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INTEGRAL INEQUALITIES FOR ASYMMETRIZED SYNCHRONOUS FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. In this paper we establish some integral inequalities for the product of asymmetrized synchronous/asynchronous functions. Some examples for integrals of monotonic functions, including power, logarithmic and sin functions are also provided.

1. Introduction

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{1.1}$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],t\in[a,b].$$

It is obvious that for any function f we have $\breve{f}+\tilde{f}=f$.

If f is convex on $[a,b]$, then for any $t_1,\ t_2\in[a,b]$ and $\alpha,\beta\geq0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\breve{f}(\alpha t_1+\beta t_2)&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(a+b-\alpha t_1-\beta t_2)\right]\\&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(\alpha(a+b-t_1)+\beta(a+b-t_2))\right]\\&\leq\frac{1}{2}\left[\alpha f(t_1)+\beta f(t_2)+\alpha f(a+b-t_1)+\beta f(a+b-t_2)\right]\\&=\frac{1}{2}\alpha\left[f(t_1)+f(a+b-t_1)\right]+\frac{1}{2}\beta\left[f(t_2)+f(a+b-t_2)\right]\\&=\alpha\breve{f}(t_1)+\beta\breve{f}(t_2),\end{aligned}$$

which shows that $\breve{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=t^3$. We have [6]

$$\breve{f}_0(t):=\frac{1}{2}\left[t^3+(a+b-t)^3\right]=\frac{3}{2}(a+b)t^2-\frac{3}{2}(a+b)^2t+\frac{1}{2}(a+b)^3$$

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for any $t \in \mathbb{R}$.

Since the second derivative $\left(\breve{f}_0\right)''(t) = 3(a+b)$, $t \in \mathbb{R}$, then $\breve{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2} > 0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2} < 0$. Therefore if $a < 0 < b$ with $\frac{a+b}{2} > 0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\breve{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity [6], see also [9] for an equivalent definition.

Definition 1. *We say that the function $f : [a,b] \to \mathbb{R}$ is symmetrized convex (concave) on the interval $[a,b]$ if the symmetrical transform $\breve{f}$ is convex (concave) on $[a,b]$.*

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the closed convex cone of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a,b] \subsetneq SCon[a,b]. \tag{1.2}$$

Also, if $[c,d] \subset [a,b]$ and $f \in SCon[a,b]$, then this does not imply in general that $f \in SCon[c,d]$.

We have the following result [6], [9] :

Theorem 1. *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex and integrable on the interval $[a,b]$. Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

We also have [6]:

Theorem 2. *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x \in [a,b]$ we have the bounds*

$$f\left(\frac{a+b}{2}\right) \le \breve{f}(x) \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

For a monograph on Hermite-Hadamard type inequalities see [8].

In a similar way, we can introduce the following concept as well:

Definition 2. *We say that the function $f : [a,b] \to \mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a,b]$ if the anti-symmetrical transform $\tilde{f}$ is monotonic nondecreasing (nonincreasing) on the interval $[a,b]$.*

If f is monotonic nondecreasing on $[a,b]$, then for any $t_1, t_2 \in [a,b]$ we have

$$\begin{aligned}\tilde{f}(t_2) - \tilde{f}(t_1) &= \frac{1}{2}[f(t_2) - f(a+b-t_2)] - \frac{1}{2}[f(t_1) - f(a+b-t_1)] \\ &= \frac{1}{2}[f(t_2) - f(t_1)] + \frac{1}{2}[f(a+b-t_1) - f(a+b-t_2)] \\ &\ge 0,\end{aligned}$$

which shows that $f : [a,b] \to \mathbb{R}$ is asymmetrized monotonic nondecreasing on the interval $[a,b]$.

Consider the real numbers $a < b$ and define the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = t^2$. We have

$$\tilde{f}_0(t) := \frac{1}{2}\left[t^2 - (a+b-t)^2\right] = (a+b)t - \frac{1}{2}(a+b)^2$$

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and $\left(\tilde{f}_0\right)'(t) = a + b$, therefore $f : [a, b] \to \mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a, b]$ provided $\frac{a+b}{2} > 0\,(< 0)$. So, if we take $a < 0 < b$ with $\frac{a+b}{2} > 0$, then f is asymmetrized monotonic nondecreasing on $[a, b]$ but not monotonic nondecreasing on $[a, b]$.

If we denote by $\mathcal{M}^{\nearrow}[a, b]$ the closed convex cone of monotonic nondecreasing functions defined on $[a, b]$ and by $\mathcal{AM}^{\nearrow}[a, b]$ the closed convex cone of asymmetrized monotonic nondecreasing functions, then from the above remarks we can conclude that

$$\mathcal{M}^{\nearrow}[a, b] \subsetneq \mathcal{AM}^{\nearrow}[a, b]. \tag{1.5}$$

Also, if $[c, d] \subset [a, b]$ and $f \in \mathcal{AM}^{\nearrow}[a, b]$, then this does not imply in general that $f \in \mathcal{AM}^{\nearrow}[c, d]$.

We recall that the pair of functions (f, g) defined on $[a, b]$ are called *synchronous (asynchronous)* on $[a, b]$ if

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \tag{1.6}$$

for any $t,\ s \in [a, b]$. It is clear that if both functions (f, g) are monotonic nondecreasing (nonincreasing) on $[a, b]$ then they are synchronous on $[a, b]$. There are also functions that change monotonicity on $[a, b]$, but as a pair they are still synchronous. For instance if $a < 0 < b$ and $f, g : [a, b] \to \mathbb{R}$, $f(t) = t^2$ and $g(t) = t^4$, then

$$(f(t) - f(s))(g(t) - g(s)) = \left(t^2 - s^2\right)\left(t^4 - s^4\right) = \left(t^2 - s^2\right)^2\left(t^2 + s^2\right) \geq 0$$

for any $t,\ s \in [a, b]$, which show that (f, g) is synchronous.

Definition 3. *We say that the pair of functions (f, g) defined on $[a, b]$ is called asymmetrized synchronous (asynchronous) on $[a, b]$ if the pair of transforms $\left(\tilde{f}, \tilde{g}\right)$ is synchronous (asynchronous) on $[a, b]$, namely*

$$\left(\tilde{f}(t) - \tilde{f}(s)\right)(\tilde{g}(t) - \tilde{g}(s)) \geq (\leq)\, 0 \tag{1.7}$$

for any $t, s \in [a, b]$.

It is clear that if f, g are asymmetrized monotonic nondecreasing (nonincreasing) on $[a, b]$ then they are asymmetrized synchronous on $[a, b]$.

One of the most important results for synchronous (asynchronous) and integrable functions f, g on $[a, b]$ is the well-known *Čebyšev's inequality*:

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \geq (\leq)\, \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1.8}$$

For integral inequalities of Čebyšev's type, see [1]-[5], [7], [10]-[18] and the references therein.

Motivated by the above results, we establish in this paper some inequalities for asymmetrized synchronous (asynchronous) functions on $[a, b]$. Some examples for power, logarithm and sin functions are provided as well.

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2. Main Results

We have the following result:

Theorem 3. *Assume that f, g are asymmetrized synchronous (asynchronous) and integrable functions on $[a,b]$. Then*

$$\int_a^b \tilde{f}(t)\, g(t)\, dt \geq (\leq)\, 0. \tag{2.1}$$

Proof. We consider only the case of symmetrized synchronous and integrable functions.

1. By the Čebyšev's inequality (1.8) for $\left(\tilde{f}, \tilde{g}\right)$ we get

$$\frac{1}{b-a}\int_a^b \tilde{f}(t)\, \tilde{g}(t)\, dt \geq \frac{1}{b-a}\int_a^b \tilde{f}(t)\, dt \frac{1}{b-a}\int_a^b \tilde{g}(t)\, dt. \tag{2.2}$$

We have

$$\int_a^b \tilde{f}(t)\, dt = \frac{1}{2}\left[\int_a^b f(t)\, dt - \int_a^b f(a+b-t)\, dt\right] = 0$$

since, by the change of variable $s = a+b-t$, $t \in [a,b]$,

$$\int_a^b f(a+b-t)\, dt = \int_a^b f(s)\, ds.$$

Also,

$$\begin{aligned}
\int_a^b \tilde{f}(t)\, \tilde{g}(t) &= \frac{1}{4}\int_a^b [f(t) - f(a+b-t)]\,[g(t) - g(a+b-t)]\, dt \\
&= \frac{1}{4}\int_a^b [f(t)\, g(t) + f(a+b-t)\, g(a+b-t)]\, dt \\
&\quad - \frac{1}{4}\int_a^b [f(t)\, g(a+b-t) + f(a+b-t)\, g(t)]\, dt \\
&= \frac{1}{4}\left[\int_a^b f(t)\, g(t)\, dt + \int_a^b f(a+b-t)\, g(a+b-t)\, dt\right] \\
&\quad - \frac{1}{4}\left[\int_a^b f(t)\, g(a+b-t)\, dt + \int_a^b f(a+b-t)\, g(t)\, dt\right] \\
&= \frac{1}{2}\left(\int_a^b f(t)\, g(t)\, dt - \int_a^b f(a+b-t)\, g(t)\, dt\right) \\
&= \int_a^b \tilde{f}(t)\, g(t)\, dt
\end{aligned} \tag{2.3}$$

since, by the change of variable $s = a+b-t$, $t \in [a,b]$, we have

$$\int_a^b f(a+b-t)\, g(a+b-t)\, dt = \int_a^b f(t)\, g(t)\, dt$$

and

$$\int_a^b f(t)\, g(a+b-t)\, dt = \int_a^b f(a+b-t)\, g(t)\, dt.$$

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By (2.2) we then get the desired result (2.1).

2. An alternative proof is as follows. Since $\left(\tilde{f},\tilde{g}\right)$ are synchronous, then

$$\left[\tilde{f}\left(t\right)-\tilde{f}\left(\frac{a+b}{2}\right)\right]\left[\tilde{g}\left(t\right)-\tilde{g}\left(\frac{a+b}{2}\right)\right]\geq 0$$

for any $t\in\left[a,b\right]$, which is equivalent to

$$\tilde{f}\left(t\right)\tilde{g}\left(t\right)\geq 0 \text{ for any } t\in\left[a,b\right], \tag{2.4}$$

or to

$$\left[f\left(t\right)-f\left(a+b-t\right)\right]\left[g\left(t\right)-g\left(a+b-t\right)\right]\geq 0 \text{ for any } t\in\left[a,b\right].$$

This is a property of interest for asymmetrized synchronous functions.

If we integrate the inequality (2.4) and use the identity (2.3) we get the desired result (2.1). □

Remark 1. *The inequality (2.1) can be written in an equivalent form as*

$$\int_a^b f\left(t\right)g\left(t\right)dt\geq\int_a^b f\left(a+b-t\right)g\left(t\right)dt,$$

or as

$$\int_a^b f\left(t\right)g\left(t\right)dt\geq\int_a^b \breve{f}\left(t\right)g\left(t\right)dt.$$

Theorem 4. *If both f, g are asymmetrized monotonic nondecreasing (nonincreasing) and integrable functions on $\left[a,b\right]$, then*

$$\frac{1}{4}\left|f\left(b\right)-f\left(a\right)\right|\left|g\left(b\right)-g\left(a\right)\right|\geq\frac{1}{b-a}\int_a^b\tilde{f}\left(t\right)g\left(t\right)dt\geq 0, \tag{2.5}$$

and

$$\frac{1}{2}\min\left\{\left|f\left(b\right)-f\left(a\right)\right|\frac{1}{b-a}\int_a^b\left|g\left(t\right)\right|dt,\left|g\left(b\right)-g\left(a\right)\right|\frac{1}{b-a}\int_a^b\left|f\left(t\right)\right|dt\right\} \tag{2.6}$$
$$\geq\frac{1}{b-a}\int_a^b\tilde{f}\left(t\right)g\left(t\right)dt\geq 0.$$

Proof. Assume that both f, g are asymmetrized monotonic nondecreasing and integrable functions on $\left[a,b\right]$, then they are asymmetrized synchronous and by (2.1) we get the second inequality in (2.5).

We also have

$$\tilde{f}\left(a\right)\leq\tilde{f}\left(t\right)\leq\tilde{f}\left(b\right)$$

for any $t\in\left[a,b\right]$, namely

$$-\frac{1}{2}\left[f\left(b\right)-f\left(a\right)\right]\leq\frac{1}{2}\left[f\left(t\right)-f\left(a+b-t\right)\right]\leq\frac{1}{2}\left[f\left(b\right)-f\left(a\right)\right],$$

for any $t\in\left[a,b\right]$, which implies that $\frac{1}{2}\left[f\left(b\right)-f\left(a\right)\right]\geq 0$ and

$$\frac{1}{2}\left|f\left(t\right)-f\left(a+b-t\right)\right|\leq\frac{1}{2}\left[f\left(b\right)-f\left(a\right)\right] \tag{2.7}$$

for any $t\in\left[a,b\right]$.

Similarly, we have $\frac{1}{2}\left[g\left(b\right)-g\left(a\right)\right]\geq 0$ and

$$\frac{1}{2}\left|g\left(t\right)-g\left(a+b-t\right)\right|\leq\frac{1}{2}\left[g\left(b\right)-g\left(a\right)\right] \tag{2.8}$$

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for any $t \in [a, b]$.

If we multiply (2.7) and (2.8), then we get

$$\begin{aligned}\frac{1}{4}&[f(t) - f(a+b-t)][g(t) - g(a+b-t)] \\ &= \frac{1}{4}|[f(t) - f(a+b-t)][g(t) - g(a+b-t)]| \\ &\leq \frac{1}{4}[f(b) - f(a)][g(b) - g(a)]\end{aligned} \tag{2.9}$$

for any $t \in [a, b]$.

Since

$$\begin{aligned}0 &\leq \int_a^b \tilde{f}(t) g(t) dt = \frac{1}{4}\int_a^b [f(t) - f(a+b-t)][g(t) - g(a+b-t)] dt \\ &\leq \frac{1}{4}[f(b) - f(a)][g(b) - g(a)](b-a),\end{aligned}$$

where for the last inequality we used (2.9), hence we get the first inequality in (2.5).

Also, we observe that

$$0 \leq \int_a^b \tilde{f}(t) g(t) dt = \int_a^b \left|\tilde{f}(t) g(t)\right| dt \leq \frac{1}{2}[f(b) - f(a)]\int_a^b |g(t)| dt$$

and since

$$\int_a^b \tilde{f}(t) g(t) dt = \int_a^b f(t) \tilde{g}(t) dt,$$

then also

$$\int_a^b f(t) \tilde{g}(t) dt \leq \frac{1}{2}[g(b) - g(a)]\int_a^b |f(t)| dt$$

and the inequality (2.6) is also proved. □

Remark 2. *If the functions f, $g : [a, b] \to \mathbb{R}$ are either both of them nonincreasing or nondecreasing on $[a, b]$, then they are integrable and we have the inequalities (2.5) and (2.6).*

We have the following refinement of the inequality in (2.1).

Theorem 5. *Assume that f, g are asymmetrized synchronous and integrable functions on $[a, b]$. Then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \tilde{f}(t) g(t) dt \\ &\geq \left|\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right| |\tilde{g}(t)| dt - \frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right| dt \frac{1}{b-a}\int_a^b |\tilde{g}(t)| dt\right| \geq 0.\end{aligned} \tag{2.10}$$

Proof. By the continuity property of modulus, we have

$$\begin{aligned}\left[\tilde{f}(t) - \tilde{f}(s)\right][\tilde{g}(t) - \tilde{g}(s)] &= \left|\left[\tilde{f}(t) - \tilde{f}(s)\right][\tilde{g}(t) - \tilde{g}(s)]\right| \\ &= \left|\tilde{f}(t) - \tilde{f}(s)\right| |\tilde{g}(t) - \tilde{g}(s)| \\ &\geq \left|\left|\tilde{f}(t)\right| - \left|\tilde{f}(s)\right|\right| |\tilde{g}(t) - \tilde{g}(s)| \\ &= \left|\left(\left|\tilde{f}(t)\right| - \left|\tilde{f}(s)\right|\right)(\tilde{g}(t) - \tilde{g}(s))\right|\end{aligned}$$

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for any $t, s \in [a, b]$.

Taking the double integral mean on $[a, b]^2$ and using the properties of the integral versus the modulus, we have

$$\begin{aligned} &\frac{1}{(b-a)^2}\int_a^b\int_a^b \left[\tilde{f}(t)-\tilde{f}(s)\right][\tilde{g}(t)-\tilde{g}(s)]\,dtds \\ &\geq \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\left|\tilde{f}(t)\right|-\left|\tilde{f}(s)\right|\right)(|\tilde{g}(t)|-|\tilde{g}(s)|)\,dtds\right|. \end{aligned} \tag{2.11}$$

Since, by Korkine's identity we have

$$\begin{aligned} &\frac{1}{(b-a)^2}\int_a^b\int_a^b \left[\tilde{f}(t)-\tilde{f}(s)\right][\tilde{g}(t)-\tilde{g}(s)]\,dtds \\ &= 2\left[\frac{1}{b-a}\int_a^b \tilde{f}(t)\,\tilde{g}(t)\,dt - \frac{1}{b-a}\int_a^b \tilde{f}(t)\,dt\frac{1}{b-a}\int_a^b \tilde{g}(t)\,dt\right] \\ &= \frac{2}{b-a}\int_a^b \tilde{f}(t)\,\tilde{g}(t)\,dt \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\left|\tilde{f}(t)\right|-\left|\tilde{f}(s)\right|\right)(|\tilde{g}(t)|-|\tilde{g}(s)|)\,dtds \\ &= 2\left[\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right||\tilde{g}(t)|\,dt - \frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right|dt\frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt\right], \end{aligned}$$

hence by (2.11) we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \tilde{f}(t)\,\tilde{g}(t)\,dt \\ &\geq \left|\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right||\tilde{g}(t)|\,dt - \frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right|dt\frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt\right|. \end{aligned}$$

By using the identity (2.3) we get the desired result (2.10). □

Remark 3. *We remark that, if* $\left(\tilde{f}, g\right)$ *are synchronous, then by a similar argument to the one above for* $g \leftrightarrow \tilde{g}$ *we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt \\ &\geq \left|\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right||g(t)|\,dt - \frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right|dt\frac{1}{b-a}\int_a^b |g(t)|\,dt\right| \geq 0. \end{aligned} \tag{2.12}$$

Also, since

$$\frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt = \frac{1}{b-a}\int_a^b f(t)\,\tilde{g}(t)\,dt,$$

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then if we assume that $(f, \tilde{g})$ *are synchronous we also have*

$$\frac{1}{b-a}\int_a^b \tilde{f}(t)\, g(t)\, dt \geq \left|\frac{1}{b-a}\int_a^b |f(t)|\,|\tilde{g}(t)|\, dt - \frac{1}{b-a}\int_a^b |f(t)|\, dt \frac{1}{b-a}\int_a^b |\tilde{g}(t)|\, dt\right| \geq 0. \tag{2.13}$$

Now, if f *and* g *have the same monotonicity, then* $\left(\tilde{f}, \tilde{g}\right)$, $\left(\tilde{f}, g\right)$, $(f, \tilde{g})$ *are synchronous and we have*

$$\frac{1}{b-a}\int_a^b \tilde{f}(t)\, g(t)\, dt \geq \max\left\{\left|C\left(\tilde{f}, \tilde{g}\right)\right|, \left|C\left(\tilde{f}, g\right)\right|, |C(f, \tilde{g})|\right\} \geq 0, \tag{2.14}$$

where

$$C(h, \ell) := \frac{1}{b-a}\int_a^b |h(t)\, \ell(t)|\, dt - \frac{1}{b-a}\int_a^b |h(t)|\, dt \frac{1}{b-a}\int_a^b |\ell(t)|\, dt$$

provided h *and* ℓ *are integrable on* $[a, b]$.

We say that the function $h : [a, b] \to \mathbb{R}$ is *H-r-Hölder continuous* with the constant $H > 0$ and power $r \in (0, 1]$ if

$$|h(t) - h(s)| \leq H\, |t - s|^r \tag{2.15}$$

for any t, $s \in [a, b]$. If $r = 1$ we call that h is *L-Lipschitzian* when $H = L > 0$.

Theorem 6. *Assume that* f, g *are asymmetrized synchronous with* f *is* H_1*-*r_1*-Hölder continuous and* g *is* H_2*-*r_2*-Hölder continuous on* $[a, b]$. *Then*

$$\frac{1}{4(r_1 + r_2 + 1)} H_1 H_2 (b-a)^{r_1 + r_2} \geq \frac{1}{b-a}\int_a^b \tilde{f}(t)\, g(t)\, dt \geq 0. \tag{2.16}$$

If particular, if f *is* L_1*-Lipschitzian and* g *is* L_2*-Lipschitzian, then*

$$\frac{1}{12} L_1 L_2 (b-a)^2 \geq \frac{1}{b-a}\int_a^b \tilde{f}(t)\, g(t)\, dt \geq 0. \tag{2.17}$$

Proof. From (2.3) we have

$$\begin{aligned}
0 &\leq \int_a^b \tilde{f}(t)\, g(t)\, dt = \frac{1}{4}\int_a^b [f(t) - f(a+b-t)]\,[g(t) - g(a+b-t)]\, dt \\
&= \frac{1}{4}\int_a^b |[f(t) - f(a+b-t)]\,[g(t) - g(a+b-t)]|\, dt \\
&\leq \frac{1}{4} H_1 H_2 \int_a^b |2t - a - b|^{r_1 + r_2}\, dt = \frac{2^{r_1 + r_2}}{4} H_1 H_2 \int_a^b \left|t - \frac{a+b}{2}\right|^{r_1 + r_2} dt \\
&= \frac{2}{2^{2 - r_1 - r_2}} H_1 H_2 \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right)^{r_1 + r_2} dt = \frac{2}{2^{2 - r_1 - r_2}} H_1 H_2 \frac{\left(\frac{b-a}{2}\right)^{r_1 + r_2 + 1}}{r_1 + r_2 + 1} \\
&= \frac{1}{4(r_1 + r_2 + 1)} H_1 H_2 (b-a)^{r_1 + r_2 + 1},
\end{aligned}$$

which is equivalent to the desired result (2.16). □

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3. Some Examples

Consider the identity function $\ell : [a,b] \to \mathbb{R}$ defined by $\ell(t) = t$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we have

$$\frac{1}{4}(b-a)[g(b)-g(a)] \geq \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) g(t)\,dt \geq \max\left\{\left|C_{1,\ell}(g)\right|, \left|C_{2,\ell}(g)\right|, \left|C_{3,\ell}(g)\right|\right\} \geq 0, \tag{3.1}$$

where

$$C_{1,\ell}(g) := \frac{1}{b-a}\int_a^b \left|\left(t - \frac{a+b}{2}\right)\tilde{g}(t)\right| dt - \frac{1}{4}\int_a^b |\tilde{g}(t)|\,dt,$$

$$C_{2,\ell}(g) := \frac{1}{b-a}\int_a^b \left|\left(t - \frac{a+b}{2}\right)g(t)\right| dt - \frac{1}{4}\int_a^b |g(t)|\,dt$$

and

$$C_{3,\ell}(g) := \frac{1}{b-a}\int_a^b |t\tilde{g}(t)|\,dt - \frac{1}{b-a}\int_a^b |t|\,dt \frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt.$$

If g is monotonic nondecreasing and L-*Lipschitzian* on $[a,b]$, then by (2.17) we get

$$\frac{1}{12}L(b-a)^2 \geq \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) g(t)\,dt \ (\geq 0). \tag{3.2}$$

Consider the power function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f(t) = t^p$ with $p > 0$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we get

$$\frac{1}{4}(b^p - a^p)[g(b)-g(a)] \geq \frac{1}{b-a}\int_a^b \left[\frac{t^p - (a+b-t)^p}{2}\right] g(t)\,dt \geq \max\left\{\left|C_{1,p}(g)\right|, \left|C_{2,p}(g)\right|, \left|C_{3,p}(g)\right|\right\} \geq 0, \tag{3.3}$$

where

$$C_{1,p}(g) := \frac{1}{b-a}\int_a^b \left|\frac{t^p - (a+b-t)^p}{2}\right| |\tilde{g}(t)|\,dt - \frac{1}{b-a}\int_a^b \left|\frac{t^p - (a+b-t)^p}{2}\right| dt \frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt,$$

$$C_{2,p}(g) := \frac{1}{b-a}\int_a^b \left|\frac{t^p - (a+b-t)^p}{2}\right| |g(t)|\,dt - \frac{1}{b-a}\int_a^b \left|\frac{t^p - (a+b-t)^p}{2}\right| dt \frac{1}{b-a}\int_a^b |g(t)|\,dt$$

and

$$C_{3,p}(g) := \int_a^b t^p |\tilde{g}(t)|\,dt - \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt.$$

If g is monotonic nondecreasing and L-*Lipschitzian* on $[a,b]$, then by (2.17) we get

$$\frac{p}{12}L(b-a)^2 \begin{cases} b^{p-1} \text{ if } p \geq 1 \\ \\ a^{p-1} \text{ if } p \in (0,1) \end{cases} \geq \frac{1}{b-a}\int_a^b \left[\frac{t^p - (a+b-t)^p}{2}\right] g(t)\,dt \ (\geq 0). \tag{3.4}$$

Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$, $f=\ln$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we have

$$\begin{aligned}\frac{1}{4}\ln\left(\frac{b}{a}\right)[g(b)-g(a)] &\geq \frac{1}{2(b-a)}\int_a^b \ln\left(\frac{t}{a+b-t}\right)g(t)\,dt \\ &\geq \max\left\{\left|C_{1,\ln}(g)\right|,\left|C_{2,\ln}(g)\right|,\left|C_{3,\ln}(g)\right|\right\}\geq 0,\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}C_{1,\ln}(g) &:= \frac{1}{b-a}\int_a^b\left|\ln\left(\frac{t}{a+b-t}\right)^{1/2}\right|\left|\tilde{g}(t)\right|dt \\ &\quad -\frac{1}{b-a}\int_a^b\left|\ln\left(\frac{t}{a+b-t}\right)^{1/2}\right|dt\frac{1}{b-a}\int_a^b\left|\tilde{g}(t)\right|dt,\end{aligned}$$

$$C_{2,\ln}(g) := \frac{1}{b-a}\int_a^b\left|\ln t\right|\left|\tilde{g}(t)\right|dt-\frac{1}{b-a}\int_a^b\left|\ln t\right|dt\frac{1}{b-a}\int_a^b\left|\tilde{g}(t)\right|dt$$

and

$$\begin{aligned}C_{1,\ln}(g) &:= \frac{1}{b-a}\int_a^b\left|\ln\left(\frac{t}{a+b-t}\right)^{1/2}\right|\left|g(t)\right|dt \\ &\quad -\frac{1}{b-a}\int_a^b\left|\ln\left(\frac{t}{a+b-t}\right)^{1/2}\right|dt\frac{1}{b-a}\int_a^b\left|g(t)\right|dt.\end{aligned}$$

If g is monotonic nondecreasing and L-*Lipschitzian* on $[a,b]$, then by (2.17) we get

$$\frac{1}{6a}L(b-a)^2\geq\frac{1}{b-a}\int_a^b\ln\left(\frac{t}{a+b-t}\right)g(t)\,dt\ \ (\geq 0).\tag{3.6}$$

Consider the function $f:[a,b]\subset\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$, $f=\sin$. If g is monotonic nondecreasing, then by (2.5) we have

$$\frac{1}{2}\sin\left(\frac{b-a}{2}\right)[g(b)-g(a)]\geq\frac{1}{b-a}\int_a^b\sin\left(t-\frac{a+b}{2}\right)g(t)\,dt\geq 0.\tag{3.7}$$

If g is monotonic nondecreasing and L-*Lipschitzian* on $[a,b]$, then by (2.17) we get

$$\begin{aligned}&\frac{1}{12}L(b-a)^2\times\begin{cases}\cos b \text{ if } -\frac{\pi}{2}\leq a<b\leq 0,\\ \max\{\cos a,\cos b\}\ \text{ if } -\frac{\pi}{2}\leq a<0<b\leq\frac{\pi}{2},\\ \cos a \text{ if } 0\leq a<b\leq\frac{\pi}{2}\end{cases} \\ &\geq\frac{1}{b-a}\cos\left(\frac{a+b}{2}\right)\int_a^b\sin\left(t-\frac{a+b}{2}\right)g(t)\,dt\ \ (\geq 0).\end{aligned}\tag{3.8}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa