h)p_h, \delta\,\beta \cdot \nabla\, v_{h)}\right| \le \varkappa\,\|p_h\|^2 + 2L\|v_h\|^2 + \ \mu\ \|\beta.\nabla v_h\|^2,$$
(4.39)

Where $\varkappa = (L + n)$, we get,

$$\|p_h\|^2 + \|v_h\|^2 \le \mathcal{B}\{\|q\|^2 + \|v_h\|^2\} \qquad (4.40)$$

where $\mathcal{B} = \frac{1}{\varkappa}\max\{2L\}$

Theorem 4.2: (the stability for Saturation)

Let a solution $s \in \varphi_{h,r} \subset \eta$ and λ_4 is a constant and independent h then

$$\|s_{h,t}\|^2 \le \|s_h\|^2 exp^{(-\lambda_1 z)} + \lambda_4 exp^{(-\lambda_1 z)}\int_0^t exp^{(\lambda_1 z)}\ \{\|\beta.\nabla s_h(z)\|^2 + \|v_h(z)\|^2\}dz,$$

Proof: Rewrite (3.21) and put $\varphi = s_h$ we have,

$$(\emptyset s_{h,t}, s_h) + (\emptyset s_{t,h}, \delta\,\beta \cdot \nabla s_h) + (a(s_h)\nabla.\ s_h, \nabla.s_h) + (v_h\nabla.(f(s_h), s_h)$$
$$+(v_h\nabla.(f(s_h), \delta\,\beta \cdot \nabla\, s_h) + (f(s_h)\nabla.v_h, s_h) + (f(s_h)\nabla.v_h, \delta\,\beta \cdot \nabla\, s_h) +$$
$$(q\_s, s_h) + (q\_s, \delta\,\beta \cdot \nabla\, s_h) = 0;\ \ s(x,0) = s^0(x)\ \ \forall s_h \in \eta\ , \qquad (4.41)$$

we get,

$$(\emptyset s_{h,t}, s_h) \le \frac{1}{2}\ \|\emptyset\|\ \frac{d}{dt}\ \|s_{h,t}\|^2, \qquad (4.42)$$

through the use of Cauchy – Schwartz and Young's-inequality, we obtain;

$$(\emptyset s_{h,t}, \delta\, \beta \cdot \nabla s_h) \leq \|\emptyset\| \, \|s_h\| \, \|\delta\, \beta \cdot \nabla s_h\|, \quad (4.43)$$

$$\leq \frac{z}{2} \|\emptyset\| \, \|s_{h,t}\|^2 + \frac{1}{2z} \delta \, \| \beta . \nabla s_h \|^2 \quad (4.45)$$

Let $\tau = \frac{z}{2} \|\emptyset\|$ and $z = 2$ we have,

$$\leq \tau \, \|s_h\|^2 + \frac{1}{4} \delta \, \| \beta . \nabla s_h\|^2, \quad (4.46)$$

$$|(a(s_h)\nabla . \, s_h, \nabla . s_h)| \leq \frac{1}{4} \| a(s_h) \|^2 + \| \nabla s_h\|^2 , \quad (4.47)$$

$$\leq c_1 \| \nabla s_h\|^2 + c_2 \| \nabla s_h\|^2 \leq J \| \nabla s_h\|^2 \quad (4.48)$$

Where $J = c_1 + c_2$,

$$|(v_h \nabla . (f(s_h), s_h)| \leq \frac{1}{4} \| v_h \|^2 \| \nabla f(s_h)\|^2 + \| s_h\|^2, \quad (4.49)$$

$$\leq c_3 \| s_h\|^2 + \| s_h\|^2 \leq \gamma \| \nabla s_h\|^2, \quad (4.50)$$

Where $\gamma = c_3 + 1$, and,

$$| (v_h \nabla . (f(s_h), \delta\, \beta \cdot \nabla s_h)| \leq \frac{1}{4} \| v_h \|^2 \| \nabla f(s_h)\|^2 + \delta \, \| \beta . \nabla s_h\|^2 , \quad (4.51)$$

$$\leq c_3 \| s_h\|^2 + \frac{1}{4} \delta \, \| \beta . \nabla s_h\|^2, \quad (4.52)$$

$$|(f(s_h)\nabla . v_h, s_h)| \leq \frac{1}{4} \| \nabla f(s_h)\|^2 \, \|\nabla v_h \|^2 + \| s_h\|^2, \quad (4.53)$$

$$\leq \frac{1}{16} \| f(s_h)\|^2 + \frac{1}{4} \|\nabla v_h \|^2 + \| s_h\|^2 \leq c_4 \| s_h\|^2 + c_5 \| v_h \|^2, \quad (4.54)$$

and,

$$|(f(s_h)\nabla . v_h, \delta\, \beta \cdot \nabla s_h)| \leq \frac{1}{4} \| \nabla f(s_h)\|^2 \, \|\nabla v_h \|^2 + \delta \, \| \beta . \nabla s_h\|^2, \quad (4.55)$$

$$\leq \frac{1}{16} \| \nabla f(s_h)\|^2 + \frac{1}{4} \|\nabla v_h \|^2 + \delta \, \| \beta . \nabla s_h\|^2, \quad (4.56)$$

$$\leq c_6 \| s_h\|^2 + c_7 \| v_h \|^2 + \delta \, \| \beta . \nabla s_h\|^2, \quad (4.57)$$

$$|(q_{-}s, s_h)| \leq c_8 \| s_h\|^2 , \quad (4.58)$$

$$|(q_{-}s, \delta\, \beta \cdot \nabla s_h)| \leq c_9 \, \delta \, \| \beta . \nabla s_h\|^2, \quad (4.59)$$

became equation (4.42),

$$\frac{1}{2} \|\emptyset\| \frac{d}{dt} \|s_{h,t}\|^2 + \tau \, \|s_h\|^2 + \frac{1}{4} \delta \, \| \beta . \nabla s_h\|^2 + J \| \nabla s_h\|^2 + \gamma \| \nabla s_h\|^2 +$$

$$c_3 \| s_h\|^2 + \frac{1}{4} \delta \, \| \beta . \nabla s_h\|^2 + c_4 \| s_h\|^2 + c_5 \| v_h \|^2 + c_6 \| s_h\|^2 + c_7 \| v_h \|^2$$

$$+ \delta \, \| \beta . \nabla s_h\|^2 + c_8 \| s_h\|^2 + c_9 \, \delta \, \| \beta . \nabla s_h\|^2 \leq 0 , \quad (4.60)$$

$$\frac{d}{dt} \|s_{h,t}\|^2 + \lambda_1 \|s_h\|^2 \leq \lambda_2 \| \beta . \nabla s_h\|^2 + \lambda_3 \| v_h \|^2$$

where $\lambda_1 = \frac{2}{\|\emptyset\|}(\tau + J + \gamma + c_3 + c_4 + c_6 + c_8), \lambda_2 = \frac{2}{\|\emptyset\|}((\frac{1}{4} + \frac{1}{4} + 1 + c_9)\, \delta)$ and $\lambda_3 = \frac{2\, c_7}{\|\emptyset\|}$,

The integral component is multiplied by both sides of the inequality above factor $\exp(\lambda_1 z)$ and then integrate from 0 to t, we have

$$\|s_{h,t}\|^2 \leq \|s_h\|^2 exp^{(-\lambda_1 z)} + \lambda_4 exp^{(-\lambda_1 z)} \int_0^t exp^{(\lambda_1 z)} \{\| \beta . \nabla s_h(z)\|^2 + \| v_h(z)\|^2 \}dz,$$

(4.61)

Where $\lambda_4 = \max\{\lambda_1, \lambda_2\}$.

CONCLUSIONS

Conclusions and Upcoming Projects

We have created a sequential solution technique for fluid fluxes that are miscible in porous medium using the Petrov Galerkin method to solve the saturation transport equation and the pressure and Darcy velocity equations can be solved using the Petrov Galerkin mixed FEM. the stability for pressure, saturation, and velocity, in that order.

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