Global Dynamics of Generalized Second-Order Beverton–Holt Equations of Linear and Quadratic Type

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Abstract. We investigate second-order generalized Beverton–Holt difference equations of the form

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \dots,$$

where f is a function nondecreasing in both arguments, the parameter a is a positive constant, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers in the domain of f. We will discuss several interesting examples of such equations and present some general theory. In particular, we will investigate the local and global dynamics in the event f is a certain type of linear or quadratic polynomial, and we explore the existence problem of period-two solutions.

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1 Introduction and Preliminaries

Consider the following second-order difference equation:

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \dots. \tag{1}$$

Here f is a continuous function nondecreasing in both arguments, the parameter a is a positive real number, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers in the domain of f. Equation (1) is a generalization of the first-order Beverton–Holt equation

$$x_{n+1} = \frac{ax_n}{1 + x_n}, \quad n = 0, 1, \dots, \tag{2}$$

where $a > 0$ and $x_0 \geq 0$. The global dynamics of Equation (2) may be summarized as follows, see [9, 15]:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a \leq 1 \\ a-1 & \text{if } a > 1 \text{ and } x_0 > 0. \end{cases} \tag{3}$$

Many variations of Equation (2) have been studied. German biochemist Leonor Michaelis and Canadian physician Maud Menten used the model in their study of enzyme kinetics in 1913; see [20]. Additionally, Jacques Monod, a French biochemist, happened upon the model empirically in his study of microorganism growth around 1942; see [20]. It was not until 1957 that fisheries scientists Ray Beverton and Sidney Holt used the model in their study of population dynamics, see [1, 9]. The so-called Monod differential equation [20] is given by

$$\frac{1}{N} \cdot \frac{dN}{dt} = \frac{rS}{a + S}, \tag{4}$$

where $N(t)$ is the concentration of bacteria at time t, $\frac{dN}{dt}$ is the growth rate of the bacteria, $S(t)$ is the concentration of the nutrient, r is the maximum growth rate of the bacteria, and a is a half-saturation

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constant (when $S = a$, the right-hand side of Equation (4) equals $r/2$). Based on experimental data, the following system of two differential equations for the nutrient S and bacteria N, as presented in [20], is given by

$$\frac{dS}{dt} = -\frac{1}{\gamma}N\frac{rS}{a+S}, \quad \frac{dN}{dt} = N\frac{rS}{a+S}, \tag{5}$$

where the constant γ is called the growth yield. Both Equation (4) and System (5) contain the function $f(x) = rx/(a+x)$ known as the Monod function, Michaelis-Menten function, Beverton–Holt function, or Holling function of the first kind; see [1, 5, 9, 11].

One possible two-generation population model based on Equation (2),

$$x_{n+1} = \frac{a_1 x_n}{1+x_n} + \frac{a_2 x_{n-1}}{1+x_{n-1}}, \quad n = 0, 1, \ldots, \tag{6}$$

where $a_i > 0$ for $i = 1, 2$ and $x_{-1}, x_0 \geq 0$, was considered in [18]. The global dynamics of Equation (6) may be summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a_1 + a_2 \leq 1 \\ a_1 + a_2 - 1 & \text{if } a_1 + a_2 > 1 \text{ and } x_0 + x_{-1} > 0. \end{cases}$$

This result was extended in [5] to the case of a k-generation population model based on Equation (2) of the form

$$x_{n+1} = \sum_{i=0}^{k-1} \frac{a_i x_{n-i}}{1+x_{n-i}}, \quad n = 0, 1, \ldots, \tag{7}$$

where $a_i \geq 0$ for $i = 0, 1, \ldots, k-1$, $\sum\limits_{i=0}^{k-1} a_i > 0$, and $x_{1-k}, \ldots, x_0 \geq 0$. It was shown that the global dynamics of Equation (7) may be given precisely by (3), where $a = \sum\limits_{i=0}^{k-1} a_i$ and we consider all initial conditions positive.

The simplest model of Beverton–Holt type which exhibits two coexisting attractors and the Allee effect is the sigmoid Beverton–Holt (or second-type Holling) difference equation

$$x_{n+1} = \frac{a x_n^2}{1+x_n^2}, \quad n = 0, 1, \ldots, \tag{8}$$

where $a > 0$ and $x_0 \geq 0$. The dynamics of Equation (8) may be concisely summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a < 2 \text{ or } (a \geq 2 \text{ and } x_0 < \overline{x}_-) \\ \overline{x}_- & \text{if } a \geq 2 \text{ and } x_0 = \overline{x}_- \\ \overline{x}_+ & \text{if } a \geq 2 \text{ and } x_0 > \overline{x}_-, \end{cases} \tag{9}$$

where $\overline{x}_-$ and $\overline{x}_+$ are the two positive equilibria when $a \geq 2$; see [1, 5]. One possible two-generation population model based on Equation (8),

$$x_{n+1} = \frac{a_1 x_n^2}{1+x_n^2} + \frac{a_2 x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{10}$$

where $a_i > 0$ for $i = 1, 2$ and $x_{-1}, x_0 \geq 0$, was considered in [4]. However, the summary of the global dynamics of Equation (10) is not an immediate extension of the global dynamics of Equation (8) as given in

(9); see [4]. Equation (10) can have up to three equilibrium solutions and up to three period-two solutions. In the case when Equation (10) has three equilibrium solutions and three period-two solutions, the zero equilibrium, the larger positive equilibrium, and one period-two solution are attractors with substantial basins of attraction, which together with the remaining equilibrium and the global stable manifolds of the saddle-point period-two solutions exhaust the first quadrant of initial conditions. This behavior happens when the coefficient a_2 is in some sense dominant to a_1; see [4]. Such behavior is typical for other models in population dynamics such as

$$x_{n+1} = \frac{a_1 x_n}{1+x_n} + \frac{a_2 x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \ldots$$

and

$$x_{n+1} = a_1 x_n + \frac{a_2 x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \ldots,$$

which were also investigated in [4]. In the case of a k-generation population model based on the sigmoid Beverton–Holt difference equation with $k > 2$, one can expect to have attractive period-k solutions as well as chaos.

The first model of the form given in Equation (1), where f is a linear function in both variables (that is, $f(u, v) = cu + dv$ for $c, d, u, v \geq 0$) was considered in [19] to describe the global dynamics in part of the parametric space. Here we will extend the results from [19] to the whole parametric space. In this paper we will then restrict ourselves to the case when $f(u, v)$ is a quadratic polynomial, which will give similar global dynamics to that presented for Equation (10). The corresponding dynamic scenarios will be essentially the same for any polynomial function of the type $f(u, v) = cu^k + du^m$ where $c, d \geq 0$ and m, k are positive integers. Higher values of m and k may only create additional equilibria and period-two solutions but should replicate the global dynamics seen in the quadratic case presented in this paper. The global dynamics of some higher-order transcendental-type generalized Beverton-Holt equation was considered in [3].

Let the function $F : [0, \infty)^2 \to [0, a)$ be defined as follows:

$$F(u, v) = \frac{af(u, v)}{1 + f(u, v)}. \tag{11}$$

Then Equation (1) becomes $x_{n+1} = F(x_n, x_{n-1})$ for all $n = 0, 1, \ldots,$ where $F(u, v)$ is nondecreasing in both of its arguments.

The following theorem from [2] immediately applies to Equation (1).

Theorem 1 *Let I be a set of real numbers and $F : I \times I \to I$ be a function which is nondecreasing in the first variable and nondecreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = F(x_n, x_{n-1}), \quad x_{-1}, x_0 \in I, \quad n = 0, 1, \ldots, \tag{12}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of Equation (12) converges to either an equilibrium, a period-two solution, or to a singular point on the boundary. It should be noticed that Theorem 1 is specific for second-order difference equations and does not extend to difference equations of order higher than two. Furthermore, the powerful theory of monotone maps in the plane [16, 17] can be applied to Equation (1) to determine the boundaries of the basins of attraction of the equilibrium solutions and period-two solutions. Finally, when $f(u,v)$ is a polynomial function, all computation needed to determine the local stability of all equilibrium solutions and period-two solutions is reduced to the theory of counting the number of zeros of polynomials in a given interval, as given in [12]. This theory will give more precise results than the global attractivity and global asymptotic stability results in [7, 8]. However, in the case of difference equations of the form

$$x_{n+1} = \frac{ag(x_n, x_{n-1}, \ldots, x_{n+1-k})}{1 + g(x_n, x_{n-1}, \ldots, x_{n+1-k})}, \quad n = 0, 1, \ldots, \quad k \geq 1,$$

where $a > 0$ and g is nondecreasing in all its arguments, Theorem 1 does not apply for $k > 2$, but the results from [7, 8, 13] can give global dynamics in some regions of the parametric space.

The following theorem from [10] is often useful in determining the global attractivity of a unique positive equilibrium.

Theorem 2 *Let $I \subseteq [0, \infty)$ be some open interval and assume that $F \in C[I \times I, (0, \infty)]$ satisfies the following conditions:*
(i) $F(x, y)$ is nondecreasing in each of its arguments;
(ii) Equation (12) has a unique positive equilibrium point $\overline{x} \in I$ and the function $F(x, x)$ satisfies the ***negative feedback condition****:*

$$(x - \overline{x})(F(x, x) - x) < 0 \text{ for every } x \in I \backslash \{\overline{x}\}.$$

Then every positive solution of Equation (12) with initial conditions in I converges to $\overline{x}$.

The following result from [4] will be used to describe the global dynamics of Equation (1).

Theorem 3 *Assume that difference equation (12) has three equilibrium points $U_1 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$ where the equilibrium points $\bar{x}_0$ and $\bar{x}_{NE}$ are locally asymptotically stable. Further, assume that there exists a minimal period-two solution $\{\Phi_1, \Psi_1\}$ which is a saddle point such that $(\Phi_1, \Psi_1) \in int(Q_2(E_{SW}))$. In this case there exist four continuous curves $\mathcal{W}^s(\Phi_1, \Psi_1)$, $\mathcal{W}^s(\Psi_1, \Phi_1), \mathcal{W}^u(\Phi_1, \Psi_1), \mathcal{W}^u(\Psi_1, \Phi_1)$, where $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1)$ are passing through the point E_{SW}, and are graphs of decreasing functions. The curves $\mathcal{W}^u(\Phi_1, \Psi_1), \mathcal{W}^u(\Psi_1, \Phi_1)$ are the graphs of increasing functions and are starting at E_0. Every solution which starts below $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$ in the North-east ordering converges to E_0 and every solution which starts above $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$ in the North-east ordering converges to E_{NE}, i.e. $\mathcal{W}^s(\Phi_1, \Psi_1) = \mathcal{C}_1^+ = \mathcal{C}_2^+$ and $\mathcal{W}^s(\Psi_1, \Phi_1) = \mathcal{C}_1^- = \mathcal{C}_2^-$.*

This paper is organized as follows. The next section deals with the local stability of equilibrium solutions and period-two solutions of the general second-order difference equation (12), where $F(u,v)$ is nondecreasing in both of its arguments. In view of the results for monotone maps in [16, 17] and their applications to second-order difference equations in [4, 5], the local dynamics of the equilibrium solutions and period-two solutions will determine the global dynamics in hyperbolic cases and some nonhyperbolic cases as well. The third section will provide some examples of global dynamic scenarios of Equation (1) when the function $f(u,v)$ is either linear in both variables or linear in one variable and quadratic in the other variable. The obtained results will be interesting from a modeling point of view as they show that the appearance of period-two solutions with substantial basins of attraction (sets which contain open subsets) is controlled by the coefficient of the x_{n-1} term that is affected by the size of the grandparents' population. The same phenomenon was observed in the case of Equation (10).

2 Local Stability

In this section we provide general conditions to determine the local stability of equilibrium solutions and period-two solutions.

It is clear that $x_n \leq a$ for all $n \geq 1$. In light of Theorem 1, since all solutions are bounded, if there are no singular points on the boundary of the domain of F, it immediately follows that all solutions to Equation (1) converge to an equilibrium or a period-two solution.

An equilibrium $\overline{x}$ of Equation (1) satisfies

$$\overline{x}(1 + f(\overline{x}, \overline{x})) = af(\overline{x}, \overline{x}). \tag{13}$$

Clearly $\overline{x}_0 = 0$ is an equilibrium point if and only if $(0, 0)$ is in the domain of f and $f(0, 0) = 0$.

The linearized equation of Equation (1) about an equilibrium $\overline{x}$ is

$$z_{n+1} = F_u(\overline{x}, \overline{x})z_n + F_v(\overline{x}, \overline{x})z_{n-1}, \quad n = 0, 1, \ldots.$$

Since f is a nondecreasing function, it follows that $F_u(\overline{x}, \overline{x}) \geq 0, F_v(\overline{x}, \overline{x}) \geq 0$. Therefore, if

$$\lambda(\overline{x}) = F_u(\overline{x}, \overline{x}) + F_v(\overline{x}, \overline{x}) = \frac{a(f_u(\overline{x}, \overline{x}) + f_v(\overline{x}, \overline{x}))}{(1 + f(\overline{x}, \overline{x}))^2}, \tag{14}$$

then in view of Corollary 2 of [13] we may conclude that

$$\overline{x} \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } \lambda(\overline{x}) < 1 \\ \text{nonhyperbolic} & \text{if } \lambda(\overline{x}) = 1 \\ \text{unstable} & \text{if } \lambda(\overline{x}) > 1. \end{cases}$$

Further, Theorem 2.13 of [15] implies that if $\overline{x}$ is unstable, then

$$\overline{x} \text{ is} \begin{cases} \text{a repeller} & \text{if } \delta(\overline{x}) > 1 \\ \text{nonhyperbolic} & \text{if } \delta(\overline{x}) = 1 \\ \text{a saddle point} & \text{if } \delta(\overline{x}) < 1, \end{cases}$$

where

$$\delta(\overline{x}) = F_v(\overline{x}, \overline{x}) - F_u(\overline{x}, \overline{x}) = \frac{a(f_v(\overline{x}, \overline{x}) - f_u(\overline{x}, \overline{x}))}{(1 + f(\overline{x}, \overline{x}))^2}. \tag{15}$$

Let (ϕ, ψ) be a period-two solution of Equation (1). The Jacobian matrix of the corresponding map $T = G^2$, where $G(u, v) = (v, F(v, u))$ and F is given by Equation (11), is given in Theorem 12 of [6]. The linearized equation evaluated at (ϕ, ψ) is

$$\lambda^2 - TrJ_T(\phi, \psi)\lambda + DetJ_T(\phi, \psi) = 0,$$

where

$$TrJ_T(\phi, \psi) = D_2F(\psi, \phi) + D_1F(F(\psi, \phi), \psi) \cdot D_1F(\psi, \phi) + D_2F(F(\psi, \phi), \psi)$$

and

$$DetJ_T(\phi, \psi) = D_2F(F(\psi, \phi), \psi) \cdot D_2F(\psi, \phi).$$

3 Examples

In this section we present four examples of different forms of Equation (1) where the transition function $f(u,v)$ is linear or quadratic polynomial in its variables which effects the global dynamics.

3.1 Linear-Linear: $f(u,v) = cu + dv$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1})}{1 + cx_n + dx_{n-1}}, \quad n = 0, 1, \dots, \tag{16}$$

where $c \geq 0$ and $d > 0$. If $d = 0$, then Equation (16) becomes Equation (2) after a reduction of parameters. By Equation (13) we know that $\overline{x}_0 = 0$ is always a fixed point and $\overline{x}_+ = \frac{a(c+d)-1}{c+d}$ is a unique positive fixed point for $a(c+d) > 1$.

Since $\lambda(\overline{x}_0) = a(c+d)$, we have that

$$\overline{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } a(c+d) < 1 \\ \text{nonhyperbolic} & \text{if } a(c+d) = 1 \\ \text{unstable} & \text{if } a(c+d) > 1. \end{cases}$$

Further, notice that

$$\lambda(\overline{x}_+) = \frac{a(c+d)}{\left(1 + \left(\frac{a(c+d)-1}{c+d}\right) \cdot (c+d)\right)^2} = \frac{1}{a(c+d)} < 1$$

for all values of parameters for which $\overline{x}_+$ exists. Therefore

$$\overline{x}_+ = \frac{a(c+d)-1}{c+d} \text{ is always locally asymptotically stable.}$$

Note that there is an exchange in stability from $\overline{x}_0$ to $\overline{x}_+$ as the parametric value $a(c+d)$ passes through 1.

We next search for period-two solutions. Suppose there exists such a solution $\{\psi, \phi, \psi, \phi, \dots\}$ with $\phi \neq \psi$. Then $\{\psi, \phi\}$ satisfies the following system:

$$\begin{cases} \psi &= \dfrac{af(\phi,\psi)}{1+f(\phi,\psi)} &= \dfrac{a(c\phi + d\psi)}{1 + c\phi + d\psi} \\ \phi &= \dfrac{af(\psi,\phi)}{1+f(\psi,\phi)} &= \dfrac{a(c\psi + d\phi)}{1 + c\psi + d\phi} \end{cases}. \tag{17}$$

Notice that

$$\psi - \phi = \frac{a(d-c)(\psi - \phi)}{(1 + c\phi + d\psi)(1 + c\psi + d\phi)},$$

whence we deduce that $d > c$ and $(1 + c\phi + d\psi)(1 + d\psi + d\phi) = a(d-c)$. Now

$$\psi + \phi = \frac{a\left((c+d)(\psi + \phi) + 2(c\phi + d\psi)(c\psi + d\phi)\right)}{a(d-c)},$$

or equivalently,

$$2c(\psi + \phi) + 2(c\phi + d\psi)(c\psi + d\phi) = 0.$$

Since $\psi + \phi > 0$, it must be the case that $c = 0$, and then $2d^2\psi\phi = 0$ so that one of either ϕ or ψ equals zero. Without loss of generality assume $\phi = 0$. But then $\psi = \frac{ad\psi}{1+d\psi}$, and hence $\psi = \frac{ad-1}{d} = \overline{x}_+$. Thus the only non-equilibrium solution of System (17) is the period-two solution $\{\overline{x}_+, 0, \overline{x}_+, 0, \dots\}$, which exists for $ad > 1$ and $c = 0$. Now we formulate our main result about the global dynamics of Equation (16).

Theorem 4 *Consider Equation* (16).

(a) *If $a(c+d) \leq 1$, then $\overline{x}_0 = 0$ is a global attractor of all solutions.*

(b) *If $c = 0$ and $ad > 1$, then there exists a period-two solution $\left\{\frac{ad-1}{d}, 0, \frac{ad-1}{d}, 0, \dots\right\}$. $\overline{x}_+$ is a global attractor of all solutions with positive initial conditions. Any solution with exactly one initial condition equal to zero will converge to the period-two solution.*

(c) *If $c > 0$ and $a(c+d) > 1$, $\overline{x}_+$ is a global attractor of all nonzero solutions.*

Proof.

(a) If $a(c+d) \leq 1$, then $\overline{x}_0 = 0$ is the only equilibrium, and no period-two solutions exist. By Theorem 1 all solutions must converge to zero.

(b) Suppose $c = 0$ and $ad > 1$, and consider $I = (0, \infty)$. Notice that

$$F(x, x) = \frac{adx}{1+dx} \gtrless x \iff \overline{x}_+ \gtrless x,$$

and therefore by Theorem 2 we have that all solutions with initial conditions in I converge to $\overline{x}_+$. Now suppose one initial condition is zero, so without loss of generality assume $x_{-1} = 0$ and $x_0 > 0$. Then $x_1 = 0$ and

$$x_2 = \frac{adx_0}{1+dx_0} \gtrless x_0 \iff \frac{ad-1}{d} = \overline{x}_+ \gtrless x_0.$$

Further, one can show $x_2 \lesseqgtr \overline{x}_+ \iff x_0 \lesseqgtr \overline{x}_+$. By induction, $\lim_{k\to\infty} x_{2k} = \overline{x}_+$ and $x_{2k-1} = 0$ for all $k = 0, 1, \dots$. Thus all solutions with exactly one initial condition equal to zero will converge to the period-two solution $\{\overline{x}_+, 0, \overline{x}_+, 0, \dots\}$.

(c) When $c > 0$ and $a(c+d) > 1$, $\overline{x}_+$ is locally asymptotically stable while $\overline{x}_0$ is unstable. As in the proof of (b) we can employ Theorem 2 to show that all solutions with positive initial conditions must converge to $\overline{x}_+$. Since $c > 0$ and $d > 0$, if $x_0 + x_{-1} > 0$, then $x_1 = F(x_0, x_{-1}) > 0$ (and also $x_2 > 0$), so the solution eventually has consecutive positive terms and must converge to $\overline{x}_+$.

□

3.2 Translated Linear-Linear: $f(u, v) = cu + dv + k$

We briefly consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1} + k)}{1 + cx_n + dx_{n-1} + k}, \quad n = 0, 1, \dots, \tag{18}$$

where $c \geq 0$, $d \geq 0$, $c+d>0$, and $k>0$. We notice in this example $f(0,0)=k>0$, so the origin cannot be an equilibrium. More specifically, an equilibrium of Equation (18) must satisfy

$$(c+d)\overline{x}^2 + (k+1-a(c+d))\,\overline{x} - ak = 0$$

Since $c+d>0$ and $ak>0$ by Descartes' Rule of Signs it must be the case that there exists a unique positive equilibrium $\overline{x}_+$.

Theorem 5 *Consider Equation* (18) *such that $c+d>0$ and $k>0$. The unique positive equilibrium $\overline{x}_+$ is a global attractor.*

Proof. The result follows from a straightforward application of Theorem 1.4.8 of [14]. □

3.3 Quadratic-Linear: $f(u,v) = cu^2 + dv$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n^2 + dx_{n-1})}{1 + cx_n^2 + dx_{n-1}}, \quad n = 0, 1, \ldots. \tag{19}$$

Remark 1 For the analysis that follows, we will consider Equation (19) with $c>0$ and $d>0$. Notice that when $c=0$ Equation (19) is a special case of Equation (16), and the global dynamics for this case is discussed in Theorem 4. When $d=0$ Equation (19) is essentially Equation (8), the dynamics of which may be seen in (9).

An equilibrium solution of Equation (19) satisfies

$$c\overline{x}^3 + d\overline{x}^2 + \overline{x} = ac\overline{x}^2 + ad\overline{x}$$

so that all nonzero equilibria satisfy

$$c\overline{x}^2 + (d-ac)\overline{x} + (1-ad) = 0, \tag{20}$$

whence we easily deduce the possible solutions

$$\overline{x}_\pm = \frac{ac - d \pm \sqrt{(d-ac)^2 + 4c(ad-1)}}{2c},$$

which are real if and only if $R = (d-ac)^2 + 4c(ad-1) \geq 0$.
Notice that

$$R \geq 0 \iff d^2 - 2acd + a^2c^2 + 4acd - 4c \geq 0 \iff (ac+d)^2 \geq 4c. \tag{21}$$

Here we have that

$$\lambda(\overline{x}) = \frac{a(2c\overline{x} + d)}{(1 + c\overline{x}^2 + d\overline{x})^2}.$$

Theorem 6 *Equation* (19) *always has the zero equilibrium $\overline{x}_0 = 0$, and*

$$\overline{x}_0 \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } ad < 1 \\ \text{nonhyperbolic} & \text{if } ad = 1 \\ \text{a repeller} & \text{if } ad > 1. \end{cases}$$

Proof. The proof follows from the fact that $\lambda(\overline{x}_0) = \delta(\overline{x}_0) = ad$. □

The next result gives the local stability of positive equilibrium solutions.

Theorem 7 *Assume $c > 0$ and $d > 0$.*

(1) Suppose either
(a) $d \geq ac$ and $1 \geq ad$, or
(b) $d < ac$, $1 > ad$, and $R < 0$.
Then Equation (19) has no positive equilibria.

(2) Suppose either
(a) $1 < ad$, or
(b) $d < ac$ and $1 = ad$.
Then Equation (19) has the positive equilibrium solution $\overline{x}_+$, and it is locally asymptotically stable.

(3) Suppose $d < ac$, $1 > ad$, and $R = 0$. Then Equation (19) has the positive equilibrium solution $\overline{x}_\pm$, and it is nonhyperbolic of stable type (that is one characteristic value is $\lambda_1 = \pm 1$ and the other $|\lambda_1| < 1$).

(4) Suppose $d < ac$, $1 > ad$, and $R > 0$. Then Equation (19) has two positive equilibria, $\overline{x}_+$ and $\overline{x}_-$; $\overline{x}_+$ is locally asymptotically stable, and $\overline{x}_-$ is a saddle point.

Proof. The existence of positive equilibria follows from Descartes' Rule of Signs. Using Equation (14), notice that

$$\lambda(\overline{x}) = \frac{a(2c\overline{x}+d)}{(1+c\overline{x}^2+d\overline{x})^2} = \frac{a(2c\overline{x}+d)}{(a(c\overline{x}+d))^2} = \frac{2c\overline{x}+d}{a(c\overline{x}+d)^2} = \frac{1}{a(c\overline{x}+d)} + \frac{c\overline{x}}{a(c\overline{x}+d)^2}.$$

Further, for the parametric values for which $\overline{x}_+$ exists,

$$\begin{aligned}
\lambda(\overline{x}_+) \leq 1 &\iff \frac{c\overline{x}_+}{a(c\overline{x}_+ + d)^2} \leq \frac{a(c\overline{x}_+ + d) - 1}{a(c\overline{x}_+ + d)} \\
&\iff c\overline{x}_+ \leq (c\overline{x}_+ + d)\left(a(c\overline{x}_+ + d) - 1\right) = (c\overline{x}_+ + d)(c\overline{x}_+^2 + d\overline{x}_+) \\
&\iff c \leq (c\overline{x}_+ + d)^2 \\
&\iff 4c \leq (2c\overline{x}_+ + 2d)^2 = (ac + d + \sqrt{R})^2,
\end{aligned}$$

which is true by (21). Thus if $R > 0$, $\overline{x}_+$ is locally asymptotically stable, and if $R = 0$, $\overline{x}_\pm$ is nonhyperbolic. In the latter case the characteristic equation of the linearization of Equation (19) about $\overline{x}_\pm$, $y^2 = F_u(\overline{x}_\pm, \overline{x}_\pm)y + F_v(\overline{x}_\pm, \overline{x}_\pm)$, reduces to $acy^2 - (ac-d)y - d = 0$, which has characteristic values $y_1 = 1$ and $y_2 = -\frac{d}{ac}$, where $-1 < y_2 < 0$ since $ac > d$. Thus in this case $\overline{x}_\pm$ is nonhyperbolic of stable type. When $\overline{x}_-$ exists, then

$$\begin{aligned}
\lambda(\overline{x}_-) > 1 &\iff 4c > (ac + d - \sqrt{R})^2 \\
&\iff 4c + (ac+d)\sqrt{R} > (ac+d)^2 \\
&\iff (ac+d)\sqrt{R} > (ac+d)^2 - 4c = R \\
&\iff (ac+d)^2 > R = (ac+d)^2 - 4c,
\end{aligned}$$

which is true since $c > 0$. To show more specifically that $\overline{x}_-$ is a saddle point when $R > 0$, we must show that $\delta(\overline{x}_-) < 1$, where δ is defined by Equation (15). Notice

$$\delta(\overline{x}_-) = \frac{a(d - 2c\overline{x}_-)}{(1 + c\overline{x}_-^2 + d\overline{x}_-)^2} = \frac{a(d-2c\overline{x}_-)}{(a(c\overline{x}_- + d))^2} = \frac{4(d - 2c\overline{x}_-)}{a(2c\overline{x}_- + 2d)^2} = \frac{4\left(2d - ac + \sqrt{R}\right)}{a(ac + d - \sqrt{R})^2},$$

and so we have that

$$\begin{aligned}\delta(\overline{x}_-) < 1 &\iff 4\left(2d - ac + \sqrt{R}\right) < a\left(ac + d - \sqrt{R}\right)^2\\ &\iff (2 + a(ac+d))\sqrt{R} < a(ac+d)^2 - 4d.\end{aligned}$$

The right-hand side of the latter inequality is positive since $a(ac+d)^2 - 4d > 4ac - 4d = 4(ac-d) > 0$ by assumption. But then

$$\begin{aligned}\delta(\overline{x}_-) < 1 &\iff (2 + a(ac+d))^2\left((ac+d)^2 - 4c\right) < \left(a(ac+d)^2 - 4d\right)^2\\ &\iff 3a^3c^2d + 6a^2cd^2 + 3ad^3 - 3a^2c^2 - 2acd - 3d^2 - 4c < 0\\ &\iff (ad-1)\left(3d^2 + 3a^2c^2 + 2c(3ad+2)\right) < 0,\end{aligned}$$

which is automatically true since the latter factor is strictly positive and $ad < 1$. Thus indeed $\overline{x}_-$ is a saddle point when it exists for $R > 0$. □

Theorem 8 *There exist no minimal period-two solutions to Equation* (19) *if* $c, d > 0$.

Proof. Suppose there exist $\phi, \psi > 0$ with $\phi \neq \psi$ such that

$$\begin{cases} \psi &= \dfrac{af(\phi,\psi)}{1+f(\phi,\psi)} &= \dfrac{a(c\phi^2 + d\psi)}{1 + c\phi^2 + d\psi} \\ \\ \phi &= \dfrac{af(\psi,\phi)}{1+f(\psi,\phi)} &= \dfrac{a(c\psi^2 + d\phi)}{1 + c\psi^2 + d\phi} \end{cases}. \tag{22}$$

From System (22) we notice that

$$\psi - \phi = \frac{a(\psi - \phi)(d - c(\psi + \phi))}{(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi)},$$

whence it immediately follows that $(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi) = a(d - c(\psi + \phi))$. But then

$$\psi + \phi = \frac{2(c\phi^2 + d\psi)(c\psi^2 + d\phi) + c(\psi^2 + \phi^2) + d(\psi + \phi)}{d - c(\psi + \phi)}.$$

Thus we have that necessarily

$$2\phi\psi = \frac{2a^2(c\phi^2 + d\psi)(c\psi^2 + d\phi)}{a(d - c(\psi+\phi))} = a\left((\psi + \phi) - \frac{c(\psi^2 + \phi^2) + d(\psi + \phi)}{d - c(\psi + \phi)}\right) > 0$$

since both $\psi, \phi > 0$. But this implies that

$$\begin{aligned}&(\psi + \phi)(d - c(\psi + \phi)) > c(\psi^2 + \phi^2) + d(\psi + \phi)\\ \iff& d(\psi + \phi) - c(\psi + \phi)^2 > c(\psi^2 + \phi^2) + d(\psi + \phi)\\ \iff& 0 > c(\psi^2 + \phi^2) + c(\psi + \phi)^2,\end{aligned}$$

a clear contradiction since $c > 0$.

Now suppose there exists a period-two solution $\{\phi, \psi, \phi, \psi, \ldots\}$ with $\phi \neq \psi$ but $\phi\psi = 0$. Suppose without loss of generality that $\phi = 0$. Now

$$\begin{cases} \psi &= \dfrac{af(0,\psi)}{1+f(0,\psi)} &= \dfrac{ad\psi}{1+d\psi} \\ 0 &= \dfrac{af(\psi,0)}{1+f(\psi,0)} &= \dfrac{ac\psi^2}{1+c\psi^2} \end{cases},$$

which immediately leads to the contradiction $\psi = \phi = 0$ for $c > 0$. Thus Equation (19) has no minimal period-two solutions. □

The next result describes the global dynamics of Equation (19).

Theorem 9 *Consider Equation* (19) *under the condition* $c > 0$ *and* $d > 0$.

(1) Suppose either
(a) $d \geq ac$ *and* $1 \geq ad$, *or*
(b) $d < ac$, $1 > ad$, *and* $R < 0$.
Then $\overline{x}_0$ *is a global attractor of all solutions.*

(2) Suppose either
(a) $1 < ad$, *or*
(b) $d < ac$ *and* $1 = ad$.
Then $\overline{x}_+$ *is a global attractor of all nonzero solutions.*

(3) Suppose $d < ac$, $1 > ad$, *and* $R = 0$. *Then Equation* (19) *has the equilibria* $\overline{x}_0 = 0$, *which is locally asymptotically stable, and* $\overline{x}_\pm$, *which is nonhyperbolic of stable type. There exists a continuous curve* $\mathcal{C}$ *passing through* $E = (\overline{x}_\pm, \overline{x}_\pm)$ *such that* $\mathcal{C}$ *is the graph of a decreasing function. The set of initial conditions* $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ *is the union of two disjoint basins of attraction, namely* $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E)$, *where* $E_0 = (\overline{x}_0, \overline{x}_0)$,

$$\mathcal{B}(E_0) = \{(x_{-1}, x_0) : (x_{-1}, x_0) \prec_{ne} (x, y) \text{ for some } (x, y) \in \mathcal{C}\}, \text{ and}$$
$$\mathcal{B}(E) = \{(x_{-1}, x_0) : (x, y) \prec_{ne} (x_{-1}, x_0) \text{ for some } (x, y) \in \mathcal{C}\} \cup \mathcal{C}.$$

(4) Suppose $d < ac$, $1 > ad$, *and* $R > 0$. *Then Equation* (19) *has the equilibria* $\overline{x}_0 = 0$, *which is locally asymptotically stable,* $\overline{x}_-$, *which is a saddle point, and* $\overline{x}_+$, *which is locally asymptotically stable. There exist two continuous curves* $\mathcal{W}^s(E_-)$ *and* $\mathcal{W}^u(E_-)$, *both passing through* $E_- = (\overline{x}_-, \overline{x}_-)$, *such that* $\mathcal{W}^s(E_-)$ *is the graph of a decreasing function and* $\mathcal{W}^u(E_-)$ *is the graph of an increasing function. The set of initial conditions* $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ *is the union of three disjoint basins of attraction, namely* $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_-) \cup \mathcal{B}(E_+)$, *where* $E_0 = (\overline{x}_0, \overline{x}_0)$, $E_+ = (\overline{x}_+, \overline{x}_+)$, $\mathcal{B}(E_-) = \mathcal{W}^s(E_-)$,

$$\mathcal{B}(E_0) = \{(x_{-1}, x_0) : (x_{-1}, x_0) \prec_{ne} (x, y) \text{ for some } (x, y) \in \mathcal{W}^s(E_-)\}, \text{ and}$$
$$\mathcal{B}(E_+) = \{(x_{-1}, x_0) : (x, y) \prec_{ne} (x_{-1}, x_0) \text{ for some } (x, y) \in \mathcal{W}^s(E_-)\}$$

Proof. (1) The proof in this case follows from Theorems 1, 7, and 8 along with the fact that $\overline{x}_0 = 0$ is the sole equilibrium of Equation (19).

(2) The proof used to show that all solutions with positive initial conditions converge to $\overline{x}_+$ follows from an application of Theorem 2 (as used above in the proof of Theorem 4). Notice that $x_1 = F(x_0, x_{-1}) > 0$ if either $x_0 > 0$ or $x_{-1} > 0$ (and similar for x_2), so $I = (0, \infty)$ is an attracting and invariant interval. Thus all nonzero solutions must converge to $\overline{x}_+$.

(3) The proof follows from an application of Theorems 1-4 of [17] applied to the *cooperative* second iterate of the map corresponding to Equation (19). The proof is completely analogous to the proof of Theorem 5 in [4], so we omit the details.

(4) The proof follows from an immediate application of Theorem 5 in [4]. □

3.4 Linear-Quadratic: $f(u, v) = cu + dv^2$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1}^2)}{1 + cx_n + dx_{n-1}^2}, \quad n = 0, 1, \ldots. \tag{23}$$

Remark 2 For the analysis that follows, we will consider Equation (23) with $c > 0$ and $d > 0$. Notice that when $d = 0$ Equation (23) reduces to Equation (2), a special case of Equation (16). When $c = 0$ Equation (23) is essentially Equation (8) with delay.

An equilibrium of (23) satisfies

$$d\overline{x}^3 + c\overline{x}^2 + \overline{x} = ac\overline{x} + ad\overline{x}^2$$

so that all nonzero equilibria satisfy

$$d\overline{x}^2 + (c - ad)\overline{x} + (1 - ac) = 0, \tag{24}$$

whence we easily deduce the possible solutions

$$\overline{x}_\pm = \frac{ad - c \pm \sqrt{(c - ad)^2 + 4d(ac - 1)}}{2d},$$

which are real if and only if $R = (c - ad)^2 + 4d(ac - 1) \geq 0$.
Notice that

$$R \geq 0 \iff c^2 - 2acd + a^2d^2 + 4acd - 4d \geq 0 \iff (ad + c)^2 \geq 4d. \tag{25}$$

Here we have that

$$\lambda(\overline{x}) = \frac{a(c + 2d\overline{x})}{(1 + c\overline{x} + d\overline{x}^2)^2}.$$

Theorem 10 *Equation* (23) *always has the zero equilibrium* $\overline{x}_0 = 0$, *and*

$$\overline{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } ac < 1 \\ \text{nonhyperbolic} & \text{if } ac = 1 \\ \text{unstable} & \text{if } ac > 1. \end{cases}$$

Proof. The proof follows from the fact that $\lambda(\overline{x}_0) = ac$. □

Theorem 11 *Consider Equation* (23) *and assume* $c > 0$ *and* $d > 0$.

(1) Suppose either
(a) $c \geq ad$ and $1 \geq ac$, or
(b) $c < ad$, $1 > ac$, and $R < 0$.
Then Equation (23) has no positive equilibria.

(2) Suppose either
(a) $1 < ac$, or
(b) $c < ad$ and $1 = ac$.
Then Equation (23) has the positive equilibrium solution $\overline{x}_+$, and it is locally asymptotically stable.

(3) Suppose $c < ad$, $1 > ac$, and $R = 0$. Then Equation (23) has the positive equilibrium solution $\overline{x}_\pm$, and it is nonhyperbolic of stable type.

(4) Suppose $c < ad$, $1 > ac$, and $R > 0$. Then Equation (23) has two positive equilibria, $\overline{x}_+$ and $\overline{x}_-$; $\overline{x}_+$ is locally asymptotically stable, and $\overline{x}_-$ is unstable.
Let $K = a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d$.
(i) If $K < 0$, then $\overline{x}_-$ is a saddle point.
(ii) If $K > 0$, then $\overline{x}_-$ is a repeller.
(iii) If $K = 0$, then $\overline{x}_-$ is nonhyperbolic of unstable type (that is one characteristic value is $\lambda_1 = \pm 1$ and the other $|\lambda_1| > 1$).

Proof. Much of the analysis is similar to the considerations in the proof of Theorem 7. Notice that

$$\lambda(\overline{x}) = \frac{a(c + 2d\overline{x})}{(1 + c\overline{x} + d\overline{x}^2)^2} = \frac{a(c + 2d\overline{x})}{(a(c + d\overline{x}))^2} = \frac{c + 2d\overline{x}}{a(c + d\overline{x})^2} = \frac{1}{a(c + d\overline{x})} + \frac{d\overline{x}}{a(c + d\overline{x})^2}.$$

For the parametric values for which $\overline{x}_+$ exists,

$$\begin{aligned}
\lambda(\overline{x}_+) \leq 1 &\iff \frac{d\overline{x}_+}{a(c + d\overline{x}_+)^2} \leq \frac{a(c + d\overline{x}_+) - 1}{a(c + d\overline{x}_+)} \\
&\iff d\overline{x}_+ \leq (c + d\overline{x}_+)\left(a(c + d\overline{x}_+) - 1\right) = (c + d\overline{x}_+)(c\overline{x}_+ + d\overline{x}_+^2) \\
&\iff d \leq (c + d\overline{x}_+)^2 \\
&\iff 4d \leq (2c + 2d\overline{x}_+)^2 = (ad + c + \sqrt{R})^2,
\end{aligned}$$

which is true by (25). Thus if $R > 0$, $\overline{x}_+$ is locally asymptotically stable, and if $R = 0$, $\overline{x}_\pm$ is nonhyperbolic. In the latter case the characteristic equation of the linearization of Equation (23) about $\overline{x}_\pm$, $y^2 = F_u(\overline{x}_\pm, \overline{x}_\pm)y + F_v(\overline{x}_\pm, \overline{x}_\pm)$, reduces to $ady^2 - cy + c - ad = 0$, which has characteristic values $y_1 = 1$ and $y_2 = \frac{c - ad}{ad}$, where $-1 < y_2 < 0$ since $ad > c$. Thus in this case $\overline{x}_\pm$ is nonhyperbolic of stable type.
When $\overline{x}_-$ exists,

$$\begin{aligned}
\lambda(\overline{x}_-) > 1 &\iff 4d > (ad + c - \sqrt{R})^2 \\
&\iff 4d + (ad + c)\sqrt{R} > (ad + c)^2 \\
&\iff (ad + c)\sqrt{R} > (ad + c)^2 - 4d = R \\
&\iff (ad + c)^2 > R = (ad + c)^2 - 4d
\end{aligned}$$

which is true since $d > 0$. To more specifically classify $\overline{x}_-$, we must calculate $\delta(\overline{x}_-)$. Notice

$$\delta(\overline{x}_-) = \frac{a(2d\overline{x}_- - c)}{(1 + c\overline{x}_- + d\overline{x}_-^2)^2} = \frac{a(2d\overline{x}_- - c)}{(a(c + d\overline{x}_-))^2} = \frac{4(2d\overline{x}_- - c)}{a(2c + 2d\overline{x}_-)^2} = \frac{4\left(ad - 2c - \sqrt{R}\right)}{a(ad + c - \sqrt{R})^2},$$

and so we have that

$$\begin{aligned}\delta(\overline{x}_-) \gtrless 1 &\iff 4\left(ad - 2c - \sqrt{R}\right) \gtrless a\left(ad + c - \sqrt{R}\right)^2 \\ &\iff (a(ad+c)-2)\sqrt{R} \gtrless a(ad+c)^2 - 4ad + 4c = aR + 4c.\end{aligned}$$

Notice that $R > 0$ automatically implies $a(ad + c) > 2$, as

$$0 < (ad+c)^2 - 4d < a^2d^2 + 2acd + a^2d^2 - 4d = 2d\,(a(ad+c)-2)$$

since $c < ad$. Therefore we may square both sides to obtain

$$\begin{aligned}\delta(\overline{x}_-) \gtrless 1 &\iff (a(ad+c)-2)^2 R \gtrless (aR+4c)^2 \\ &\iff R\left(a^2(ad+c)^2 - 4a(ad+c) + 4\right) \gtrless a^2R^2 + 8acR + 16c^2 \\ &\iff R\left(a^2R - 4ac + 4\right) \gtrless a^2R^2 + 8acR + 16c^2 \\ &\iff R(1-3ac) - 4c^2 \gtrless 0 \\ &\iff a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d \gtrless 0.\end{aligned}$$

Thus if

$$K = a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d, \tag{26}$$

$K < 0$ implies $\overline{x}_-$ is a saddle point and $K > 0$ implies it is a repeller. If $K = 0$, $\overline{x}_-$ is nonhyperbolic, and we expect in such case to be nonhyperbolic of unstable type. Indeed one can show that in the event $K = 0$, the characteristic equation of the linearization of Equation (23) about $\overline{x}_-$, $y^2 = F_u(\overline{x}_-, \overline{x}_-)y + F_v(\overline{x}_-, \overline{x}_-)$, has roots $y_1 = -1$ and $y_2 = F_u(\overline{x}_-, \overline{x}_-) + 1 > 1$, which immediately shows the desired result. □

The investigation of the existence of periodic solutions of Equation (23) is an interesting one that involves a thorough analysis of potential parametric cases. This analysis will reveal the potential for the existence of several nonzero periodic solutions. The juxtaposition of Equation (19) with Equation (23) illustrates an interesting phenomenon in which, loosely speaking, the dominance of the delay term x_{n-1} contributes to the possibility of periodic solutions arising.

A minimal period-two solution $\{\phi, \psi, \phi, \psi, \ldots\}$ with $\phi, \psi > 0$ and $\phi \neq \psi$ must satisfy

$$\begin{cases} \psi &= \dfrac{af(\phi,\psi)}{1+f(\phi,\psi)} = \dfrac{a(c\phi + d\psi^2)}{1 + c\phi + d\psi^2} \\ \\ \phi &= \dfrac{af(\psi,\phi)}{1+f(\psi,\phi)} = \dfrac{a(c\psi + d\phi^2)}{1 + c\psi + d\phi^2} \end{cases}. \tag{27}$$

Eliminating either ψ or ϕ from System (27) we obtain

$$\left(d\phi^2 + (c-ad)\phi + (1-ac)\right)h(\phi) = 0, \quad \text{or} \quad \left(d\psi^2 + (c-ad)\psi + (1-ac)\right)h(\psi) = 0,$$

where

$$\begin{aligned}h(x) = &-d^3x^6 + d^2(c+2ad)x^5 - d(c^2 + 2d + 3acd + a^2d^2)x^4 + d(c + 3ac^2 + 2ad + 3a^2cd)x^3 \\ &- (c^2 + ac^3 + d + 2acd + 3a^2c^2d + a^3cd^2)x^2 + ac(1+ac)(2c+ad)x - a^2c^2(1+ac).\end{aligned} \tag{28}$$

Since $dx^2 + (c - ad)x + (1 - ac) \neq 0$ for any x that is not a solution of the equilibrium equation (24), minimal period-two solutions must be the solutions of the equation

$$h(x) = 0. \tag{29}$$

Theorem 12 *Any real solutions of Equation* (29) *are positive numbers for* $c, d > 0$, *and there exist up to three minimal period-two solutions of Equation* (23). *Furthermore, let* K *be as defined in Equation* (26), *and define the following expressions:*

$$\begin{aligned} J &= 4a^5cd^4 - 8a^4c^2d^3 + 12a^3c^3d^2 - 24a^3cd^3 - 8a^2c^4d + 28a^2c^2d^2 - a^2d^3 + 4ac^5 + 4ac^3d \\ &\quad + 32acd^2 + 4c^4 + 8c^2d + 4d^2 \\ \Delta_1 &= 6d^6 \\ \Delta_2 &= d^{10}\left(8a^2d^2 - 16acd - 7c^2 - 24d\right) \\ \Delta_3 &= -2d^{12}\left(8a^5cd^5 + 13a^4c^2d^4 + 10a^3c^3d^3 - 44a^3cd^4 + 4a^2c^4d^2 - 34a^2c^2d^3 - 4a^2d^4 - 19ac^5d \right. \\ &\quad \left. +14ac^3d^2 + 44acd^3 + 6c^6 + 7c^4d + 5c^2d^2 + 16d^3\right) \end{aligned}$$

$$\begin{aligned} \Delta_4 &= c^2d^{13}\left(-16a^9cd^8 - 12a^8c^2d^7 + 24a^7c^3d^6 + 152a^7cd^7 - 68a^6c^4d^5 + 80a^6c^2d^6 + 8a^6d^7 + 48a^5c^5d^4 \right. \\ &\quad -164a^5c^3d^5 - 464a^5cd^6 - 60a^4c^6d^3 + 20a^4c^4d^4 - 180a^4c^2d^5 - 64a^4d^6 + 56a^3c^7d^2 - 332a^3c^5d^3 \\ &\quad +388a^3c^3d^4 + 488a^3cd^5 - 48a^2c^8d + 272a^2c^6d^2 + 255a^2c^4d^3 + 152a^2c^2d^4 + 136a^2d^5 + 24ac^9 \\ &\quad \left. +8ac^7d + 124ac^5d^2 + 180ac^3d^3 - 152acd^4 + 24c^8 + 68c^6d + 32c^4d^2 - 44c^2d^3 - 32d^4\right) \end{aligned}$$

$$\begin{aligned} \Delta_5 &= 2c^4d^{13}J\left(3a^8c^2d^6 + 2a^7cd^6 - 18a^6c^2d^5 - a^6d^6 + 6a^5c^5d^3 + 10a^5c^3d^4 - 8a^5cd^5 - 10a^4c^4d^3 \right. \\ &\quad +44a^4c^2d^4 + 6a^4d^5 + 54a^3c^5d^2 - 25a^3c^3d^3 - 6a^3cd^4 + 3a^2c^8 - 8a^2c^6d + 35a^2c^4d^2 - 39a^2c^2d^3 \\ &\quad \left. -9a^2d^4 + 6ac^7 + 2ac^5d + 4ac^3d^2 + 14acd^3 + 3c^6 + 10c^4d + 11c^2d^2 + 4d^3\right) \end{aligned}$$

$$\Delta_6 = a^2c^6d^{14}(ac+1)KJ^2.$$

(1) If $\Delta_i > 0$ *for all* $2 \le i \le 6$ *then Equation* (29) *has six real roots. Consequently, Equation* (23) *has three minimal period-two solutions.*
(2) If $\Delta_j \le 0$ *for some* $2 \le j \le 5$ *and* $\Delta_i > 0$ *for* $i \ne j$, *then Equation* (29) *has two distinct real roots and two pairs of conjugate imaginary roots. Consequently, Equation* (23) *has one minimal period-two solution.*
(3) If $\Delta_i \le 0$, $\Delta_{i+1} \ge 0$ *(such that at least one of these is strict) for some* $2 \le i \le 4$, *and if* $\Delta_6 < 0$, *then Equation* (29) *has three pairs of conjugate imaginary roots. Consequently, Equation* (23) *has no minimal period-two solutions.*

Proof. The proof of the first statement follows from Descartes' Rule of Signs.
Let disc(h) denote the 12×12 discrimination matrix as defined in [12]:

$$\mathrm{disc}(h) = \begin{bmatrix} a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 \end{bmatrix}.$$

Here a_k equals the coefficient of the degree-k term of h as defined in Equation (28); that is, $a_6 = -d^3$, $a_5 = d^2(c + 2ad)$, $a_4 = -d(c^2 + 2d + 3acd + a^2d^2)$, $a_3 = d(c + 3ac^2 + 2ad + 3a^2cd)$, $a_2 = -(c^2 + ac^3 + d + 2acd + 3a^2c^2d + a^3cd^2)$, $a_1 = ac(1+ac)(2c+ad)$, and $a_0 = -a^2c^2(1+ac)$. Let Δ_k denote the determinant of the submatrix of disc(h) formed by its first $2k$ rows and $2k$ columns for $k = 1, 2, \ldots, 6$. Then the values of Δ_k are listed above, and the veracity of the statements above may now be verified by employing Theorem 1 of [12]. Notice that $\Delta_1 > 0$ for all $d > 0$. □

Remark 3 The parametric conditions discussed above do not exhaust all of the parametric space but cover a substantial region of parameters for which Equation (23) possesses hyperbolic dynamics.

We will use the sufficient conditions provided in Theorems 10, 11, and 12 to obtain some global dynamic scenarios discussed in [4]. We will not investigate the dynamics of Equation (23) when it has one or no positive fixed point since in such cases the dynamics should be similar to the dynamics of Equation (19) discussed in Theorem 9. The following theorem relies on results from [4] and summarizes potential hyperbolic dynamic scenarios for Equation (23) in the event it possesses three fixed points and zero, one, or three pairs of hyperbolic period-two points. In particular, Theorem 3 is applicable to case (ii) of the following result. See also the statement and proof of Theorem 11 in [4].

Theorem 13 *Consider Equation* (23) *and assume* $0 < c < ad, ac < 1$ *such that* $R > 0$.

(i) If $\Delta_i > 0$ *for all* $2 \leq i \leq 6$ *then Equation* (23) *has three equilibria* $\overline{x}_0 < \overline{x}_- < \overline{x}_+$*, where* $\overline{x}_0$ *and* $\overline{x}_+$ *are locally asymptotically stable and* $\overline{x}_-$ *is a repeller, and three minimal period-two solutions* $\{\phi_1, \psi_1\}$, $\{\phi_2, \psi_2\}$, *and* $\{\phi_3, \psi_3\}$. *Here* $(\phi_1, \psi_1) \prec_{ne} (\phi_2, \psi_2) \prec_{ne} (\phi_3, \psi_3)$, $\{\phi_1, \psi_1\}$ *and* $\{\phi_3, \psi_3\}$ *are saddle points, and* $\{\phi_2, \psi_2\}$ *is locally asymptotically stable. The global behavior of Equation* (23) *is described by Theorem 8 of [4]. In this case there exist four continuous curves* $\mathcal{W}^s(\phi_1, \psi_1)$, $\mathcal{W}^s(\psi_1, \phi_1)$, $\mathcal{W}^s(\phi_3, \psi_3)$, $\mathcal{W}^s(\psi_3, \phi_3)$ *that have endpoints at* $E_- = (\overline{x}_-, \overline{x}_-)$ *and are graphs of decreasing functions. Every solution which starts below* $\mathcal{W}^s(\phi_1, \psi_1) \cup \mathcal{W}^s(\psi_1, \phi_1)$ *in the northeast ordering converges to* $E_0 = (\overline{x}_0, \overline{x}_0)$ *and every solution which starts above* $\mathcal{W}^s(\phi_3, \psi_3) \cup \mathcal{W}^s(\psi_3, \phi_3)$ *in the northeast ordering converges to* $E_+ = (\overline{x}_+, \overline{x}_+)$. *Every solution which starts above* $\mathcal{W}^s(\phi_1, \psi_1) \cup \mathcal{W}^s(\psi_1, \phi_1)$ *and below* $\mathcal{W}^s(\phi_3, \psi_3) \cup \mathcal{W}^s(\psi_3, \phi_3)$ *in the northeast ordering converges to* $\{\phi_2, \psi_2\}$. *For example, this happens for* $a = 1$, $c = \frac{389}{2176}$, *and* $d = \frac{249}{64}$.

(ii) *If* $\Delta_j \leq 0$ *for some* $2 \leq j \leq 5$ *and* $\Delta_i > 0$ *for* $i \neq j$, *then Equation* (23) *has three equilibria* $\overline{x}_0 < \overline{x}_- < \overline{x}_+$, *where* $\overline{x}_0$ *and* $\overline{x}_+$ *are locally asymptotically stable and* $\overline{x}_-$ *is a repeller, and one period-two solution* $\{\phi_1, \psi_1\}$, *which is a saddle point. The global behavior of Eq.* (23) *is described by Theorem 7 of [4]. In this case there exist four continuous curves* $\mathcal{W}^s(\phi_1,\psi_1), \mathcal{W}^s(\psi_1,\phi_1), \mathcal{W}^u(\phi_1,\psi_1), \mathcal{W}^u(\psi_1,\phi_1)$, *where* $\mathcal{W}^s(\phi_1,\psi_1), \mathcal{W}^s(\psi_1,\phi_1)$ *have endpoints at* $E_- = (\overline{x}_-, \overline{x}_-)$ *and are graphs of decreasing functions. The curves* $\mathcal{W}^u(\phi_1,\psi_1), \mathcal{W}^u(\psi_1,\phi_1)$ *are graphs of increasing functions and start at* $E_0 = (\overline{x}_0, \overline{x}_0)$. *Every solution which starts below* $\mathcal{W}^s(\phi_1,\psi_1) \cup \mathcal{W}^s(\psi_1,\phi_1)$ *in the northeast ordering converges to* E_0 *and every solution which starts above* $\mathcal{W}^s(\phi_1,\psi_1) \cup \mathcal{W}^s(\psi_1,\phi_1)$ *in the northeast ordering converges to* $E_+ = (\overline{x}_+, \overline{x}_+)$. *For example, this happens for* $a = 1$, $c = \frac{1}{5}$, *and* $d = \frac{237}{64}$.

(iii) *If* $\Delta_i \leq 0$ *and* $\Delta_{i+1} \geq 0$ *(such that at least one of these is strict) for some* $2 \leq i \leq 4$, *and if* $\Delta_6 < 0$, *then Eq.* (23) *has three equilibria* $\overline{x}_0 < \overline{x}_- < \overline{x}_+$, *where* $\overline{x}_0$ *and* $\overline{x}_+$ *are locally asymptotically stable and* $\overline{x}_-$ *is a saddle point, and no period-two solution. The global behavior of Equation* (23) *is described by Theorem 5 of [4] or Theorem 9 case (4). For example, this happens for* $a = 1$, $c = \frac{493}{1024}$, *and* $d = \frac{157}{48}$.

Equation (23) exhibits global dynamics similar to that of Equation (10), which was investigated in [4]. Therefore, we pose the following conjecture.

Conjecture 1 *There exists a topological conjugation between the maps in Equations* (10) *and* (23).

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