Automorphism Graph of the Cartesian product of Cyclic Graph of Order3

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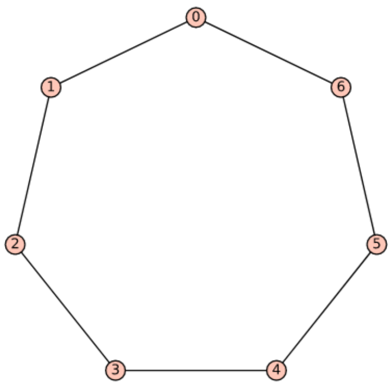
ABSTRACT
In this paper we introduce the computing the automorphism graph of Cartesian product of cyclic graph of order three its isomorphic to $\mathrm{Aut}(\Gamma \boxdot \Gamma) \cong (D_6 \times D_6) \rtimes C_2$.

Keywords: automorphism, Cartesian, cyclic

1. INTRODUCTION AND PRELIMINARY

Suppose that A graph $\Gamma = (V, E)$ is a set of vertices, V, so as set of edges, E. The set of all vertices is denoted by $V(\Gamma)$ and the set of all edges is denoted by $E(\Gamma)$[1]. It is well known for any two vertices is connected in graph is edge and denoted by $\{a, b\}$. For the following example we can computing the set of all vertices and edges,



$\Gamma = C_7$

$V(\Gamma) = \{0,1,2,3,4,5,6\}$,

$$E(\Gamma) = \{\{0,1\},\{0,5\},\{0,6\},\{1,7\},\{1,2\},\{2,8\},\{2,3\},\{3,9\},\{3,4\},\{4,10\},\{4,5\},\{5,11\},\{6,10\},\{6,8\},\{7,6\},\{7,9\},\{8,10\},\{9,11\},\{1,7\},\{1,2\}\}$$

Let Γ be a finite graph [2], [3], the automorphism graph is define the isomorphism from a graph G to itself and denoted by Aut(Γ), the automorphism graph of a graph Γ, is a set whose elements are automorphism $\sigma: \Gamma \to \Gamma$, and where the group [4]multiplication is composition of automorphism. [5]In other words, its group structure is obtained as a subgroup of Sym(G). the group of all permutations on G. Thus, an automorphism ρ of graph Γis a structure-preserving permutation$\rho_V$on $V(\Gamma)$ along with a (consistent) permutation $\rho_E$on $E(\Gamma)$We may write $\rho = (\rho_V, \rho_E)$.[5], [6]

It is well known, for any permutation we can write it by the following:

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

which maps 1 to 3, 2 to4, and so on, has the disjoint cycle form

$$\rho = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{pmatrix}$$

From above example $\Gamma = C_7$ we can computing the automorphim graph and we will present the $\mathrm{Aut}(C_7) \cong D_{14}$ is a dihedral group of order 14, the Dihedral group denoted by D_{2n} is a finite group of order 2n and generate by two elements a and b. [3], [7]

Where the element a of order n and the element b have order 2, the representation of the group is define by :

$$D_{2n} = \langle a, b | a^n = b^2 = e | bab = a^{-1} \rangle$$

The automorpism $Aut(\Gamma(C_3))$, the homomorphism graph is define by:

$$\rho: V(\Gamma(C_3)) \rightarrow V(\Gamma(C_3))$$

$$\rho_1(V(\Gamma(C_7))): \begin{cases} 1 \rightarrow 1 \\ 2 \rightarrow 2 \\ 3 \rightarrow 3 \end{cases} = (1)(2)(3) = \lambda_1,$$

$$\rho_2(V(\Gamma(C_7))): \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 3 \\ 3 \rightarrow 1 \end{cases} = (123) = \lambda_2,$$

$$\rho_3(V(\Gamma(C_7))): \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 1 \\ 3 \rightarrow 2 \end{cases} = (132) = \lambda_3,$$

The above elements of rotation of degree $\frac{2\pi}{3}$, now, by reflexive elements we obtain on the following:

$$\rho_4(V(\Gamma(C_7))): \begin{cases} 1 \rightarrow 1 \\ 2 \rightarrow 3 \\ 3 \rightarrow 2 \end{cases} = (1)(23) = \mu_1,$$

$$\rho_5(V(\Gamma(C_7))): \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 2 \\ 3 \rightarrow 1 \end{cases} = (2)(13) = \mu_2,$$

$$\rho_4(V(\Gamma(C_7))): \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 1 \\ 3 \rightarrow 3 \end{cases} = (12)(3) = \mu_3,$$

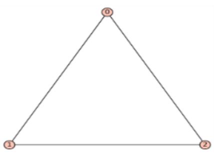
For structural representation of any a finite groups, we can from using the Cayley tables. A Cayley table lists all the elements of a finite group and results of group operation between all possible pair of elements of the group.

	λ_1	λ_2	λ_3	μ_1	μ_2	μ_3
λ_1	λ_1	λ_2	λ_3	μ_1	μ_2	μ_3
λ_2	λ_2	λ_3	λ_1	μ_2	μ_3	μ_1
λ_3	λ_3	λ_1	λ_2	μ_3	μ_1	μ_2
μ_1	μ_1	μ_3	μ_2	λ_1	λ_2	λ_3
μ_2	μ_2	μ_1	μ_3	λ_3	λ_1	λ_2
μ_3	μ_3	μ_2	μ_1	λ_2	λ_3	λ_1

The cartesian product graph is define by, Let Γ_1 and Γ_2 be a finite graph, the Cartesian product $\Gamma_1 \boxdot \Gamma_2$ of graphs such that:[1], [6]

- the vertex set of $\Gamma_1 \boxdot \Gamma_2$ is the Cartesian product $V(\Gamma_1) \times V(\Gamma_2)$; and
- for any two vertices (u, v) and (u', v') is adjacent in $\Gamma_1 \boxdot \Gamma_2$ if and only if either
  - $u = u'$ and v is adjacent to v' in Γ_2, or
  - $v = v'$ and u is adjacent to u' in Γ_1.

```
sage: G = graphs.CycleGraph(3)
sage: G.show()
```



```
sage: G = graphs.CycleGraph(3)
sage: C = G.cartesian_product(G)
sage: C.show()
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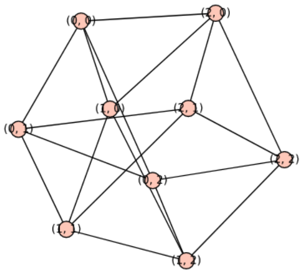

The set of all vertices are {(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)} and the set of edges are:

(0,0)~{(0,1), (0,2), (1,0), (2,0)}

$$(0,1)\sim\{(0,0),(0,2),(1,1),(2,1)\}$$
$$(0,2)\sim\{(0,1),(0,0),(1,2),(2,2)\}$$
$$(1,0)\sim\{(0,0),(2,0),(1,2),(1,1)\}$$
$$(1,1)\sim\{(0,1),(2,1),(1,0),(1,2)\}$$
$$(1,2)\sim\{(1,0),(1,1),(0,2),(2,2)\}$$
$$(2,0)\sim\{(2,1),(2,2),(1,0),(0,0)\}$$
$$(2,1)\sim\{(2,2),(2,0),(1,1),(0,1)\}$$
$$(2,2)\sim\{(2,0),(2,1),(1,2),(0,2)\}$$

2. MAIN RESULTS

In this section, we will prove that, the following theorem.

2.1 Definition

Suppose that $\mathcal{H}$ and $\mathcal{K}$ are groups and an action $\phi: \mathcal{K} \to \mathrm{Aut}(\mathcal{H})$ of $\mathcal{K}$ on $\mathcal{H}$ by automorphisms, the corresponding semi-direct product $\mathcal{H} \rtimes_\phi \mathcal{K}$.

2.2 Theorem

The automorphism group of cycle graph be isomorphic to Dihedral group.

2.1 Theorem

Let Γ be finite graph and isomorphic to cycle graph of order 3, the automorphism group of graph Γ is given by the following:

$$\mathrm{Aut}(\Gamma \boxdot \Gamma) \cong (D_6 \times D_6) \rtimes C_2$$

Proof:

Clear that, the quotient group $\left[{}^{(D_6 \times D_6) \rtimes C_2}/_{D_6 \times D_6}\right] = 2$, this means the subgroup $\mathcal{H} = D_6 \times D_6$ is normal subgroup of group $(D_6 \times D_6) \rtimes C_2$, this is sufficient to prove that group $(D_6 \times D_6) \rtimes C_2$ is the result of a semi-product product. Suppose that $\mathcal{K} = C_2$.and an action $\phi: C_2 \to \mathrm{Aut}(D_6 \times D_6)$ of C_2 on $(D_6 \times D_6)$ by automorphisms, the corresponding semi-direct product $(D_6 \times D_6) \rtimes_\phi C_2$. By GAP program we compute the $\mathrm{Aut}(D_6 \times D_6)$ has 72 elements and generators of {(1,4)(2,5)(3,6), (5,6), (2,3)(5,6), (4,6,5), (1,3,2) }

```
gap> d:=Dihedral Group (IsPermGroup,6);
x:=Direct Product (d,d);
aut:=Automorphism Group (x);
Group ([ (1,2,3), (2,3) ])
gap> c:=Cyclic Group (IsPermGroup,2);
Group ([ (1,2) ])
gap> x:=Direct Product (d,d);
ss:=Structure Description(s); Group ([ (1,2,3), (2,3), (4,5,6), (5,6) ])
gap> aut:=Automorphism Group (x);
<group of size 72 with 5 generators>
gap> h:=All Homomorphisms (c,aut);
[ [ (1,2) ] -> [ Identity Mapping ( Group([ (1,2,3), (2,3), (4,5,6), (5,6) ]) ) ], [ (1,2) ] -> [ ^(2,3) ],
 [ (1,2) ] -> [ ^(5,6) ], [ (1,2) ] -> [ ^(1,2) ], [ (1,2) ] -> [ ^(4,5) ], [ (1,2) ] -> [ ^(1,3) ],
  [ (1,2) ] -> [ ^(4,6) ], [ (1,2) ] -> [ ^(2,3)(5,6) ], [ (1,2) ] -> [ ^(2,3)(4,5) ],
  [ (1,2) ] -> [ ^(1,2)(5,6) ], [ (1,2) ] -> [ ^(2,3)(4,6) ], [ (1,2) ] -> [ ^(1,2)(4,5) ],
  [ (1,2) ] -> [ ^(1,3)(5,6) ], [ (1,2) ] -> [ ^(1,2)(4,6) ], [ (1,2) ] -> [ ^(1,3)(4,6) ],
  [ (1,2) ] -> [ ^(1,3)(4,5) ], [ (1,2) ] -> [ ^(1,4)(2,5)(3,6) ], [ (1,2) ] -> [ ^(1,4)(2,6)(3,5) ],
  [ (1,2) ] -> [ ^(1,6)(2,4)(3,5) ], [ (1,2) ] -> [ ^(1,5)(2,6)(3,4) ], [ (1,2) ] -> [ ^(1,6)(2,5)(3,4) ]
   , [ (1,2) ] -> [ ^(1,5)(2,4)(3,6) ] ]
gap> s:=SemidirectProduct(c,h[22],x);
Group([ (7,8), (6,7,8), (4,5), (3,4,5), (1,2)(3,7)(4,6)(5,8) ])
gap> ss:=StructureDescription(s);
"(S3 x S3) : C2"
```

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