Exploring Fuzzy Algebraic Concepts: From Theory to Applications

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ABSTRACT
Uncertainty pervades our understanding of the world, presenting formidable chal- lenges to traditional mathematical frameworks. Fuzzy algebraic concepts offer a specialized approach to address this uncertainty, with roots tracing back to Zadeh's seminal work in 1965. This paper provides a comprehensive review of the evolution of fuzzy algebra. It covers various aspects of the fuzzification of algebraic structures, such as fuzzy subrings and fuzzy ideals, the characterization of fuzzy zero divisors, and applications of fuzzy polynomials and fuzzy matrices. Through real-life case studies in domains like medical diagnosis and agricultural analysis, the study underscores the versatility and importance of fuzzy algebra in addressing uncertainty. Aimed at researchers and practitioners, this paper serves as a valuable resource for navigating the complexities of uncertainty using fuzzy algebraic concepts.

Keywords: Fuzzy subring, Fuzzy zero divisor, Fuzzy polynomial, Fuzzy matrix

1. INTRODUCTION

Uncertainty is a fundamental aspect of life, posing a significant challenge for human understanding. Fuzzy sets offer a tailored mathematical approach to tackle this com- plexity and navigate through uncertainties. Zadeh's [58] pioneering work in 1965 kick started the study of fuzzy sets, sparking widespread interest among researchers and leading to a plethora of applications across diverse scientific fields. The literature on fuzzy set theory and its applications has since burgeoned, with thousands of papers spanning disciplines like economics, psychology, artificial intelligence, network analysis, and decision-making [9, 15].

Fuzzification of algebraic structures is a process that extends classical algebraic concepts to accommodate uncertainty or imprecision. It involves enriching traditional algebraic definitions and operations with degrees of membership or grades of truth, reflecting the fuzzy nature of real-world data or systems. The fuzzification of algebraic structures commenced in 1971 with Rosenfeld's work [49]. Unlike traditional groups, a fuzzy subgroup relaxes the requirement of closure under the group operation, per- mitting elements to possess partial membership in the group. Subsequently, numerous researchers delved into expanding their concepts within the domain of fuzzy algebra [3, 4, 17, 45].

Subsequently, researchers extended fuzzification to other algebraic structures. Fuzzy subrings, for example, generalize the notion of rings by allowing elements to have varying degrees of adherence to the ring properties, such as distributivity and associativity. Liu's [37] introduction of fuzzy subrings and fuzzy ideals in 1982 marked a pivotal moment, further advanced by researchers such as [1, 2, 11, 23, 48], who significantly broadened and deepened the study within this domain.

Continuing this trajectory, the exploration of zero divisors in rings emerges as a fundamental aspect in understanding algebraic properties and structures. Zero divisors delineate rings that are not integral domains, thus playing a crucial role in the classification of various algebraic systems. However, the concept of fuzzy zero divisors extends beyond traditional zero divisors, accommodating uncertain or graded adherence to their defining properties within fuzzy algebraic structures. This expansion provides a framework to analyze elements exhibiting varying degrees of zero divisor characteristics, particularly in systems characterized by fuzziness or uncertainty. The study of fuzzy zero divisors of the ring is initiated by Ray [47]. Melliani [41] introduced another approach to defining fuzzy zero divisors using fuzzy points and also introduced a ring without fuzzy zero divisors, called an Integral ring. Ayub [7] presented the concept of fuzzy zero divisors for non-zero and non-unit elements of a ring. The study of fuzzy divisors can be extended to [30, 56].

The study of fuzzy polynomials intersects with the broader exploration of fuzzy algebraic structures, including the characterization of fuzzy zero divisors. The application of fuzzy polynomials spans across engineering and fuzzy mathematics, offering versatile utility. Eslami [24] laid the groundwork for the exploration of fuzzy polynomials, which has since been enriched by diverse methodologies presented by different authors. Barhoi [10] introduced an approach to defining fuzzy polynomials utiliz- ing fuzzy triangular numbers, elucidating a connection between crisp polynomials and their fuzzy counterparts. Additionally, Melliani [41] formulated fuzzy polynomi- als based on fuzzy points, contributing to the varied approaches within this domain. Fuzzy polynomials are predominantly employed in neural networks, with Zarandi [59] utilizing fuzzy polynomial neural networks (FPNN) to predict concrete's compressive strength.

The theory of fuzzy matrices is pivotal in diverse scientific and engineering domains, initially introduced by Thomason [55]. Building on this foundation, Hashimoto [27] introduced the concept of nilpotent fuzzy matrices, while Kim [33] investigated properties such as adjoints and determinants of fuzzy matrices. Additionally, Atanassov [6] introduced intuitionistic fuzzy matrices, which are an extension of conventional fuzzy matrices that deal with membership and non-membership values. The versatility of fuzzy matrices is demonstrated in practical applications in various fields. For example, Meenakshi [40] used fuzzy matrices to calculate diagnosis scores based on symptoms, diseases, and patient data in medical diagnosis. Similarly, Sun [22] utilized fuzzy matrices in agriculture to identify suitable crops for specific land patches.

This paper comprises eight sections. Section 2 discusses various structures of fuzzy subrings, including intuitionistic fuzzy subrings, Q-fuzzy subrings using t-norm T, anti Q-fuzzy subrings using t-conorm C, bipolar fuzzy subrings, and picture fuzzy subrings. Section 3 provides a comprehensive review of fuzzy polynomials and fuzzy polynomial subrings. Section 4 examines the concept of fuzzy matrices as studied by various authors, covering controllable fuzzy matrices, intuitionistic fuzzy matrices, and their associated determinant and adjoint properties. The primary focus of this literature is to provide a source of inspiration for future research on fuzzy zero divisors in polynomial and matrix rings. Section 5 reviews recent developments regarding fuzzy zero divisors and their existing literature. Section 6 illustrates real-life applications in domains such as medical diagnosis and agricultural analysis, emphasizing the importance of fuzzy algebra in handling uncertainty. Section 7 presents the limitations of the study, while Section 8 discusses the conclusion and future scope of these topics.

Notation: Henceforth, X is the universal set, and R denote commutative ring with unity 1 and additive identity 0.

2. Diverse Structures Fuzzy Subrings

Ray [47] laid the groundwork for the theory of fuzzy subrings and fuzzy ideals in 1982. Since then, many researchers have engaged in extending the concept of fuzzy subring to the broader framework of abstract algebra of rings. This section delves into diverse structures of fuzzy subrings, including intuitionistic fuzzy subrings, Q-fuzzy subrings employing t-norm T, anti Q-fuzzy subrings utilizing t-conorm C, bipolar fuzzy subrings, and picture fuzzy subrings.

Definition 1 (Fuzzy Subset). [25] A mapping from a set $A \subseteq X$ to the interval [0, 1] is referred to as a fuzzy subset of A. The collection of all fuzzy subsets of A is denoted by FS(A).

Das [18] introduced the theory of level subsets, which gave new dimension to the study of fuzzy set theory.

Definition 2 (α-Level Set). [25] Let η be a fuzzy subset of $A \subseteq X$. Then, a crisp set ηα = r A η (r) α is referred to as the α- cut (or α-level set) of η.

Definition 3 (Fuzzy Subring). [37] Let η be a fuzzy subset of ring R. Then, η is called a fuzzy subring of R if and only if, for all $r, s \in R$, the following conditions hold:

(i) $\eta(r - s) \geq \min\{\eta(r), \eta(s)\}$,

(ii) $\eta(rs) \geq \min\{\eta(r), \eta(s)\}$.

Theorem 1. The characteristics function χR of ring R is the fuzzy subring of R.

Theorem 2. A non-empty subset A of ring R is a subring of R if and only if characteristics function χA is a fuzzy subring of R.

Definition 4 (Fuzzy ideal). [37] A fuzzy subset η of ring R is called a fuzzy ideal of R if and only if, for all r, $s \in R$, the following conditions hold:

(i) $\eta(r - s) \geq \min\{\eta(r), \eta(s)\}$,
(ii) $\eta(rs) \geq \max\{\eta(r), \eta(s)\}$.
Theorem 3. [20] A fuzzy subset is η is fuzzy ideal of a ring R, if and only if level subset ηt is ideal of ring R, for all $t \in \text{Im}\, \eta$.
Definition 5. [20] A fuzzy subset η is a fuzzy subring (or ideal) of ring R if and only if its level subsets ηt are subrings (or ideals) of ring R for each $t \in \text{Im}\, \eta$.
Atanassov [6] introduced the concept of intuitionistic fuzzy sets in 1986, extending the framework as a generalization of fuzzy sets. Biswas [13] contributed to this study by introducing the intuitionistic fuzzy subgroup concept, offering a more comprehensive illustration of the applications of intuitionistic fuzzy sets. Likewise, Banerjee and Basnet [8] presented intuitionistic fuzzy subrings and ideals. Mohamed [23] explores intuitionistic fuzzy ideals in BE-algebras and establishes several new results related to their structure.
Definition 6 (Intuitionistic Fuzzy Set). [6] An intuitionistic fuzzy set I of set X is defined as follows:
$I = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$
where, $\mu : X \to [0, 1]$ and $\eta : X\ [0, 1]$ represent the degree of membership and degree of non-membership functions for every $x \in X$.
Definition 7 (Intuitionistic Fuzzy Subring). [8] An intuitionistic fuzzy subset $I = \{\langle r, \mu(r), \eta(r)\rangle \mid r \in R\}$ is said to be an intuitionistic fuzzy subring of R, if for every $r, s \in R$, the following conditions hold:
(i) $\mu(r - s) \geqslant \min\{\mu(r), \mu(s)\}$,
(ii) $\mu(rs) \geqslant \min\{\mu(r), \mu(s)\}$,
(iii) $\eta(r - s) \leqslant \max\{\eta(r), \eta(s)\}$,
(iv) $\eta(rs) \leqslant \max\{\eta(r), \eta(s)\}$.
Example 1. Let μ and η be fuzzy subsets of Z_2 defined as,

$$\mu(r) = \begin{cases} 0.7 & \text{if } r = 0, \\ 0.3 & \text{if } r = 1, \end{cases}$$

and

$$\eta(r) = \begin{cases} 0.1 & \text{if } r = 0, \\ 0.4 & \text{if } r = 1, \end{cases}$$

then $I = r, \mu(r), \eta(r)\ r\ Z_2$ is an intuitionistic fuzzy subring of Z_2.
Definition 8 (Intuitionistic Fuzzy Ideal). [8] An intuitionistic fuzzy subring $I = \{\langle r, \mu(r), \eta(r)\rangle \mid r \in R\}$ is said to be an intuitionistic fuzzy ideal of R, if for every $r, s \in R$,
(i) $\mu(r\ s) \geqslant \min \mu(r), \mu(s)$,
(ii) $\mu(rs) \geqslant \min \mu(r), \mu(s)$,
(iii) $\eta(r\ s) \leqslant \min \eta(r), \eta(s)$,
(iv) $\eta(rs) \leqslant \eta(r)$.
Anthony and Sherwood [5] employed the notion of a triangular norm to redefine fuzzy subgroups. The concept of Q-fuzzy groups was first presented by Solairaju and Nagarajan [54]. Rasuli [46] introduced the concept of Q− fuzzy subrings and anti Q− fuzzy subrings by using a t−norm T and a t−conorm C, respectively.
Definition 9 (t-norm). [46] A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called t− norm if for every $u, v, w \in [0, 1]$,
(i) $T(u, 1) = u$,
(ii) $T(u, v) \leqslant T(u, w)$ if $v \leqslant w$,
(iii) $T(u, v) = T(v, u)$,
(iv) $T(u, T(v, w)) = T(T(u, v), w)$.
Definition 10. [46] Let Q be a non-empty set. A Q− fuzzy subset η of R is said to be a Q− fuzzy subring of R with respect to the t− norm T, if for every $u, v \in [0, 1]$ and $q \in Q$,
(i) $\eta(u + v, q) \geqslant T(\eta(u, q), \eta(v, q))$,
(ii) $\eta(\ u, q) \geqslant \eta(u, q)$,
(iii) $\eta(uv, q) \geqslant T(\eta(u, q), \eta(v, q))$.
Example 2. Let η be Q− fuzzy subset of Z defined as,

$$\eta(x, q) = \begin{cases} 0.6 & if\ x \in \{\pm 1, \pm 3, \ldots\}, \\ 0.4 & if\ x \in \{0, \pm 2, \pm 4, \ldots\}. \end{cases}$$

Let T (x, y) = Tm (x, y) = min {x, y} for all x, y ∈ Z. Then η is Q− fuzzy subring of Z with respect to t− norm T.

Definition 11 (t-conorm). [46] A mapping C : [0, 1] × [0, 1] → [0, 1] is called t− conorm if for every u, v, w ∈ [0, 1],

(i) C (u, 0) = u,
(ii) C (u, v) ⩽ C (u, w) if v ⩽ w,
(iii) C (u, v) = C (v, u) ,
(iv) C (u, C (v, w)) = C (C (u, v) , w).

Definition 12. [46] Let Q be a non-empty set. A Q− fuzzy subset μ of R is said to be an anti Q− fuzzy subring of R with respect to the t− conorm C, if for every u, v ∈ [0, 1] and q ∈ Q,

(i) η (u + v, q) ⩽ C (η (u, q) , η (v, q))
(ii) η (u, q) ⩽ η (u, q)
(iii) η (uv, q) ⩽ C (η (u, q) , η (v, q)).

Zhang [60] initiated the conceptualization of bipolar fuzzy sets. The study of bipo- lar fuzzy sets involves considerations of both positive and negative membership values. In a bipolar fuzzy set, one membership value is confined to the interval [0, 1], repre- senting positive membership, while the other belongs to [1, 0], signifying negative − membership. This key distinction stands as a notable departure from intuitionistic fuzzy sets, where membership and non-membership values both fall within the range [0, 1]. In a fuzzy set with bipolar values, a membership degree of 0 implies that ele- ments are irrelevant to the corresponding property. A membership degree in the range (0, 1] indicates that elements somewhat satisfy the property. On the other hand, a membership degree in the range [-1, 0) suggests that elements somewhat satisfy the implicit counter-property.

Definition 13. [60] [Bipolar Fuzzy Set] Let X be a non-empty set. The bipolar fuzzy set A of set X is defined as follows:

A = {⟨ x, μ (x) , η (x)⟩ | x ∈ X} ,

where μ : X [0, 1] and η : X [0, 1] represent the degree of positive and negative ∈ membership functions, respectively, for every x∈ X.

Definition 14. [38] [Bipolar Fuzzy Subring] A bipolar fuzzy subset B = {⟨ r, μ (r) , η (r)⟩ | r ∈ R} is called a bipolar fuzzy subring of R, if for every r, s ∈ R,

(i) μ (r − s) ⩾ min {μ (r) , μ (s)} ,
(ii) μ (rs) ⩾ min {μ (r) , μ (s)} ,
(iii) η (r − s) ⩽ max {η (r) , η (s)} ,
(iv) η (rs) ⩽ max {η (r) , η (s)}.

Example 3. Let B = {⟨ r, μ (r) , η (r)⟩ | r ∈ Z_3} be bipolar fuzzy set of Z_3 defined by

$$\mu(r) = \begin{cases} 0.5 & if\ r = 0, \\ 0.1 & if\ r = 1, \\ 0.5 & if\ r = 2, \end{cases}$$

and

$$\eta(r) = \begin{cases} -0.3 & if\ r = 0, \\ -0.7 & if\ r = 1, \\ -0.4 & if\ r = 2. \end{cases}$$

Then, B is a bipolar fuzzy subring of Z_3.

In the realm of intuitionistic fuzzy sets, every element in the set X is characterized by two components: the degree of membership and the degree of non-membership. The picture fuzzy set serves as a broader framework, extending the intuitionistic fuzzy set by incorporating three components for each element in X: the degree of positive mem- bership, the degree of neutral membership, and the degree of negative membership. This enriched model of fuzzy sets was introduced by Cuong [16].

Definition 15 (Picture Fuzzy Set). [16] Let X be a non-empty set. A picture fuzzy set A of set X is defined as follows:

A = {⟨ x, μ (x) , η (x) , γ (x)⟩ | x ∈ X} ,

where μ : X → [0, 1], η : X→ [0, 1], and γ : X [0, 1] represent the degree of positive membership, the degree of neutral membership, and the degree of negative membership, respectively, for every $x \in X$.

Definition 16 (Picture Fuzzy Subring). [21] A picture fuzzy subset $P = \{\langle r, \mu(r), \eta(r), \gamma(r)\rangle \mid r \in R\}$ is said to be picture fuzzy subring of R, if for every $r, s \in R$,

(i) $\mu(r-s) \geqslant \min\{\mu(r), \mu(s)\}, \eta(r-s) \geqslant \min\{\eta(r), \eta(s)\}, \gamma(r-s) \leqslant \max\{\gamma(r), \gamma(s)\}$,

(ii) $\mu(rs) \geqslant \min\{\mu(r), \mu(s)\}, \eta(rs) \geqslant \min\{\eta(r), \eta(s)\}, \gamma(r-s) \leqslant \max\{\gamma(r), \gamma(s)\}$.

Example 4. Let $P = \{\langle \mu(r), \eta(r), \gamma(r)\rangle \mid r \in Z\}$ be a picture fuzzy set of Z defined by

$$\mu(x) = \begin{cases} 0.4 & \text{if } r = 0, \\ 0.2 & \text{if } r /= 0, \end{cases}$$

$$\eta(r) = \begin{cases} 0.4 & \text{if } r = 0, \\ 0.15 & \text{if } r /= 0, \end{cases}$$

$$\gamma(r) = \begin{cases} 0.2 & \text{if } r = 0, \\ 0.3 & \text{if } r /= 0. \end{cases}$$

Then, P is a picture fuzzy subring of Z.

3. Fuzzy Polynomial

Fuzzy polynomials hold significant importance in both engineering and fuzzy mathematics. This section undertakes a comprehensive investigation into the study of fuzzy polynomials, covering diverse aspects such as their representation using fuzzy numbers, their extension to fuzzy polynomials involving triangular fuzzy numbers, and the intricate relationship existing between crisp polynomials and their fuzzy polynomials. Furthermore, the section explores advancements in the study of fuzzy polynomial sub-rings, as proposed by Melliani [41], fuzzy polynomial ideals, as introduced by Kim [33], and the extension to intuitionistic fuzzy polynomial ideals, as investigated by Sharma [53]. This multifaceted exploration sheds light on the wide-ranging applications and theoretical developments surrounding fuzzy polynomials within various mathematical contexts and their relevance to engineering disciplines.

Definition 17 (Fuzzy Number). [19] A fuzzy subset μ of set X is called fuzzy number if

(i) μ is normal, i.e., there exist $x_0 \in X$ such that $\mu(x_0) = 1$,

(ii) μ is fuzzy convex, i.e.,

$\mu(\lambda x1 + (1-\lambda) x2) \geqslant \min\{\mu(x_1), \mu(x_2)\}$,

for all $x_1, \in x_2$ X and $\lambda \in [0, 1]$,

(iii) $\mu(x)$ is upper semi continuous,

(iv) support of $\mu = \{x \in R \mid \mu(x) > 0\}$ is bounded.

Rouhparvar [50] introduced the concept of fuzzy polynomials, wherein the coefficients are represented as fuzzy numbers.

Definition 18. [50] A polynomial of the type,

$A_0 + A_1x + A_2x^2 + ... + A_nx^n = 0$,

where $x \in X$, the coefficients $A_0, A_1, A_2, ...A_n$ are fuzzy numbers, is called fuzzy polynomial equation.

Fuzzy polynomials can be numerically solved through a variety of methods, with the Newton-Raphson method, ranking method, modified Adomian decomposition method, and fuzzy neural network method emerging as the most popular techniques. Each of these methods is characterized by distinctive algorithms, contributing to their effectiveness in accurately determining the real roots of fuzzy polynomials.

Barhoi [10] presented one approach to defining fuzzy polynomial using fuzzy triangular numbers and established a relation between crisp polynomial and fuzzy polynomial.

Definition 19 (Traingular Fuzzy Set). [10] The fuzzy set μ on set A is said to be triangular if the membership function is given by

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c. \end{cases}$$

A triangular fuzzy set is denoted by A = [a, b, c].

Definition 20. [10] Let $A = [a_1, b_1, c_1]$ and $B = [a_2, b_2, c_2]$ be two traingular fuzzy numbers.

(i) Addition: $A + B = [a_1, b_1, c_1] + [a_2, b_2, c_2] = [a_1 + a_2, b_1 + b_2, c_1 + c_2]$.

(ii) Multiplication:A−B=$[a_1,b_1,c_1]-[a_2,b_2,c_2]=[a_1-a_2,b_1-b_2,c_1-c_2]$.
(iii) Scalar Multiplication: Fork≠ 0 we have k $\tilde{A}=[ka_1,kb_1,kc_1]$.
(iv) Multiplication:

$$\tilde{A} * \tilde{B} = [a_1, b_1, c_1] * [a_2, b_2, c_2] = [\min\{a_i b_i\},\ \text{product of mid point},\ \max\{a_i b_i\}].$$

Definition 21. [10] A polynomial of the form

$$\tilde{f}\ \tilde{X} = \sum_i \tilde{A}_i \tilde{X}^i = \tilde{A}_0 + \tilde{A}_1\tilde{X} + \tilde{A}_2\tilde{X}^2 + \tilde{A}_3\tilde{X}^3 + \ldots,$$

where $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \ldots$ and $\tilde{X}$ are traingular fuzzy numbers.

Example 5. *Let $f(x) = 5x + 6$. Then $f(2) = (5 \times 2) + 6 = 16$. Ex ten ding this crisp polynomial to a fuzzy polynomial, we have the following expression $f\ \tilde{X} = \tilde{A}_0 + \tilde{A}_1\tilde{X}$,*

where $\tilde{A}_0 = [6, 6, 6]$, $\tilde{A}_1 = [5, 5, 5]$, $\tilde{X} = [1, 2, 3]$. Hence $f\ \tilde{2} = [6, 6, 6] + [5, 5, 5] * [1, 2, 3] = [6, 6, 6] + [5, 10, 15] = [11, 16, 21]$. The middle value is the same as the crisp value.

Example 6. *Consider polynomial with more than one variable $f(x, y) = 3xy^2 + 5x + 6y + 7$. Then $f(2, 3) = 3 \times 2 \times 3^2 + (5 \times 2) + (6 \times 3) + 7 = 89$. Now extending this crisp polynomial to fuzzy polynomial we have $f\ \tilde{X}, \tilde{Y} = \tilde{3}\tilde{X}\tilde{Y}^2 + \tilde{5}\tilde{X} + \tilde{6}\tilde{Y} + \tilde{7}$ and*

$$f\ \tilde{2}, \tilde{3} = [3, 3, 3] * [1, 2, 3] * [2, 3, 4]^2 + [5, 5, 5] * [1, 2, 3] + [6, 6, 6] * [2, 3, 4] + [7, 7, 7]$$

$$= [12, 54, 144] + [24, 35, 46] = [36, 89, 190].$$

Similarly, we get a fuzzy value in the middle, which is the same as the crisp value. The study of fuzzy polynomial subrings involves investigating the algebraic properties of the polynomials. Eslami [24] pioneered the study of the fuzzy polynomial ring. Expanding on this work, Melliani [41] defined fuzzy polynomials using fuzzy points as coefficients and established that F_η (R) [X] constitutes the fuzzy polynomial ring. Here, η denotes a fuzzy subring of R, and F_η (R) represents the set encompassing all fuzzy points of η within the ring R.

Definition 22. [24] Let S be a commutative ring with unity. Let R = S $[x_1, x_2, \ldots x_n]$ be the polynomial ring in the indeterminates $x_1, x_2, \ldots x_n$ over S. Let η be fuzzy subring over R. For all

$$P = \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \ldots \sum_{i_1=0}^{m_1} c_{i_1 \ldots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in R,$$

where $c_{i_1 \ldots i_n} \in S$. Let

$$\eta(P) = \min\{\min\{\eta(c_{i_1 \ldots i_n}) \mid i_j = 0 \text{ to } m_j, j = 1 \text{ to } n\}, \min\{t_j \mid j = 1 \text{ to } n\}\}$$

then η is called fuzzy polynomial ring over R. where

$$t_j = \begin{cases} \eta(x_j) & \text{if } \eta(x_j), \text{ is non trivial} \\ 1 & \text{otherwise.} \end{cases}$$

Definition 23. [41] A fuzzy polynomial ring in one determinate on F_η (R) is a set of sequences $(a_{t1}, a_{t2}, a_{t3} \ldots) = (a_{tk})_{k\in N}$ with $(a_{tk}) \in F_\eta(R)$ such that there exist n ∈ N such that ∀p ≥ n $a_{tp} = 0_s$ with t_i, s ∈ (0, 1]. It is denoted by F_η (R) [X].

Let P = $(a_{tk})_{k\in N}$ with $a_{tp} = 0_s$ for all p≥n and Q = $(b_{sk})_{k\in N}$ with $b_{tp} = 0_s$ for all p≥m. The ring operations on the fuzzy polynomial ring are as follows:

Addition:

$P + Q = (a+b)_{t_k \wedge s_k}$ with $(a+b)_{t_p \ s_p} = 0_s$ for all $p \geq max\{n, m\}$

Multiplication:

$P \times Q = (d_{\beta_k})_{k \in N}$ with $d_{\beta_k} = \sum_{i+j=p} a_{t_i} b_{s_j}$, $\beta_k = \min_{0 \geq i,j \geq k} \{t_i, s_j\}$ and $d_{\beta_p} = 0_s$ for all $p > m + n$.

Theorem 4. *[41] $(F_\eta(R)[X], +, \times)$ is commutative ring.*

Proof. Let $P, Q, R \in F_\eta(R)[X]$ and zero element of $F_\eta(R)[X]$ is $(0_s, 0_s, 0_s, \dots)$ for some $\eta_* \subseteq \{0\} \cup Q^+[x]$.

(i) **Closure:** $P + Q = a_{t_i} + b_{s_i} = (a+b)_{t_i \wedge s_i} \in F_\eta(R)[X]$. Hence it is closed under addition.

(ii) **Associativity:**

$$\begin{aligned}
(P+Q)+R &= (a_{t_i} + b_{s_i}) + c_{k_i} \\
&= (a+b)_{t_i \wedge s_i} + c_{k_i} \\
&= ((a+b)+c)_{(t_i \wedge s_i)\wedge k_i} \\
&= (a+(b+c))_{t_i \wedge (s_i \wedge k_i)} \\
&= a_{t_i} + (b_{s_i} + c_{k_i}) \\
&= P + (Q+R).
\end{aligned}$$

(iii) **Additive inverse:** $-P = -a_{t_i}$ such that $P + (-P) = a_{t_i} + (-a_{t_i}) = (a-a)_{t_i} = 0_{t_i}$.

(iv) **Commutative:** $P + Q = a_{t_i} + b_{s_i} = (a+b)_{t_i \wedge s_i} = (b+a)_{s_i \wedge t_i} = b_{s_i} + a_{t_i} = Q + P$.

Hence $F_\eta(R)[X]$ is additive abelian group. Also $P \times Q = \sum_i a_{t_i} b_{s_i} = d_{\beta_k} \in F_\eta(R)[X]$. We have

$$P \times (Q \times R) = \sum_i a_{t_i}(b_{s_i} c_{k_i}) = \sum_i (a_{t_i} b_{s_i}) c_{k_i} = (P \times Q) \times R.$$

Hence multiplication is associative and

$$\begin{aligned}
P \times (Q+R) &= \sum a_{t_i}(b_{s_i} + c_{k_i}) \\
&= \sum [(a_{t_i} b_{s_i}) + (a_{t_i} c_{k_i})] \\
&= \sum_i a_{t_i} b_{s_i} + \sum_i a_{t_i} c_{k_i} \\
&= (P \times Q) + (P \times R).
\end{aligned}$$

Therefore multiplication is distributive over addition. The identity is given by $I = (1_1, 0_s, 0_s, 0_s \dots)$ since

$$P \times I = (a_{t_0}, a_{t_1}, \dots a_{t_n}, 0_s, 0_s \dots) \times (1_1, 0_s, 0_s, 0_s \dots) = (a_{t_0}, a_{t_1}, \dots a_{t_n}, 0_s, 0_s \dots) = P.$$

Denote $X^0 = (1_1, 0_s, 0_s, 0_s \dots)$; $X^1 = (0_s, 1_1, 0_s, 0_s \dots)$; $X^2 = (0_s, 0_s, 1_1, 0_s \dots)$ and

$$X^n = (\underbrace{0_s, 0_s, \dots 0_s}_{n\text{times}}, 1_1, 0_s, \dots)$$

Then,

$$\begin{aligned}
P &= (a_{t_0}, a_{t_1}, a_{t_2}, \dots a_{t_n}, 0_s, 0_s, \dots) \\
&= a_{t_0}(1_1, 0_s, 0_s, \dots) + a_{t_1}(0_s, 1_1, 0_s, \dots) + \dots + a_{t_n}(\underbrace{0_s, 0_s, \dots, 0_s}_{n\text{times}}, 1_1, 0_s, \dots) \\
&= a_{t_0} + a_{t_1} X + a_{t_2} X^2 + \dots + a_{t_n} X^n.
\end{aligned}$$

Definition 24. *[41] $P \in F_\eta(R)[X]$ is said to be fuzzy polynomial on $F_\eta(R)$ if there exist $a_{t_i} \in F_\eta(R)$ such that $P = \sum_{i=0}^{n} a_{t_i} X^i$.*

Definition 25. *[41] A polynomial $P = a_{t_0} + a_{t_1} X + a_{t_2} X^2 + \ldots + a_{t_n} X^n$ is said to be non zero polynomial if there exist non zero coefficients $a_{t_0}, a_{t_1}, a_{t_2}, \ldots, a_{t_n}$.*

Definition 26 (Fuzzy Degree). *[41] Let $P = a_{t_0} + a_{t_1} X + a_{t_2} X^2 + \ldots + a_{t_n} X^n \in F_\eta(R)[X]$. The fuzzy degree of P is denoted by $deg(P)$ or d^0 and is defined as the maximal number n such that $a_{t_n} \quad 0_{t_n}$. In this a_{t_n} is called as leading coefficient of P.*

Definition 27 (Zero of the Fuzzy Polynomial). *[41] The $\alpha_s \in F_\eta(R)$ is said to be zero of the polynomial $P \in F_\eta(R)[X]$ iff $P(\alpha_s) = \sum_{i=0}^{n} a_{t_i} \alpha^i = 0_\beta, \beta \leq s$.*

Let $I(b_t) = \{P \in F_\eta(R)[X] \mid P(b_t) = 0_s, s \leq t\}$. It can be easily shown that $I(b_t)$ is ideal of $F_\eta(R)[X]$.

Definition 28 (Algebraic and Transcendent Fuzzy Point). *[41] The fuzzy point $b_t \in F_\eta(R)$ is said be algebraic fuzzy point if $I(b_t) \quad \{0\}$. Otherwise, b_t is called a transcendent fuzzy point.*

Theorem 5. *Let R be a ring. Then R is called integral domain iff $F_\eta(R)[X]$ is integral ring.*

Kim [35] introduced the concept of a fuzzy polynomial ideal, denoted as η_x, within the polynomial ring $R[x]$. This particular fuzzy polynomial ideal is induced by a fuzzy ideal η existing in the ring R, thereby establishing a relationship between fuzzy ideals in the base ring and the corresponding induced fuzzy polynomial ideals.

Theorem 6. *[35] Let $\eta : R \to [0, 1]$ be a fuzzy ideal of R, and consider $\eta_x : R[x] \to [0, 1]$, a fuzzy subset of $R[x]$ defined by $\eta_x(f(x)) = \min_i \{\eta(a_i)\}$ for any polynomial $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ in $R[x]$. Then η_x is fuzzy ideal of $R[x]$.*

Theorem 7. [35] Let $\eta : R \to [0, 1]$ be a fuzzy ideal of R. Then the set $D = \{f(x) \in R[x] \mid \eta_x(f(x)) = \eta_x(0)\}$ is a subring of $R[x]$.

The intuitionistic fuzzy polynomial ideal Ax of a polynomial ring R [x] induced by intuitionistic fuzzy ideal A of a ring R was introduced by Sharma [53].

Theorem 8. *[53] Let $I = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in R\}$ be intuitionstic fuzzy ideal of R, and let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be any polynomial in $R[x]$. Define an intuitionstic fuzzy ideal,*

$$I_x = \{\langle f(x), \mu_x(f(x)), \eta_x(f(x))\rangle \mid f(x) \in R[x]\}$$

of $R[x]$ by $\mu_x(f(x)) = \min_i \{\mu(a_i)\}$ and $\eta_x(f(x)) = \max_i \{\eta(a_i)\}$. Then I_x is an intuitionstic fuzzy ideal of $R[x]$.

Theorem 9. *[53] Let I be an intuitionstic fuzzy ideal of R. Then the set*

$$S = \{f(x) \in R[x] \mid \mu_x(f(x)) = \mu_x(0), \eta_x(f(x)) = \eta_x(0)\}$$

is subring of $R[x]$.

4. Fuzzy Matrix

The concept of fuzzy matrices was initially introduced by Thomason [55], who also studied the convergence of powers of the fuzzy matrix. In this section, we explore the fuzzy matrix concept as studied by various authors, encompassing controllable fuzzy matrices, intuitionistic fuzzy matrices, and their associated determinant and adjoint properties. This comprehensive study holds significant importance in various areas of science and engineering.

Definition 29 (Fuzzy Matrix). *[55] A fuzzy matrix is a matrix with its elements from the unit interval* $[0, 1]$.

Following are the operations on fuzzy matrices, Let $A = [a_{ij}]_{m\times k}$, $B = [b_{ij}]_{k\times n}$ and $C = [c_{ij}]_{k\times n}$ be any two fuzzy matrices with $a_{ij}, b_{ij}, c_{ij} \in [0, 1]$ then,

(i) $B + C = [d_{ij}]$ where $d_{ij} = \max\{b_{ij}, c_{ij}\}$.

(ii) $AB = \sum_{p=1} a_{ip}b_{pj}$ where $a_{ip}b_{pj} = \min\{a_{ip}, b_{pj}\}$.

(iii) $B \leqslant C$ if $b_{ij} \leqslant c_{ij}$ for all i and j.

For fuzzy square matrix $A = [a_{ij}]_{m\times m}$, we have

(i) $A^t = [a_{ji}]$ (transpose of fuzzy matrix).

(ii) $A^k = [a^k_{ij}]$, A^{k} $, A$ $A = A$, $k = 0, 1,$ 2, ... (power of fuzzy matrix).

(iii) $A^0 = I_{m\times m}$ where $I_{m\times m}$ is identity matrix.

Definition 30. [28] If A is a fuzzy square matrix of order m × m, then
(i) A is symmetric if and only if $A = A^t$.
(ii) A is reflexive if and only if $A \geqslant I_n$.
(iii) A is transitive if and only if $A^2 \leqslant A$.
Hence A is called reflexive, symmetric, & transitive (or idempotent) if and only if $A^2 = A$.
Kim [33] presented several properties related to the determinant and adjoint of a fuzzy square matrix.
Definition 31 (Determinant of fuzzy matrix). [33] The determinant of a fuzzy matrix A is denoted by |A| and defined as,

$$|A| = \sum_{\sigma\in S_k} a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{k\sigma(k)}$$

where S_k is the permutation group of all permutations.

Example 7. *Let* $A = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.6 & 0.1 \\ 0.2 & 0.4 & 0.3 \end{bmatrix}$ *be fuzzy matrix then* $|A| = 0.3\begin{vmatrix} 0.6 & 0.1 \\ 0.4 & 0.3 \end{vmatrix} + 0.2\begin{vmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{vmatrix} + 0.5\begin{vmatrix} 0.5 & 0.6 \\ 0.2 & 0.4 \end{vmatrix} = 0.3\,(0.3 + 0.1) + 0.2\,(0.3 + 0.1) + 0.5\,(0.4 + 0.2) = 0.3\,(0.3) + 0.2\,(0.3) + 0.5\,(0.4) = 0.3 + 0.2 + 0.4 = 0.4.$

Definition 32 (Adjoint of fuzzy matrix). [33] The adjoint of a fuzzy square matrix is denoted by Adj A and defined as, Adj A = $[|A_{ij}|]_{m\times m}$, where $|A_{ij}|$ is a determinant obtained by deleting ith row and jth column of matrix A.
Kim and Roush [32] conducted a study on the canonical form of an idempo- tent matrix in the years 1980. In a related context, Hashimoto [27] investigated the canonical form of a transitive matrix in the year 1983.
Theorem 10. [27] If R is idempotent fuzzy matrix (transitive, strongly transitive), then there exist a permutation matrix P such that the matrix $T = [t_{ij}] = P R P'$ satisfies $t_{ij} \geqslant t_{ji}$, for $i > j$.
In the above matrix, T is referred to as the canonical form of an idempotent fuzzy matrix (transitive, strongly transitive).
Xin [57] studied the controllable fuzzy matrix as below:
Definition 33. [57] [Controllable Fuzzy Matrix] A fuzzy matrix R is said to be controllable from below (above), if there exist a permutation matrix P such that
$T = [t_{ij}] = P R P'$ satisfies $t_{ij} \geqslant t_{ji}$ ($t_{ij} \leqslant t_{ji}$) for $i > j$.
A fuzzy matrix $R = [r_{ij}]$ is said to be controlled from below (above), if $r_{ij} \geqslant r_{ji}$ ($r_{ij} \leqslant r_{ji}$) for $i > j$.
Intuitionistic fuzzy matrix deals with both membership and non membership value. Atanassov [6] has first introduced the concept of intuitionistic fuzzy matrix which is the extension of fuzzy matrix.
Definition 34 (Intuitionistic Fuzzy Matrix). [43] Let $A = [a_{ij}]_{m\ n}$ and $B = [b_{ij}]_{m\times n}$ be two fuzzy matrices such that $a_{ij} + b_{ij} \leqslant 1$ for every $i \leqslant m$ and $j \leqslant n$. The pair ⟨A, B⟩ is called intuitionistic fuzzy matrix.

Example 8. *Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ be two fuzzy matrices such that*

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad b_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

then intuitionistic fuzzy matrix $\langle A, B\rangle$ =

$$\begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle & .. & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle & .. & \langle 0,1\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle 0,1\rangle & \langle 0,1\rangle & .. & \langle 1,0\rangle \end{bmatrix}_{n\times n}.$$

5. Fuzzy Zero Divisor

In this section, various methodologies for defining fuzzy zero divisors are examined. The introduction of the concept of fuzzy zero divisors for rings was initially put forth by Ray [47] in 2004. Ray also introduced the notions of fuzzy characteristics and fuzzy unit within the ring, extending the idea of fuzzy order for group elements introduced by Kim [34].

Definition 35 (Fuzzy order of element of group). [34] Let μ be a fuzzy subgroup ∈ of G. For a given x [0, 1], if there exists a smallest positive integer n such that $\mu(x^n) = \mu(e)$, then x has a fuzzy order of n with respect to μ. If no such n exists, x is of infinite order with respect to μ.

For any x, y ∈ G, let the fuzzy order of elements x and y be denoted by FOμ(x) and FOμ(y), respectively. If O(x) and O(y) are the orders of elements x and y in the group G, respectively, then O(x) = O(y) does not imply that FOμ(x) = FOμ(y), as shown in the following example.

Example 9. *Let $G = \langle a, b \mid a^2 = b^2 = (ab)^2 = e\rangle$ be Klein four group. Define $\mu : G \to [0,1]$ such that $\mu(e) = \mu(ab) = t_0$, $\mu(a) = \mu(b) = t_1$ with $t_0 > t_1$. Then $O(a) = O(ab) = 2$ and $FO_\mu(a) = 2$, $FO_\mu(ab) = 1$.*

Definition 36 (Fuzzy Characteristics of Ring). [47] Let η be a fuzzy subring of a ring R. If η(mx) = η(0) for all x in R and the smallest positive integer m, then m is said to be the fuzzy characteristic of the ring R with respect to η. If m = 0 is the only integer such that η(mx) = η(0) for all x in R, then the fuzzy characteristic of ring R is 0 (or infinity).

Example 10. Let η be a fuzzy subring of Z_6 defined by

$$\eta(x) = \begin{cases} 0.9 & \text{if } x \in \{0,3\}, \\ 0.2 & \text{otherwise.} \end{cases}$$

Then η (3x) = η (0) for all x Z_6 . Hence, Z_6 has a fuzzy characteristic of 3.

In general, the fuzzy characteristic is not necessarily equal to the ring characteristic. However, if the set x R η(x) = η(0) = 0, then FCη(R) is equal to the characteristic of the ring R.

Theorem 11. [47] If η is a fuzzy ideal and e is the identity of ring R, then FCη(R) = FOη(e).

Definition 37. *[47] [Fuzzy Zero Divisor] Let η be a fuzzy subring of R, and let a, b ∈ R. If η(ab) = η(0) and η(a) ≠ η(0), η(b) ≠ η(0), then a is referred to as the left fuzzy divisor of b, and b is referred to as the right fuzzy divisor of a with respect to η.*

If $\{x \in R \mid \eta(x) = \eta(0)\} = 0$ then the fuzzy zero divisors with respect to η are equal to zero divisors of ring.

Theorem 12. *[47] Let η be a fuzzy ideal of a ring R, and let a, b ∈ R. If η has no fuzzy zero divisors with respect to η and η(a) = η(0), η(b) = η(0), then $FO_\eta(a) = FO_\eta(b)$ in the additive group of R.*

Theorem 13. *[47] Let η be a fuzzy ideal of a ring R with no fuzzy zero divisors with respect to η and $FC_\eta(R) = 0$. If η (a) η (0) for some a ∈ R, then $FC_\eta(R)$ is prime number.*

Corollary 1. *[47] Let η be a fuzzy ideal of the ring R with no fuzzy zero divisors with respect to η then $FC_\eta(R) = 0$ if and only if $FO_\eta(a)$ is infinite for every a ∈ R with η (a)' = η (0). Otherwise $FC_\eta(R)$ and $FO_\eta(a)$ is prime p for every a ∈ R with η (a) = η (0).*

Definition 38 (Fuzzy Unit of Ring). *[47] Suppose η be a fuzzy subring of R with identity 1 such that 0 ≠ η (1) η (0) . An element a ∈ R with η (a) ≠ η (0) is said to be fuzzy unit of R denoted by $FU_\eta(R)$ if there exist b ∈ R with η (b) ≠ η (0) such that η (ab) = η (ba) = η (1).*

Example 11. *Let η be a subring of Z_6 defined by*

$$\eta(x) = \begin{cases} 1 & \textit{if } x \in \{0, 2, 4\}, \\ 0.5 & \textit{otherwise.} \end{cases}$$

Thus, the element $3 \in \mathbb{Z}_6$ *with* $\eta(3) \neq \eta(0)$, *is a fuzzy unit of* $\mathbb{Z}_6$, *as there exist an element* $5 \in \mathbb{Z}_6$ *such that* $\eta(3 \cdot 5) = \eta(5 \cdot 3) = \eta(15) = \eta(3) = \eta(1)$.

Definition 39. *[42] Let* η_G *and* η_H *be fuzzy subsets of the sets* G *and* H *respectively, then the product of* η_G *and* η_H, *denoted by* $\eta_G \times \eta_H$ *and defined by*

$$(\eta_G \times \eta_H)(x, y) = \min\{\eta_G(x), \eta_H(x)\}$$

for all $x \in G$ *and* $y \in H$.

Theorem 14. *[47] If* η_R *and* η_S *are fuzzy subrings of the rings* R *and* S, *respectively, then*

$$FC_{\eta_R \times \eta_S}(R \times S) = \begin{cases} 0 & \textit{if } FC_{\eta_R}(R) = 0 \textit{ and } FC_{\eta_S}(S) = 0, \\ p & \textit{if } FC_{\eta_R}(R) \neq 0 \textit{ and } FC_{\eta_S}(S) \neq 0, \end{cases}$$

where $p = lcm\{FC_{\eta_R}, FC_{\eta_S}\}$.

Pu and Liu [44] introduced the concept of fuzzy points. Building upon this idea, Melliani [41] proposed the fuzzy ring $(F_\eta(R), +, \times)$ using the notion of fuzzy points, where η represents a fuzzy subring of R. Subsequently, Melliani presented the notion of zero divisors within this fuzzy ring $(F_\eta(R), +, \times)$, termed fuzzy zero divisors. Additionally, he introduced the concept of an integral ring within this framework.

Definition 40 (Fuzzy Point). *[44] Let* A *be a non empty set. For* $x \in A$ *and* $\alpha \in (0, 1]$, *define* $x_\alpha : A \to [0, 1]$ *such that*

$$x_\alpha(y) = \begin{cases} \alpha & \textit{if } x = y, \\ 0 & \textit{if } x \neq y. \end{cases}$$

Then, x_α *is called a fuzzy point(singleton).*

Definition 41. *[44] The fuzzy point* x_α *is said to be contained in a fuzzy set* η *or belongs to* η, *denoted by* $x_\alpha \in \eta$ *iff* $\eta(x) \geq \alpha$. *Generally, every fuzzy set* η *can be expressed as the union of all fuzzy points belonging to* η.

By the principal extension of Zadeh, we have

$$x_t + y_s = (x + y)_{t \wedge s},$$

$$x_t . y_s = (x . y)_{t \wedge s}.$$

The set of all fuzzy points of η of ring R is denoted by $F_\eta(R)$ and defined as

$$F_\eta(R) = \{x_\alpha \mid \eta(x) \geq \alpha, x \in R, \alpha \in (0, 1]\}.$$

Theorem 15. *[41] Let η be a fuzzy subset of R, then η is a fuzzy subring of R if and only if η_t is a subring of R for each $t \in [0, \eta(0)]$.*
Theorem 16. *[41] Let η be a fuzzy subset of R. Then η is a fuzzy subring of R if and only if, for each point $x_t, y_s \in \eta$, we have $x_t - y_s \in \eta$ and $x_t.y_s \in \eta$.*
Theorem 17. *[41] Let R ring with unity. If η is subring of R then $(F_\eta(R), +, \times)$ is ring.*

Proof. Let $x_s, y_t, z_u \in F_\eta(R)$.

(i) **Closure:** As $x_s + y_t = (x+y)_{s\wedge t} \Rightarrow F_\eta(R)$ is closed under addition.
(ii) **Associativity:**

$$\begin{aligned} x_s + (y_t + z_u) &= x_s + (y+z)_{t\wedge u} \\ &= (x + (y+z))_{s\wedge(t\wedge u)} \\ &= ((x+y)+z)_{(s\wedge t)\wedge u} \\ &= (x+y)_{s\wedge t} + z_u \\ &= (x_s + y_t) + z_u \end{aligned}$$

(iii) **Additive identity:**
As $\eta(0) \geq \eta(1) = s$ then $0_s \in F_\eta(R)$.
(iv) **Additive inverse:**
For any $x_t \in F_\eta(R)$, we have $\eta(-x) \geq \eta(x) \geq t$ then $-x_t \in F_\eta(R)$ and $x_t - x_t = (x-x)_t = 0_t$.

Hence $F_\eta(R)$ is additive abelian group. Also, $x_s.y_t = (x.y)_{s\wedge t} \in F_\eta(R)$. Hence $F_\eta(R)$ is closed under multiplication. We have

$$\begin{aligned} x_s.(y_t.z_u) &= x_s.(y.z)_{t\wedge u} \\ &= (x.(y.z))_{s\wedge(t\wedge u)} \\ &= ((x.y).z)_{(s\wedge t)\wedge u} \\ &= (x.y)_{s\wedge t}.z_u \\ &= (x_s.y_t).z_u \end{aligned}$$

.
Thus $F_\eta(R)$ is associative under multiplication.
and

$$\begin{aligned} x_s.(y_t + z_u) &= (x.(y+z))_{t\wedge(s\wedge u)} \\ &= (x.y + x.z)_{t\wedge s\wedge u} \\ &= (x.y)_{t\wedge s} + (x.z)_{t\wedge u} \end{aligned}$$

Hence multiplication is distributive with respect to addition. Hence $F_\eta(R)$ is ring.

Theorem 18. *[41] Let R be a commutative ring with unity. Let η and v be two fuzzy subrings of R such that $\eta \subset v$. Then $F_\eta(R)$ is a subring of $F_v(R)$.*
Definition 42 (Fuzzy Zero Divisor of ring $F_\eta(R)$). *[41]*
Let $a_t(\neq 0_t) \in F_\eta(R)$. Then a_t is called fuzzy zero divisor if there exist $b_s(\neq 0_s) \in F_\eta(R)$ such that $a_t.b_s = 0_{t\wedge s}$.
Definition 43 (Integral Ring). *[41] Ring $F_\eta(R)$ is said to be an integral ring if it has no zero divisors.*
Theorem 19. *[41] $F_\eta(R)$ is integral ring iff R is an integral domain.*

Ayub [7] presented an additional approach to fuzzy zero divisors, and subsequently, the concept of a fuzzy integral domain is explored.
Definition 44 (Fuzzy Zero Divisor). *[7] Let η be a fuzzy subring of R. A non zero and non unit element a is called fuzzy zero divisor over η if there exist non zero and non unit element b such that $\eta(ab) = \eta(0)$.*
Example 12. *Let η be a fuzzy subring of Z defined by*

$$\eta(x) = \begin{cases} 0.7 & \text{if } x = 2Z, \\ 0.2 & \text{otherwise.} \end{cases}$$

Then, $\{2x \mid x \in \mathbb{Z}\}$ *is the set of fuzzy zero divisors of* $\mathbb{Z}$.

Example 13. *If* η *is a fuzzy subring of R, then every zero divisor of R is a fuzzy zero divisor over* η*. However, the converse is not valid (refer Example 12) .*

Definition 45 (Fuzzy Integral Domain). *[7] Let R be an integral domain, and* η *is a fuzzy subring of R. Then* η *is called the fuzzy integral domain of R if there is no fuzzy zero divisor over* η*. The set of all fuzzy integral domains of R is denoted by FID* (R).

From now, R is assumed to be an integral domain. R^* denotes the set of all units of R. Also $\eta_* = \{x \in R \mid \eta(x) = \eta(0)\}$.

Theorem 20. *[7] Let* η *be a fuzzy subring of R. Then the following conditions are equivalent*

(i) $\eta \in FID(R)$.

(ii) $\eta(x) \neq \eta(0)$ *for all non zero and non unit* $x \in R$.

(iii) $\eta_* \subseteq \{0\} \cup R^*$.

Example 14. *Let* $\eta : \mathbb{Q}[x] \to [0, 1]$ *be a fuzzy subring of* $\mathbb{Q}[x]$ *such that*

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}, \\ 0.5 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

Clearly, η *is a fuzzy subring of* $\mathbb{Q}$ *and* $\eta_* \subseteq \{0\} \cup \mathbb{Q}^*[x]$*. Therefore,* η *is a fuzzy integral domain of* $\mathbb{Q}$.

6. Applications

A fuzzy set proves to be a valuable tool in decision-making processes. Sun [22] introduced a multi-level comprehensive fuzzy evaluation function that analyzes envi- ronmental issues, management skills, and economic benefits along with biophysical suitability. This function aids in making informed recommendations for land usage. The practical applications of fuzzy set theory extend into the medical field. Sanchez [51] developed a diagnostic model using a fuzzy matrix, which encapsulates medi- cal knowledge regarding symptoms and diseases. Meenakshi [40] further expanded on Sanchez's approach by presenting a theory of interval-valued fuzzy matrices, providing an extension to medical diagnosis methods. Meenakshi also introduced the arithmetic mean matrix of an interval-valued fuzzy matrix and directly applied Sanchez's med- ical diagnosis method to it. Kavitha [29] introduced the concept of a circulant fuzzy matrix and demonstrated its application in animal disease diagnosis.

The concept of a fuzzy zero divisor can be used to define a fuzzy zero divisor graph. Kuppan [36] initiated the study of fuzzy zero divisor graphs, which have various appli- cations in graph theory. One significant application is in pharmaceutical chemistry, where the challenge lies in providing a mathematical graphical representation for a set of chemical compounds such that distinct representations correspond to distinct com- pounds. Additionally, in computer networks, the structure of fuzzy zero divisor graphs can be applied, representing servers, hubs, and nodes as vertices and connections as edges. Moreover, fuzzy zero divisor graphs have potential applications in navigation, robotics, and coding theory.

This section presents a comprehensive examination of two case studies, centered on medical diagnosis and agricultural analysis. The initial study, conducted by Beaula [12], introduces an algorithm designed for medical diagnosis. Subsequently, the second case study elaborates on the practical implementation of (, Q) fuzzy ideals. This approach aids in the selection of effective symptoms and streamlines the diagnosis of diseases [31].

6.1 Case Study 1

Fuzzy algebra finds significant applications in the field of medicine, particularly in diagnostic models. Sanchez [52] pioneered the use of fuzzy matrices to represent medical knowledge relating symptoms to diseases. Building on this work, Meenakshi and Kali- raja [40] further developed the approach by introducing interval-valued fuzzy matrices. In 2010, Cagman et al. [14] introduced fuzzy soft matrix theory and its application in decision making.

This section presents a case study by Beaula [12] that illustrates the practical application of fuzzy matrices in medical diagnosis.

Definition 46 (Fuzzy Composition). *[12] Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ be the fuzzy matrices of order n× n. Then the sup-i composition is defined as, $A^i\ B =$* sup *$i\{A(x, y)\, B(x, y)\ \}$*

Definition 47 (Relativity Function). *[12] A relativity function between variables u and $v \in X$, denoted by $g\ \frac{u}{v}$, is defined as:*

$$g\ \frac{u}{v} = \frac{\eta_v(u) - \eta_u(v)}{\max\{\eta_v(u), \eta_u(v)\}}$$

where $\eta_v(u)$ and $\eta_u(v)$ are membership functions mapping from u to v and v to u, respectively.

6.1.1 Procedure of medical diagnosis under a fuzzy environment

Let S denote the set of symptoms associated with certain diseases, D represent the set of diseases, and P the set of patients.

1. Begin by considering the patient-symptom fuzzy matrix A.
2. Next, consider the symptom-disease fuzzy matrix B.
3. Compute $C = A^i \circ B$, where $\circ$ represents the sup i-composition operation.
4. Find the complement fuzzy matrices A^c and B^c for A and B, respectively.
5. Calculate $D = A^{ci} \circ\ B^c$ and $M = C - D$, using $-$ for the min operator and $\circ$ for the sup i–composition operation.
6. Determine the relative values and construct the comparison matrix $R = [r_{ij}]_{n\times n} = g\ \frac{x_i}{y_i}\ _{i=1,2,\ldots n}$.
7. Identify the maximum value in each row of the R matrix for ranking purposes, providing the solution to the problem at hand.

6.1.2 Illustrative Example

Let $P = \{P_1, P_2, P_3\}$ be the set of patients with symptoms $S = \{\text{High Temp.}\,(s_1), \text{Headache}\,(s_2), \text{Cough}\,(s_3)\}$, and possible diseases related to these symptoms $D = \{\text{Dengue}\,(d_1), \text{Viral Fever}\,(d_2), \text{Malaria}\,(d_3)\}$.

1. Consider the patient-symptom fuzzy matrix:

$$A = \begin{array}{c|ccc} & d_1 & d_2 & d_3 \\ \hline s_1 & 0.5 & 0.6 & 0.4 \\ s_2 & 0.2 & 0.7 & 0.3 \\ s_3 & 0.6 & 0.4 & 0.7 \end{array}$$

2. Consider the symptom-disease fuzzy matrix:

$$B = \begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline p_1 & 0.4 & 0.2 & 0.7 \\ p_2 & 0.3 & 0.6 & 0.9 \\ p_3 & 0.7 & 0.5 & 0.8 \end{array}$$

3. Compute $C = A^i \circ\ B$, where $\circ$ is sup i– composition:

$$C = \begin{array}{c|ccc} & d_1 & d_2 & d_3 \\ \hline p_1 & 0.3 & 0.4 & 0.5 \\ p_2 & 0.2 & 0.4 & 0.6 \\ p_3 & 0.5 & 0.4 & 0.6 \end{array}$$

4. The complement of fuzzy matrices A and B are: $A^{c} = \begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline p_1 & 0.5 & 0.4 & 0.6 \\ p_2 & 0.8 & 0.3 & 0.7 \\ p_3 & 0.4 & 0.6 & 0.3 \end{array}$,

$$B^{c} = \begin{array}{c|ccc} & d_1 & d_2 & d_3 \\ \hline s_1 & 0.6 & 0.8 & 0.3 \\ s_2 & 0.7 & 0.4 & 0.1 \\ s_3 & 0.3 & 0.5 & 0.2 \end{array}$$

5. Compute $D = A^{ci} \circ B^{c}$:

$$D = \begin{array}{c|ccc} & d_1 & d_2 & d_3 \\ \hline p_1 & 0.3 & 0.4 & 0.2 \\ p_2 & 0.5 & 0.6 & 0.2 \\ p_3 & 0.4 & 0.3 & 0.1 \end{array}$$

and $M = C - D = \begin{array}{c|ccc} & d_1 & d_2 & d_3 \\ \hline p_1 & 0.3 & 0.4 & 0.2 \\ p_2 & 0.2 & 0.4 & 0.2 \\ p_3 & 0.4 & 0.3 & 0.1 \end{array}$

6. Calculate relative values $g_{d_i}^{p_i}$ for all i,

$$g\frac{p_1}{d_1} = \frac{\eta_{d_1}(p_1) - \eta_{p_1}(d_1)}{\max\{\eta_{d_1}(p_1), \eta_{p_1}(d_1)\}} = \frac{0.3 - 0.3}{\max\{0.3, 0.3\}} = 0$$

Similarly we can calculate, $g_{d_2}^{p_1} = 0.5$, $g\frac{p_1}{d_3} = -0.5$, $g\frac{p_2}{d_1} = -0.5$, $g\frac{p_2}{d_2} = 0$, $g\frac{p_2}{d_3} = -0.6$, $g\frac{p_3}{d_1} = 0.5$, $g\frac{p_3}{d_2} = 0.3$ and $g\frac{p_3}{d_3} = 0$.

Hence, $R = \begin{bmatrix} 0 & 0.5 & -0.5 \\ -0.5 & 0 & -0.6 \\ 0.5 & 0.3 & 0 \end{bmatrix}$ and the maximum i^{th} row is $\begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}$

7. Therefore, it can be concluded that patient P_1 is highly susceptible to **Viral Fever**, patient P_2 is affected by **Viral Fever**, and patient P_3 is likely affected by **Dengue**.

6.2 Case Study 2

The theory of $(\in, \in\vee q)$ fuzzy ideals is employed in the domain of medical diagnosis systems. This section details the practical application of $(\in, \in\vee q)$ fuzzy ideals for the purpose of selecting effective symptoms and facilitating the diagnosis of diseases [31]. The conclusions drawn from the analysis of 15 cases are summarized to underscore the robustness and effectiveness of the application in medical diagnostics.

Definition 48. *In the defined operations, where $a \vee b$ denotes the supremum (least upper bound) and $a \wedge b$ denotes the infimum (greatest lower bound), a partially ordered set (poset) $(L; \leqslant)$ is termed a lattice if, for any elements a and b in L, both $a \vee b$ and $a \wedge b$ exist within the set L.*

Definition 49. *A fuzzy subset μ is said to be an $(\in, \in\vee q)$ fuzzy ideal of X if and only if the following conditions are hold for all $x, y \in X$.*

(i) $\mu(x \wedge y) \geqslant \mu(x) \wedge \mu(y) \wedge 0.5$

(ii) $\mu(x \vee y) \geqslant \mu(x) \wedge \mu(y) \wedge 0.5$

(iii) $x \wedge a = x$ *implies* $\mu(x) \geqslant \mu(a) \wedge 0.5$ *for all* $a \in X$.

A sixty-year-old male patient presenting with a complaint of burning micturation, the doctor considered potential underlying conditions, including Urinary Tract Infection (UTI) (P1), Renal Stone (P2), and Diabetes (P3). To confirm the diagnosis, the doctor recommended a comprehensive evaluation through blood tests, urine analysis, and an abdominal ultrasound (USG). Subsequently, the obtained reports were subjected to a process of fuzzification for further analysis and interpretation.

SN.	Test	Values	Normal Values	A_i	τ_1 (P_1)	τ_2 (P_2)	τ_2 (P_3)	$\mu(A_i)$	λ_1 $(\tau_1(P_1))$	λ_2 $(\tau_2(P_2))$	λ_3 $(\tau_3(P_3))$
1	Haemoglobin (gm %)	6.50	13-15	0.81	0.2	0	0.1	0.81	0.5	0.5	0.5
2	ESR mm	34	0-14	0.77	0.2	0.1	0.2	0.77	0.5	0.5	0.5
3	Urea (mg/dl)	27.50	20-40	0	0	0	0	0.5	0.5	0.5	0.5
4	RBSL (mb/dl)	92.50	90-110	0	0	0	0	0.5	0.5	0.5	0.5
5	Calcium (mg %)	7.69	9-10.4	0.33	0.1	0	0.1	0.5	0.5	0.5	0.5
6	Urine(R) protiens	Trace	Trace/Absent	0	0	0	0	0.5	0.5	0.5	0.5
7	Urine culture	Abnormal*	Normal/Abnormal	1	1	0	0	1	1	0.5	0.5
8	USG abdomen	Abnormal**	Normal/Abnormal	1	0.8	0	0	1	0.8	0.5	0.5

* Abnormal due to organism
** Abnormal due to internal echoes
The fuzzy sets defined for various test values are given below,

$$A_1(x) = \begin{cases} 1 & \text{if } x < 5 \\ \frac{13-x}{8} & \text{if } 5 \leqslant x \leqslant 13 \\ 0 & \text{if } x > 13 \end{cases}, \quad A_2(x) = \begin{cases} 0 & \text{if } x < 14 \\ \frac{x-14}{26} & \text{if } 14 \leqslant x \leqslant 40 \\ 1 & \text{if } x > 40 \end{cases},$$

$$A_3(x) = \begin{cases} 0 & \text{if } x < 40 \\ \frac{x-40}{60} & \text{if } 40 \leqslant x \leqslant 100 \\ 1 & \text{if } x > 100 \end{cases}, \quad A_4(x) = \begin{cases} 0 & \text{if } x < 110 \\ \frac{x-110}{90} & \text{if } 110 \leqslant x \leqslant 200 \\ 1 & \text{if } x > 200 \end{cases},$$

$$A_5(x) = \begin{cases} 1 & \text{if } x < 5 \\ \frac{9-x}{4} & \text{if } 5 \leqslant x \leqslant 9 \\ 0 & \text{if } x > 9 \end{cases}, \quad A_6(x) = \begin{cases} 1 & \text{if absent} \\ 0 & \text{if traced} \end{cases},$$

$$A_7(x) = \begin{cases} 1 & \text{if abnormal} \\ 0 & \text{if normal} \end{cases}, \quad A_8(x) = \begin{cases} 1 & \text{if abnormal} \\ 0 & \text{if normal} \end{cases},$$

The values $(\tau_i(P_i))_j$ for i = 1 to 3 and j = 1 to 8 represent the grade membership values provided by experts. These values indicate the degree to which the test is deemed useful in identifying diseases, as assessed by experts relying on their knowledge and experience.

Define fuzzy sets $\mu(A_j(x)) = A_j(x) \vee 0.5$ and $\lambda\ \tau_i(P_i)_j = \tau_i(P_i)_j \vee 0.5$ for all x and i = 1 to 3 and j = 1 to 8. Clearly μ and λ are $(\in, \in \vee q)$ -fuzzy ideals of X. Hence $\mu \cap \lambda$ is $(\in, \in \vee q)$ -fuzzy ideals of X.

Define $\gamma_i(P_i) = \bigvee_{j=1}^{8} \mu(A_j) \wedge \lambda(P_i)_j$ for all i = 1 to 3 and j = 1 to 8. Thus $\gamma_1(P_1) = 1$, $\gamma_2(P_2) = 0.1$ and $\gamma_3(P_3) = 0.2$.

Hence, the analysis indicates that urinary tract infection (UTI) has the highest grade of membership, leading to the conclusion that the patient is indeed experiencing a urinary tract infection. This finding aligns with the medical opinion provided by the doctor.

7. Limitation of study

This study faces limitations due to the absence of a standardized process for the fuzzification of algebraic structures. These limitations encompass several challenges associated with the process of fuzzification:

1. Complexity of transition: Converting crisp concepts into fuzzy ones while pre- serving their essential properties poses a significant challenge. This complexity stems from the necessity to maintain the core characteristics of the original concept while adapting it to a fuzzy framework.

2. Preservation of meaning: Fuzzifying a concept from a classical set to a fuzzy set without altering its meaning in the classical set is a difficult task.

3. Computational complexity: The fuzzification process often involves intricate computations, potentially impacting the scalability and efficiency of fuzzy systems. Managing this computational complexity is crucial to ensure the practicality and usability of fuzzy algebraic techniques.

4. Loss of precision: Fuzzification entails approximating crisp concepts with fuzzy ones, leading to a loss of precision. This loss of precision can affect the accuracy of computations and the reliability of results obtained using fuzzy algebraic structures.

5. Limited theoretical framework: The theoretical framework for fuzzy algebraic structures is still under development. This limited theoretical foundation can make it challenging to rigorously analyze and prove properties of fuzzy algebraic structures.
Addressing these limitations involves ongoing research and development efforts aimed at refining fuzzy algebraic techniques, enhancing computational methods, and establishing standardized approaches to ensure consistency and reliability in fuzzy systems.

8. Conclusion and Future Scope

A comprehensive study was conducted by analyzing research papers published in reputable journals from 1965 to 2024, focusing on the structures and applications of fuzzy subrings, fuzzy zero divisors, fuzzy polynomials, and fuzzy matrices. The examination of these papers provided valuable insights into the evolution of these concepts and their current state in research, contributing to a deeper understanding of fuzzy sets' role in ring theory.
The analysis revealed various approaches for defining and studying fuzzy subrings, fuzzy zero divisors, fuzzy polynomials, and fuzzy matrices, highlighting the complexity and challenges involved in fuzzifying algebraic structures. The study emphasized the need for standardized processes in fuzzification to ensure consistency and reliability in fuzzy systems.
Furthermore, the research highlighted the practical applications of fuzzy sub- rings/ideals and fuzzy matrices in agriculture and the medical field. These applications demonstrate the potential impact of fuzzy algebraic techniques in real-world scenarios, opening up new avenues for research and development in these fields.
In conclusion, this survey offers valuable insights into recent advances in the study of fuzzy algebraic structures and their practical applications. The findings have the potential to enrich the exploration of fuzzification in the algebraic properties of rings, particularly in areas related to zero divisors. These areas include the analysis of zero divisor graphs, the study of annihilator properties of rings, the investigation of Baer rings, principal projective rings, and projective socle rings, as well as the develop- ment of error-correcting codes in coding theory. By refining fuzzy algebraic techniques, improving computational methods, and establishing standardized approaches for fuzzification, future research holds promise for further advancing the field and expanding its practical applications.
This study lays the groundwork for future research in fuzzy algebra and its applications, providing a foundation for exploring new theoretical frameworks and practical implementations in diverse fields.

Ethics declarations

Conflicts of interest

The authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent

This article does not contain any studies with human participants, hence no informed consent is not declared.

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