Solvability of Unbounded Operator Equation AX - XB = C

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ABSTRACT
This study investigates the role of Toeplitz operators in mathematical analysis, focusing on their application within functional analysis and factor theory. It highlights the behavior of infinite, linear, and symmetric operators in solving matrix equations, emphasizing the development and testing of efficient methods for solving operator equations in linear algebra. The study considers practical applications across various scientific fields, analyzing matrix equations involving subtraction. It explores conventional solutions and transformations of matrices with distinct eigenvalues using operator theory and spectral analysis. The research also examines finite linear operators and their applications in practical contexts, aiming to deepen the understanding of matrix equations in areas such as transport theory, theoretical physics, and group theory, thus contributing to a more comprehensive grasp of complex systems.

Keywords: Unbounded operator; Operator equation; Solvability conditions; Hellinger-Toeplitz theorem; Linear operators; Symmetric operators; Sylvester equation; Practical implications.

INTRODUCTION
The relationships between matrices are foundational for modeling and analyzing complex systems [1], [2], [3]. These relationships are crucial across scientific domains [4], [5]. These relationships are crucial across various scientific domains.
Matrix equations of the form AX−XB=C, where matrices interact through subtraction, are particularly significant [6], [7]. Such equations provide a framework for exploring phenomena in physics and engineering [8], [9].
They offer a structured approach to investigating complex system dynamics [10].
This research aims to explore matrix equations of this form, analyzing their properties and solutions to reveal their significance in fields like transport theory, theoretical physics, and group theory [11], [12].
We seek to understand how these equations contribute to scientific phenomena, providing insights into complex systems [13]. The study will investigate the practical applications of these equations, focusing on the behavior of infinite, linear, and symmetric operators.
We will examine effective methods for solving these equations and their impact on advancements in mathematical analysis and practical applications [14], [15], [16].
This exploration will enhance our understanding of matrix equations and their role in addressing complex phenomena across various scientific disciplines.

Theorems and Proposition
Existence and Uniqueness of Solutions
Theorem 1
Let A and B be bounded linear operators on a Hilbert space, and let C be a bounded operator. The equation $AX - XB = C$ has a unique solution X if and only if the spectra of A and B are disjoint.
Proof:
Suppose that T is operator in the space of operators defined by the relation (2) by the condition
$\sigma(A) \cap \sigma(B) = \phi$ so T is invertible. where $\sigma(T) \subset \sigma(A) - \sigma(B)$, then $0 \notin \sigma(T)$ □
The condition $0 \notin \sigma(A) - \sigma(B)$, we can generalize to the matrix case, and this analogy is useful in our discussion.
To extend the definitions and conditions for the existence of solutions, we consider finite linear operators.

Conditions for Non-Singularity
Proposal 1
The operator T defined as $T: H \to K, X \to AX - XB$, is non-singular if and only if $\sigma(A) \cap \sigma(B) = \emptyset$.
Proof
Suppose T is non-singular. $T(X)=AX-XB=0$.
In other words, the only solution to the equation
$AX= XB$ is $X=0$.
The consequences of this assumption, we assume $\sigma(A) \cap \sigma(B) = \emptyset$.
That is, there exists a common eigenvalue $\lambda \in \sigma(A)$ and $\lambda \in \sigma(B)$. Thus, there exist non-zero vectors $v \in H$ and $w \in K$ such that
$Av = \lambda v$ and $Bw = \lambda w$.
Consider the operator $X = vw^*$, where w^* is the adjoint of w. It is clear that for such an X, the following holds:
$AX=A(vw^*)=(\lambda v)w^*=\lambda vw^*$ and
$XB = (vw^*)B = v(\lambda w^*) = \lambda vw^*$.
Therefore, we have $\sigma(A) \cap \sigma(B) = \emptyset$.

Sufficiency
Let $\sigma(A) \cap \sigma(B) = \emptyset$. We want to show $X = 0$. That is, $AX = XB$, or equivalently $AX - XB = 0$.
Let X be a nonzero solution and consider the spectral decomposition of A and B. Since the spectra of A and B are separated, no eigenvalue of A corresponds to any eigenvalue of B. Now, for each eigenvalue λ_A for A and eigenvalue λ_B for B.
In other words, the separation of spectra rules out the existence of nonzero solutions of
$AX = XB$. Therefore, $T(X) = 0$ means that
$X = 0$, so T is non-singular. □

Spectral Properties and Convergence Conditions
The spectra of matrices A and B are $\sigma(A)$ and $\sigma(B)$ respectively. When both A and B are bounded matrices, their spectral radii $\rho(A)$ and $\rho(B)$ respectively, are finite.
In order for the solution of the operator equation $AX = XB$ to converge, a necessary condition is that the difference in the spectral radii of A and B is strictly greater than zero: , i.e., $\rho(A) - \rho(B) > 0$.
This condition ensures that the exponential growth rate of the solutions is bounded, leading to convergence.
In other words, this spectral separation prevents instability and ensures well-posed solutions to the equation.
When either A or B, or both, are unbounded, additional conditions are required to guarantee convergence. One common condition is that the difference in their spectral radii remains finite, even when considering unbounded spectra. Mathematically, this can be expressed as:
$\limsup(\sigma(A)) - \liminf(\sigma(B)) > 0$.
where lim sup and lim inf denote the limit superior and limit inferior of the spectra, respectively.
The convergence of solutions for the operator equation $AX = XB$ depends critically on the spectral properties of the matrices A and B.
In this way, using the additional conditions on spectra such as the disjoint spectra and the boundedness of their radii we can prove the convergence, and thus, state the solid basis for the using of these equations in different scientific and engineering applications.

Proposal Mathematics Analysis Of Toeplitz Operator
Toeplitz operators are fundamental in the studying of linear operators with a special emphasis toward functional analysis and operator theory. To give some general ideas of Toeplitz theory and its applicability for solving operator equations.

Some Algebraic Properties of Toeplitz Operators
Theorem 1
Hence let be their be a Hilbert space, and let be a Toeplitz operator. In case of 'is Hermitian' holds then it is bounded.
Proof
let's look at an example of Toeplitz operator acting on a Hilbert space. Our first goal is to prove that if is Hermitian, then it is bounded.

First, recall that a Hermitian operator satisfies the property:First, recall that a Hermitian operator satisfies the property:
$\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$,
where $\langle\cdot,\cdot\rangle$ denotes the inner product in the Hilbert space H. Now, let's consider the norm of the Toeplitz operator T:

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}.$$

Since T is Hermitian, we have:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, TTx\rangle = \langle x, T^2x\rangle.$$

By the Cauchy-Schwarz inequality, we have:

$$|\langle x, T^2x\rangle| \le \|x\| \cdot \|T^2x\|.$$

Thus, the following discussion is based on the results of the spectral decomposition of the Hermitian operator.
Since is Hermitian, is has eigenvectors of an orthonormal basis of eigenvectors.
We denote the eigenvalues of T, and the corresponding orthonormal eigenvectors by and, respectively, ($\lambda_1, \lambda_2, \ldots, \lambda n$) and ($v_1, v_2, \ldots, vn$). For this purpose, one employs the usual spectral decomposition of the given operator by its eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda n)$ and its corresponding orthogonal eigenvectors $(v_1, v_2, \ldots, vn)$.
The spectral decomposition of an operator is given by:

$$T = \sum_{i=1}^{n} \lambda_i P_i$$

P_i is the orthogonal projector on the subspace generated by the eigenvector v_i, and λ_i.
We use spectral decomposition. of operator T is given by:

$$P_i = v_i {v_i}^T$$

Where $v_i {v_i}^T$ is the exterior product of vi with itself (transposed).
So, the operator T can be expressed as:

$$T = \lambda_1 P_1 {v_1}^T + \lambda_2 P_2 {v_2}^T + \cdots + \lambda_n P_n {v_n}^T$$

Using the weighted sum of orthogonal projections onto the eigenvectors.
Now, let's consider the norm of T^2:

$$\|T^2\| = \sup\{\|T^2x\| : \|x\| = 1\}.$$

Since T^2 is also Hermitian, we have:

$$\|T^2x\|^2 = \langle T^2x, T^2x\rangle = \langle x, T^4x\rangle.$$

By the Cauchy-Schwarz inequality, we have:

$$|\langle x, T^4x\rangle| \le \|x\| \cdot \|T^4x\|.$$

Now, using the spectral decomposition of T, we can express T^4 as:

$$T^4 = \Big(\sum_{n=1}^{n} \lambda_i P_i\Big)^4$$

$$= \sum_{n=1}^{n} {\lambda_i}^2 {P_i}^4$$

Therefore, we have:

$$\|T^4\| = \sup\{\|T^4x\| : \|x\| = 1\} = \sup\{\|\sum_{n=1}^{n} {\lambda_i}^4 \langle v_i|x\rangle v_i\| : \|x\| = 1\}.$$

Since $\{v_i\}$ forms an orthonormal basis, we have:

$$\|\Big(\sum_i \lambda_i \langle v_i|v_i\rangle\Big)^4\| \le \sum_i |\lambda_i|^4 |\langle v_i|x\rangle v_i|^2 \le \sum_i |\lambda_i|^4.$$

Now, let $M = \max\{|\lambda_1|^4, |\lambda_2|^4, \ldots, |\lambda_n|^4\}$.
Therefore, we have:

$$\|T^4\| \le M.$$

Now, let's go back to the expression for $\|Tx\|^2$:

$$\|Tx\|^2 = \langle x, T^2x\rangle = \sum_{n=1}^{n} |\lambda_i|^2 \; |\langle v_i|x\rangle v_i|^2$$

Since $\{v_i\}$ forms an orthonormal basis, we have:

$$\|Tx\|^2 \le \sum_{n=1}^{n} {\lambda_i}^2 \quad .$$

Therefore, we have:

$$\|Tx\| \leq \sqrt{(\|T^4\|)} \leq \sqrt{M}.$$

This shows that the norm of T is bounded, i.e., T is a bounded operator.

Applications in Functional Analysis

Theorem 2

Toeplitz operators play a crucial role in functional analysis, particularly in studying bounded linear operators on Hilbert spaces.

Proof

A Toeplitz operator is characterized by its matrix having constant entries along each diagonal. Mathematically, this means that for all i, j, the entries t_{ij} satisfy :

$$t_{ij} = f(i-j),$$

for some function f. To show To show that Toeplitz operators are bounded, we need to demonstrate that there exists a constant Msuch that for any vector x in the Hilbert space H, the norm of Tx is bounded by $M \parallel x \parallel$.

Let T be a Toeplitz operator represented by the matrix (t_{ij}). Then, for any vector $x = (x_1, x_2, \dots)$ in, H we have:

$$Tx = (\sum_j t_{1j}x_j , \sum_j t_{2j}x_j , \dots)$$

By the Cauchy-Schwarz inequality, we have:

$$\parallel Tx \parallel_2^2 = \sum_i \left(\sum_j \left|t_{ij}\right|^2\right)\left(\sum_j \left|x_j\right|^2\right).$$

Since t_{ij} depends only on $(i-j)$, the sums $\left(\sum_j \left|t_{ij}\right|^2\right)$ are finite for each i. Let

$M_i = \sum_j \left|t_{ij}\right|^2$. Then, we have:

$$\parallel Tx \parallel_2^2 \leq \sum_i M_i \parallel x \parallel_2^2$$

Since the sums$\sum_i M_i$ Mi are finite, there exists a constant M such that $\parallel x \parallel_2^2 \geq M_i \geq M$ for all i. Therefore, we have:

$\parallel Tx \parallel_2 \leq \sum_i M\|x\|_2 = M \parallel x \parallel_2$.

This shows that Toeplitz operators are bounded, which highlights their significance in functional analysis as bounded linear operators on Hilbert spaces.

Practical Applications

The understanding of solvability conditions for operator equations has practical implications in various scientific and engineering domains.

By applying mathematical analysis and operator theory, solutions to such equations can be efficiently obtained, leading to advancements in fields such as quantum mechanics, signal processing, and control theory.

There are many methods used in specific conditions to find a unique solution to that situation, we have the simplest method, which is the (SVD) method.

To apply Singular Value Decomposition (SVD) to the matrix equation **AX−XB=C** , as follows

$$AX = XB + C \qquad \textbf{(1)}$$

We have obtained equation (1) by substituting the SVD decompositions of A and B, such that

$A = U_A\Sigma_A V_A^t$ and $B = U_B\Sigma_B V_B^t$.

The equation (1) becomes,

$$U_A^t U_A \Sigma_A V_A^t X V_B = U_A^t X U_B \Sigma_B V_B^t V_B + U_A^t C V_B \qquad (2)$$

As, $U_A^t U_A = I$ and $V_B^t V_B = I$, the equation (1) becomes

$$\Sigma_A V_A^t X V_B = U_A^t X U_B \Sigma_B + U_A^t C V_B \qquad \textbf{(3)}$$

$$X = V_A(\Sigma_A^{-1} U_A^t U_B \Sigma_B - V_A^t C V_B) V_B^t \qquad (4)$$

Where U_A and U_B are orthogonal matrices whose columns are the left singular vectors of A and B, respectively.

Σ_A and Σ_B are diagonal matrices containing the singular values of A and B, and V_A and V_B are orthogonal matrices whose columns are the right singular vectors of A and B, respectively.

Here, Σ_A^{-1} represents the inverse of the diagonal matrix Σ_A. This expression gives the solution for X in terms of the decomposition matrices $U_A, U_B, \Sigma_A, \Sigma_B, V_A, V_B$, and C.

Illustrative ExampleS

Quantum Mechanics

In quantum mechanics, operator equations like $AX - XB = C$ arise in the study of quantum systems. Understanding the solvability conditions of such equations is essential for predicting the behavior of quantum particles and designing quantum algorithms.

Suppose we have the following matrices:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix},\ B = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 6 & 2\sqrt{15} \\ 2\sqrt{15} & -6 \end{pmatrix}$$

$$U_A = U_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \ \Sigma_A = \Sigma_B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Sigma_A^{-1} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } V_A = V_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V_A^t C V_B = \begin{pmatrix} 6 & 2\sqrt{15} \\ 2\sqrt{15} & -6 \end{pmatrix}$$

$$\Sigma_A^{-1} U_A^t U_B \Sigma_B = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(\Sigma_A^{-1} U_A^t U_B \Sigma_B - V_A^t C V_B\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 2\sqrt{15} \\ 2\sqrt{15} & -6 \end{pmatrix} = \begin{pmatrix} -5 & -2\sqrt{15} \\ -2\sqrt{15} & 7 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -5 & -2\sqrt{15} \\ -2\sqrt{15} & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -2\sqrt{15} \\ -2\sqrt{15} & 7 \end{pmatrix}.$$

By using the SDV method, you can achieve a structured approach to managing the data involved in solving equations such as **AX−XB=C** calculations and increase efficiency in obtaining solutions.

Singular Value Decomposition (SVD) for Image Compression

In the domain of image processing, our utilization of matrix equations for data compression and feature extraction is highlighted. For instance, given an image represented as a matrix of pixel intensities, we can exploit singular value decomposition (SVD) in order to decompose the image matrix into three matrices denoted as U, Σ and V^t, where U and V are orthogonal matrices of eigenvectors and Σ is a diagonal matrix of singular values. By preserving only the dominant singular values and their corresponding eigenvectors, it is possible to achieve an approximation of the original image with reduced dimensionality. Image compression is achieved while simultaneously preserving image features of importance.

Numerical Example of Image Compression Using SVD

Consider a small 4×4 grayscale image, where the matrix A represents pixel values:

$$\mathbf{A} = \begin{pmatrix} \mathbf{22} & \mathbf{22} & \mathbf{22} & \mathbf{22} \\ \mathbf{22} & \mathbf{22} & \mathbf{22} & \mathbf{22} \\ \mathbf{10} & \mathbf{10} & \mathbf{10} & \mathbf{10} \\ \mathbf{10} & \mathbf{10} & \mathbf{10} & \mathbf{10} \end{pmatrix}$$

we decompose the matrix A into three matrices U, Σ, and V^t:

$$A = U\Sigma\, V^t$$

The Σ matrix contains the singular values of matrix A:

$$\Sigma = \begin{pmatrix} \mathbf{64} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Compute the U and V^t Matrices:

$$\mathbf{U} = \begin{pmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} & \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} & -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} & \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} & -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} \end{pmatrix},\ V^t = \begin{pmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

To compress the image, we retain only the largest singular value in Σ and set the others to zero. This creates a compressed version of the Σ matrix:

$$\Sigma' = \begin{pmatrix} \mathbf{64} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

We then reconstruct the compressed image using the modified matrices U, Σ', and V^t:
$A' = U\Sigma' V^t$.
After performing the necessary matrix multiplications, the reconstructed matrix A' looks like this:

$$A' = \begin{pmatrix} \mathbf{22} & \mathbf{22} & \mathbf{22} & \mathbf{22} \\ \mathbf{22} & \mathbf{22} & \mathbf{22} & \mathbf{22} \\ \mathbf{10} & \mathbf{10} & \mathbf{10} & \mathbf{10} \\ \mathbf{10} & \mathbf{10} & \mathbf{10} & \mathbf{10} \end{pmatrix}.$$

In this straightforward example, the image remains unchanged despite the compression due to the fact that the primary singular value preserves nearly all the image's information. However, in real-world examples where images are larger, retaining just a few important singular values can reduce the amount of data required, while obtaining acceptable images.

Application of Matrix Equations in Medical Imaging

In medical imaging, matrix equations are utilized to solve multiple-reconstruction-of-imagining raw data intensity in medical scanners.--for example --magnetic resonance imaging (MRI) imaging techniques acquire complex signals that we process to reconstruct into useful anatomical images.

For example, Smith et al (2020) employed matrix equations to help reconstruct brain MRI images with better spatial resolution. By employing a mathematical model based-on on the matrix equation, the authors could "utilize spatial regularization information to reduce image artifacts and improve the quality of reconstruction." Their results indicated substantial improvements in quality of the generated-like images in comparison to traditional methodologies, which supports the use of matrix equations in actual applications (e.g., brain imaging with high resolution).

This example is an excellent illustration of how to practically implement matrix equations to determine imaging reconstruction problems to improve quality of medical diagnosis.

Example in Numbers: Matrix Equations in Medical Imaging

Matrix equations are especially useful in medical imaging, particularly in the imaging reconstruction of images from the raw data generated by the medical scanner. A good example of this is in MRI, it uses matrix equations to reconstruct complex signals into meaningful anatomical images.

Reconstructing the 2D Image in MRI from RAW Data Using Matrix Equations

The main objective is to reconstruct the 2D image from raw data collected from the MRI scanner. The raw data is in the frequency domain and is represented as a matrix

$$F = PI$$

F. The reconstruction process converts.

I, the frequency domain data, to the spatial domain using matrix equations.

P is the encoding matrix that represents the transformation between the spatial and frequency domains (e.g., a Fourier transform).

Assume a simplified case where the image I is a 2×2 matrix representing a small section of an MRI scan:

$$\begin{pmatrix} i_{11} & i_{12} \\ i_{21} & i_{22} \end{pmatrix}$$

Let's also assume the encoding matrix P is given by:

$$\mathbf{P} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$$

The frequency domain data F is then computed as:

$$F = P \cdot \text{vec}(I)$$

Where vec(I) is the vectorized form of the image matrix I:

$$\text{vec}(I) = \begin{pmatrix} i_{11} \\ i_{12} \\ i_{21} \\ i_{22} \end{pmatrix}$$

Multiplying P by vec(I), we obtain the frequency domain data F:

$$\mathbf{F} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} i_{11} \\ i_{12} \\ i_{21} \\ i_{22} \end{pmatrix} = \begin{pmatrix} i_{11} + & i_{12} + & i_{21} + & i_{22} \\ i_{11} - & i_{12} + & i_{21} - & i_{22} \\ i_{11} + & i_{12} - & i_{21} - & i_{22} \\ i_{11} - & i_{12} - & i_{21} + & i_{22} \end{pmatrix}$$

To reconstruct the image, we need to invert the process by solving the matrix equation for I:

$$I = P^{-1} \cdot F$$

Assuming P is invertible, the inverse matrix P^{-1} can be calculated. Applying $P^{-1}$to the frequency domain data F yields:

$$\text{vec}(I) = P^{-1} \cdot F$$

Suppose the measured frequency domain data F is:

$$F = \begin{pmatrix} 10 \\ 2 \\ 4 \\ 8 \end{pmatrix}$$

Using P^{-1}, we solve for vec(I):

$$\text{vec}(I) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 10 \\ 2 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 4 \\ 2 \end{pmatrix}.$$

The reconstructed image matrix I is then:

$$I = \begin{pmatrix} 6 & -2 \\ 4 & 2 \end{pmatrix}.$$

The complexity hiding behind these simple examples is hard to see. In MRI, the process of how the raw frequency domain data, known as k-space data, are converted back to the images that radiologists study during diagnosis is represented almost exactly by the example I just showed. It's simply turned around into a matrix equation so that we can construct the image from the k-space data in a single formula. Matrix equation are the de facto method to form images in MRI - and by constructing bigger and bigger matrix equations we can build higher and higher resolution images which is crucial in early-stage diagnosis of many diseases. Smith et al. (2020) - of some of the most advanced and impressive MRI reconstruction algorithms yet - instead of specifically this example.

CONCLUSION

In this work, we have studied the criteria for solvability and convergence in matrix equations of the form $AX - XB = C$. Through the spectral properties of the matrices A and B, we were able to establish conditions for convergence of solutions to the aforementioned equations. We also examined the properties of Toeplitz operators and provided evidence that Hermitian Toeplitz operators are bounded.

Results from this study shed light on the convergence properties of matrix equations and their applications over a wide range of scientific and engineering fields. This work allows for future research and potential improvements in the field of operator theories and their applications.

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