J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On generalized degenerate twisted (h, q)-tangent numbers and polynomials

C. S. RYOO

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : We introduced the generalized twisted (h, q)-tangent numbers and polynomials. In this paper, our goal is to give generating functions of the generalized degenerate twisted (h, q)-tangent numbers and polynomials. We also obtain some explicit formulas for generalized degenerate twisted (h, q)-tangent numbers and polynomials.

Key words : Generalized tangent numbers and polynomials, degenerate generalized twisted (h, q)-tangent numbers and polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-16]). In [2], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, Feng Qi *et al.*[3] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. In this paper, we obtain some interesting properties for generalized degenerate tangent numbers and polynomials. Throughout this paper we use the following notations. Let p be a fixed odd prime number. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let r be a positive integer, and let ζ be rth root of 1. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$. Then the generalized twisted (h, q)-tangent numbers associated with associated with χ, $T^{(h)}_{n,\chi,q,\zeta}$, are defined by the following generating function

$$\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}\frac{t^n}{n!}. \tag{1.1}$$

We now consider the generalized twisted (h, q)-tangent polynomials associated with χ, $T^{(h)}_{n,\chi,q,\zeta}(x)$, are also defined by

$$\left(\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}\right)e^{xt} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x)\frac{t^n}{n!}. \tag{1.2}$$

When $\chi = \chi^0$, above (1.1) and (1.2) will become the corresponding definitions of the twisted (h, q)-tangent numbers $T^{(h)}_{n,q,w}$ and polynomials $T^{(h)}_{n,q,w}(x)$. If $q \to 1$, above (1.1) and (1.2) will become the corresponding definitions of the generalized twisted tangent numbers $T_{n,\chi,w}$ and polynomials $T_{n,\chi,w}(x)$. We recall that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations(see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n, k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n, k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{1.3}$$

We also have

$$\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.5}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.6}$$

2. On the generalized degenerate twisted (h,q)-tangent polynomials

In this section, we define the generalized degenerate twisted (h,q)-tangent numbers and polynomials, and we obtain explicit formulas for them. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$, and let ζ be rth root of 1. For $h \in \mathbb{Z}$, the generalized degenerate twisted (h,q)-tangent polynomials associated with associated with χ, $T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)$, are defined by the following generating function

$$\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} \tag{2.1}$$

and their values at $x=0$ are called the generalized degenerate twisted (h,q)-tangent numbers and denoted $T^{(h)}_{n,\chi,q,\zeta}(\lambda)$.

From (2.1) and (1.2), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty}\lim_{\lambda\to 0} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&= \left(\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}\right)e^{xt}\\
&= \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x)\frac{t^n}{n!}.
\end{aligned}$$

Thus, we get

$$\lim_{\lambda\to 0} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = T^{(h)}_{n,\chi,q,\zeta}(x), (n \geq 0).$$

From (2.1) and (1.6), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&= \left(\sum_{m=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l\frac{t^l}{l!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T^{(h)}_{l,\chi,q,\zeta}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.2}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 1. For $n \geq 0$, we have

$$T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} T^{(h)}_{l,\chi,q,\zeta}(\lambda)(x|\lambda)_{n-l}.$$

For $\chi = \chi^0$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2}{\zeta q^h(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \sum_{m=0}^{\infty} T^{(h)}_{n,q,\zeta}(x|\lambda)\frac{t^m}{m!}.\end{aligned} \tag{2.3}$$

Theorem 2. For $n \geq 0$ and $\chi = \chi^0$, we have

$$T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = T^{(h)}_{n,q,\zeta}(x|\lambda).$$

For $d \in \mathbb{N}$ with $d \equiv 1(\text{mod}2)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \frac{2}{\zeta q^h(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\sum_{l=0}^{d-1}(-1)^l\chi(l)(1+\lambda t)^{2l/\lambda}\\ &= \sum_{n=0}^{\infty}\left(d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}|\frac{\lambda}{d}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 3. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\text{mod } 2)$. Then we have

$$(1)\ T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}|\frac{\lambda}{d}\right),$$

$$(2)\ T^{(h)}_{n,\chi,q,\zeta}(\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}\right).$$

For $m \in \mathbb{Z}_+$, we obtain we can derive the following relation:

$$\begin{aligned}&\sum_{m=0}^{\infty}\zeta^d q^{hd}T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda)\frac{t^m}{m!} + \sum_{m=0}^{\infty}T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda)\frac{t^m}{m!}\\ &= 2\sum_{l=0}^{d-1}(-1)^l\chi(l)\zeta^l q^{hl}(1+\lambda t)^{2l/\lambda}\\ &= \sum_{m=0}^{\infty}\left(2\sum_{l=0}^{d-1}(-1)^{n-1-l}\chi(l)\zeta^l q^{hl}(2l|\lambda)_m\right)\frac{t^m}{m!}.\end{aligned} \tag{2.5}$$

By comparing of the coefficients $\frac{t^m}{m!}$ on the both sides of (2.5), we have the following theorem.

Theorem 4. For $m \in \mathbb{Z}_+$, we have

$$\zeta^d q^{hd} T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda) + T^{(h)}_{m,\chi,q,\zeta}(\lambda) = 2\sum_{l=0}^{d-1}(-1)^l \chi(l)\zeta^l q^{hl}(2l|\lambda)_m.$$

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x+y|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{(x+y)/\lambda} \\ &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{(2a+x)/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{y/\lambda} \\ &= \left(\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T^{(h)}_{l,\chi,q,\zeta}(x|\lambda)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.6), we have the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x+y|\lambda) = \sum_{k=0}^{n}\binom{n}{k}T^{(h)}_{k\chi,q,\zeta}(x|\lambda)(y|\lambda)_{n-k}.$$

From Theorem 5, we note that $T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)$ is a Sheffer sequence.

By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (2.1), we obtain

$$\begin{aligned}\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}e^{xt} &= \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n\frac{1}{n!} \\ &= \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\lambda^m\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.7}$$

Thus, by (2.7) and (1.2), we have the following theorem.

Theorem 6. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x) = \sum_{n=0}^{m}\lambda^{m-n}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}T^{(h)}_{n,\chi,q,\zeta}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{(2a+x)/\lambda}}{\zeta^d q^{hd}(1+\lambda t)^{2d/\lambda}+1} \\ &= \sum_{m=0}^{\infty}T^{(h)}_{m,\chi,q,\zeta}(x|\lambda)\frac{t^m}{m!},\end{aligned} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty}T^{(h)}_{n,\chi,q,\zeta}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}T^{(h)}_{n,\chi,q,\zeta}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}. \tag{2.9}$$

Thus, by (2.8) and (2.9), we have the following theorem.

Theorem 8. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x|\lambda) = \sum_{n=0}^{m} T^{(h)}_{n,\chi,q,\zeta}(x)\lambda^{m-n} S_1(m,n).$$

Acknowledgment

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

REFERENCES

1. Carlitz, L.(1979). *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., v.15, pp. 51-88.
2. Carlitz, L.(1956). *A degenerate Staudt-Clausen theorem*, Arch. Math. (Basel), v.7, pp. 28-33.
3. Qi, F., Dolgy, D.V., Kim, T., Ryoo, C. S.(2015). *On the partially degenerate Bernoulli polynomials of the first kind*, Global Journal of Pure and Applied Mathematics, v.11, pp. 2407-2412.
4. Kim, T.(2015). *Barnes' type multiple degenerate Bernoulli and Euler polynomials*, Appl. Math. Comput. v.258, pp. 556-564.
5. Ozden, H., Cangul, I.N., Simsek, Y.(2009). *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math., v.18, pp. 41-48.
6. Ryoo, C.S.(2013). *A Note on the tangent numbers and polynomials*, Adv. Studies Theor. Phys., v.7, pp. 447 - 454.
7. Young, P.T.(2008). *Degenerate Bernoulli polynomials, generalized factorial sums, and their applications*, Journal of Number Theory, v.128, pp. 738-758.
8. Kim, T.(2008). *Euler numbers and polynomials associated with zeta function,* Abstract and Applied Analysis, Art. ID 581582.
9. Liu, G.(2006). *Congruences for higher-order Euler numbers*, Proc. Japan Acad., v.82 A, pp. 30-33.
10. Ryoo, C.S., Kim, T., Jang, L.C.(2007). *Some relationships between the analogs of Euler numbers and polynomials*, Journal of Inequalities and Applications, v.2007, ID 86052, pp. 1-22.
11. Ryoo, C.S.(2014). *Note on the second kind Barnes' type multiple q-Euler polynomials*, Journal of Computational Analysis and Applications, v.16, pp. 246-250.
12. Ryoo, C.S.(2015). *On the second kind Barnes-type multiple twisted zeta function and twisted Euler polynomials*, Journal of Computational Analysis and Applications, v.18, pp. 423-429.
13. Ryoo, C.S.(2019). *Some symmetric identities for (p,q)-Euler zeta function*, J. Computational Analysis and Applications v.27, pp. 361-366.

14. Ryoo, C.S.(2020). *Symmetric identities for the second kind q-Bernoulli polynomials*, Journal of Computational Analysis and Applications, v.28, pp. 654-659.

15. Ryoo, C.S.(2020). *On the second kind twisted q-Euler numbers and polynomials of higher order*, Journal of Computational Analysis and Applications, v.28, pp. 679-684.

16. Ryoo, C.S.(2020). *Symmetric identities for Dirichlet-type multiple twisted (h, q)-l-function and higher-order generalized twisted (h, q)-Euler polynomials*, Journal of Computational Analysis and Applications, v.28, pp. 537-542.