J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Orthogonal stability of a quadratic functional inequality: a fixed point approach

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Abstract. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping from an orthogonality space $(\mathcal{X}, \perp)$ into a real Banach space $(\mathcal{Y}, \|\cdot\|)$. Using fixed point method, we prove the Hyers-Ulam stability of the orthogonally quadratic functional inequality

$$\begin{aligned} \bigg\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) & \\ + f\Big(\frac{z-x-y}{2}\Big) - f(x) - f(y) \bigg\| \quad & \leq \quad \Big\| f(z) \Big\| \end{aligned} \tag{0.1}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y$, $x \perp z$ and $y \perp z$.

Keywords: Hyers-Ulam stability; quadratic functional equation; fixed point method; quadratic functional inequality; orthogonality space.

1. Introduction and preliminaries

Studying functional equations by focusing on their approximate and exact solutions conduces to one of the most substantial significant study brunches in functional equations, what we call *"the theory of stability of functional equations"*. This theory specifically analyzes relationships between approximate and exact solutions of functional equations. Actually a functional equation is considered to be *stable* if one can find an exact solution for any approximate solution of that certain functional equation. Another related and close term in this area is *superstability*, which has a similar nature and concept to the stability problem. As a matter of fact, superstability for a given functional equation occurs when any approximate solution is an exact solution too. In such this situation the functional equation is called *superstable*.

In 1940, the most preliminary form of stability problems was proposed by Ulam [58]. He gave a talk and asked the following: "when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?"

In 1941, this question that today is considered as the source of the stability theory, was formulated and solved by Hyers [26] for the Cauchy's functional equation in Banach spaces. Then the result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [47] for linear mappings by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), (\varepsilon >$

[0]2010 Mathematics Subject Classification: 39B55, 39B52, 47H10.

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$0, p \in [0,1)$). In 1994, Găvruţa [23] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by the general control function $\varphi(x,y)$ for the existence of a unique linear mapping. The first author treating the stability of the quadratic functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ was Skof [55] by proving that if f is a mapping from a normed space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ satisfying $\big\|f(x+y)+f(x-y)-2f(x)-2f(y)\big\| \le \varepsilon$, for some $\varepsilon > 0$, then there is a unique quadratic mapping $g:\mathcal{X}\to\mathcal{Y}$ such that $\big\|f(x)-g(x)\big\| \le \frac{\varepsilon}{2}$. Cholewa [13] extended the Skof's theorem by replacing $\mathcal{X}$ by an abelian group $\mathcal{G}$. The Skof's result was later generalized by Czerwik [14] in the spirit of Ulam-Hyers-Rassias. For more epochal information and various aspects about the stability of functional equations theory, we refer the reader to the monographs ([6, 11, 12, 15, 16, 20, 27, 30, 41, 42, 43, 46], [48]–[51], [54]), which also include many interesting results concerning the stability of different functional equations in many various spaces.

Assume that $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ is a real inner product space with the usual Hilbert norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. Moreover, consider the orthogonal Cauchy functional equation

$$f(x+y)=f(x)+f(y), \qquad x\perp y$$

in which $\perp$ is an abstract orthogonality relation. By the Pythagorean theorem, $f:\mathcal{X}\to\mathbb{R}$ defined by $f(x)=\|x\|^2=\langle x,x\rangle$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonally Cauchy functional equation is not equivalent to the classic Cauchy equation on the whole inner product space $(\mathcal{X}, \langle\cdot,\cdot\rangle)$.

Pinsker [44] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [56] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation was first investigated by Gudder and Strawther [25]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [52] introduced his new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Furthermore, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [53] investigated the problem in a rather more general framework.

We now recall the concept of orthogonality space in the sense of Rätz [52], and then proceed it to prove our results for the orthogonally functional inequality (0.1).

Definition 1.1. Suppose $\mathcal{X}$ is a real vector space with $\dim \mathcal{X} \ge 2$ and $\perp$ is a binary relation on $\mathcal{X}$ with the following properties:

($\mathcal{O}_1$) totality of $\perp$ for zero: $x\perp 0$, $0\perp x$ for all $x\in\mathcal{X}$;

($\mathcal{O}_2$) independence: if $x,y\in\mathcal{X}-0$, $x\perp y$, then x,y are linearly independent;

($\mathcal{O}_3$) homogeneity: if $x,y\in\mathcal{X}$, $x\perp y$, then $\alpha x\perp\beta y$ for all $\alpha,\beta\in\mathbb{R}$;

($\mathcal{O}_4$) the Thalesian property: if $\mathcal{P}$ is a 2-dimensional subspace of $\mathcal{X}$, $x\in\mathcal{P}$ and $\lambda\in\mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0\in\mathcal{P}$ such that $x\perp y_0$ and $x+y_0\perp\lambda x-y_0$.

The pair $(\mathcal{X},\perp)$ is called an orthogonality space and it becomes an orthogonality normed space when the orthogonality space equipped with a normed structure.

Some interesting examples are

(i) The trivial orthogonality on a vector space $\mathcal{X}$ defined by $(\mathcal{O}_1)$, and for non-zero elements $x, y \in \mathcal{X}$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(\mathcal{X}, \| \cdot \|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in \mathcal{X}$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [3]–[5], [10, 18, 29]).

Ger and Sikorska [24] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space $\mathcal{X}$ into a real Banach space $\mathcal{Y}$ and $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$, for all $x, y \in \mathcal{X}$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : \mathcal{X} \to \mathcal{Y}$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$, for all $x \in \mathcal{X}$.

Consider the classic quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ on the real inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$. Then the important parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + \|y\|^2$$

which holds entirely in a square norm on an inner product space, shows that $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = \|x\|^2 = \langle x, x \rangle$, is a solution for the quadratic functional equation on the whole inner product space $\mathcal{X}$, (particularly in where $x \perp y$).

The orthogonally quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x \perp y$$

was first investigated by Vajzović [59] when $\mathcal{X}$ is a Hilbert space, $\mathcal{Y}$ is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljević [19], Fochi [22], Moslehian [34, 35] and Szabó [57] generalized the Vajzović's results. See also [36, 37, 40].

The following quadratic 3-variables functional equation

$$\begin{aligned} f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) + f\Big(\frac{z-x-y}{2}\Big) \\ = f(x) + f(y) + f(z) \end{aligned} \tag{1.1}$$

has been introduced and solved by S. Farhadabadi, J. Lee and C. Park on vector spaces in [21]. It has been also shown that the functional equation (1.1) is equivalent to the classic quadratic functional equation in vector spaces. In any inner product space $(\mathcal{X}, \langle\cdot,\cdot\rangle)$, it is easy to verify that

$$\left\langle \frac{x+y+z}{2}, \frac{x+y+z}{2} \right\rangle + \left\langle \frac{x-y-z}{2}, \frac{x-y-z}{2} \right\rangle + \left\langle \frac{y-x-z}{2}, \frac{y-x-z}{2} \right\rangle + \left\langle \frac{z-x-y}{2}, \frac{z-x-y}{2} \right\rangle = \langle x,x\rangle + \langle y,y\rangle + \langle z,z\rangle$$

for all $x, y, z \in \mathcal{X}$. For this obvious reason, similar to the classic quadratic functional equation, the mapping $f(x) = \langle x, x\rangle$ can also be a solution for the 3-variables equation (1.1) on the whole inner product space $\mathcal{X}$ (particularly, for the case $x \perp y, y \perp z$ and $x \perp z$).

Fixed point theory has a basic role in applications of many considerable branches in mathematics specially in stability problems. In 1996, Isac and Rassias [28] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. In view of the fact that, we will use methods related to fixed point theory, we give briefly some useful information, a definition and a fundamental result in fixed point theory.

Definition 1.2. Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a *generalized metric* on $\mathcal{X}$ if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Theorem 1.3. ([7, 17]) *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty, \quad \forall n \geq n_0$;

(2) the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;

(3) y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} \mid d(\mathcal{J}^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, \mathcal{J} y)$ *for all $y \in \mathcal{Y}$.*

In 2003, Cădariu and Radu [7, 8, 45] exerted the above definition and fixed point theorem to prove some stability problems for the Jensen and Cauchy functional equations. During the last decade, by applying fixed point methods, stability problems of several functional equations have been extensively investigated by a number of authors (see [2, 8, 9, 31, 33, 38, 39, 45]).

Throughout this paper, $(\mathcal{X}, \perp)$ is an orthogonality space and $(\mathcal{Y}, \|\cdot\|)$ is a real Banach space.

2. Solution and Hyers-Ulam stability of the functional inequality (0.1)

In this section, we first solve the orthogonally quadratic functional inequality (0.1) by proving an orthogonal superstability proposition, and then we prove its Hyers-Ulam stability in orthogonality spaces.

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Definition 2.1. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is called an (exact) *orthogonally quadratic mapping* if

$$f(x+y) + f(x-y) = f(x) + f(y) \tag{2.1}$$

for all $x, y \in \mathcal{X}$, with $x \perp y$. And it is called an *approximate orthogonally quadratic mapping* if

$$\begin{aligned}\Big\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) \\ + f\Big(\frac{z-x-y}{2}\Big) - f(x) - f(y) \Big\| \quad &\leq \quad \Big\| f(z) \Big\|\end{aligned} \tag{2.2}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y, y \perp z$ and $x \perp z$.

Proposition 2.2. *Each approximate orthogonally quadratic mapping in the form of* (2.2) *is also an* (*exact*) *orthogonally quadratic mapping satisfying* (2.1)*.*

Proof. Assume that $f : \mathcal{X} \to \mathcal{Y}$ is an approximate orthogonally quadratic mapping satisfying (2.2).

Since $0 \perp 0$, letting $x = y = z = 0$ in (2.2), we have

$$\big\| 2f(0) \big\| \leq \big\| f(0) \big\| = 0$$

and so $f(0) = 0$.

Since $(x+y) \perp 0$ for all $x, y \in \mathcal{X}$, replacing x, y and z by $x+y, 0$ and 0 in (2.2), respectively, we conclude that

$$\Big\| 2f\Big(\frac{x+y}{2}\Big) + 2f\Big(\frac{-x-y}{2}\Big) - f(x+y) \Big\| \leq \big\| f(0) \big\| = 0,$$

which implies

$$f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{-x-y}{2}\Big) = \frac{1}{2} f(x+y) \tag{2.3}$$

for all $x, y \in \mathcal{X}$ (particularly, with $x \perp y$).

Replacing y by $-y$ in the above equality, we get

$$f\Big(\frac{x-y}{2}\Big) + f\Big(\frac{y-x}{2}\Big) = \frac{1}{2} f(x-y) \tag{2.4}$$

for all $x, y \in \mathcal{X}$ (particularly, with $x \perp y$).

Since $x \perp 0$ for all $x \in \mathcal{X}$, letting $z = 0$ in (2.2), we obtain

$$\begin{aligned}\Big\| f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{x-y}{2}\Big) + f\Big(\frac{y-x}{2}\Big) + f\Big(\frac{-x-y}{2}\Big) \\ - f(x) - f(y) \Big\| \quad &\leq \quad \big\| f(0) \big\| = 0\end{aligned}$$

and so

$$f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{-x-y}{2}\Big) + f\Big(\frac{x-y}{2}\Big) + f\Big(\frac{y-x}{2}\Big) = f(x) + f(y) \tag{2.5}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$.

It follows from (2.3), (2.4) and (2.5) that

$$\frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$ with $x \perp y$, which is the equation (2.1). Hence $f : \mathcal{X} \to \mathcal{Y}$ is an (exact) orthogonally quadratic mapping. $\square$

Theorem 2.3. *Let $\varphi : \mathcal{X}^3 \to [0,\infty)$ be a function such that $\varphi(0,0,0) = 0$ and there exists an $\alpha < 1$ with*

$$\varphi(x,y,z) \leq 4\alpha\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right) \tag{2.6}$$

for all $x,y,z \in \mathcal{X}$, with $x \perp y$, $y \perp z$ and $x \perp z$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying

$$\begin{aligned}\Bigg\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) \\ + f\Big(\frac{z-x-y}{2}\Big) - f(x) - f(y) \Bigg\| \leq \Big\| f(z) \Big\| + \varphi(x,y,z)\end{aligned} \tag{2.7}$$

for all $x,y,z \in \mathcal{X}$, with $x \perp y$, $y \perp z$ and $x \perp z$. Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\| f(x) - \mathcal{Q}(x) \big\| \leq \frac{\alpha}{1-\alpha}\varphi(x,0,0) \tag{2.8}$$

for all $x \in \mathcal{X}$.

Proof. Consider the set $\mathcal{S} := \big\{h : \mathcal{X} \to \mathcal{Y}\big\}$ and introduce the generalized metric on $\mathcal{S}$:

$$d(g,h) = \inf\Big\{\mu \in \mathbb{R}_+ : \big\|g(x) - h(x)\big\| \leq \mu\varphi(x,0,0), \quad \forall x \in \mathcal{X}\Big\},$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that $(\mathcal{S}, d)$ is complete (see [32]).

Now we consider the linear mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ such that

$$\mathcal{J}g(x) := \frac{1}{4}g(2x)$$

for all $g \in \mathcal{S}$ and all $x \in \mathcal{X}$.

Since $0 \perp 0$, letting $x = y = z = 0$ in (2.7), we have

$$2\big\|f(0)\big\| \leq \big\|f(0)\big\| + \varphi(0,0,0).$$

So $f(0) = 0$.

Since $x \perp 0$ for all $x \in \mathcal{X}$, letting $y = z = 0$ in (2.7), we get $\Big\|4f\left(\frac{x}{2}\right) - f(x)\Big\| \leq \varphi(x,0,0)$ for all $x \in \mathcal{X}$. Dividing both sides by 4, putting $2x$ instead of x and then using (2.6), we obtain

$$\Big\|\frac{1}{4}f(2x) - f(x)\Big\| \leq \frac{1}{4}\varphi(2x,0,0) \leq \alpha\varphi(x,0,0)$$

for all $x \in \mathcal{X}$, which clearly yields

$$d(\mathcal{J}f, f) \leq \alpha. \tag{2.9}$$

Let $g,h \in \mathcal{S}$ be given such that $d(g,h) = \varepsilon$. Then $\big\|g(x) - h(x)\big\| \leq \varepsilon\varphi(x,0,0)$ for all $x \in \mathcal{X}$. Hence the definition of $\mathcal{J}g$ and (2.6), result that

$$\big\|\mathcal{J}g(x) - \mathcal{J}h(x)\big\| = \Big\|\frac{1}{4}g(2x) - \frac{1}{4}h(2x)\Big\| \leq \frac{1}{4}\varepsilon\varphi(2x,0,0) \leq \alpha\varepsilon\varphi(x,0,0)$$

for all $x \in \mathcal{X}$, which implies that $d(\mathcal{J}g, \mathcal{J}h) \leq \alpha\varepsilon = \alpha d(g,h)$ for all $g,h \in \mathcal{S}$.

Thus $\mathcal{J}$ is a strictly contractive mapping with Lipschitz constant $\alpha < 1$.

According to Theorem 1.3, there exists a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

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(1) $\mathcal{Q}$ is a fixed point of $\mathcal{J}$, i.e., $\mathcal{J}\mathcal{Q} = \mathcal{Q}$, and so

$$\frac{1}{4}\mathcal{Q}(2x) = \mathcal{Q}(x) \tag{2.10}$$

for all $x \in \mathcal{X}$. The mapping $\mathcal{Q}$ is a unique fixed point of $\mathcal{J}$ in the set

$$\mathcal{M} = \big\{g \in \mathcal{S} : d(g, f) < \infty\big\}.$$

This signifies that $\mathcal{Q}$ is a unique mapping satisfying (2.10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\big\|f(x) - \mathcal{Q}(x)\big\| \le \mu\varphi(x, 0, 0)$$

for all $x \in \mathcal{X}$;

(2) $d(\mathcal{J}^n f, \mathcal{Q}) \to 0$ as $n \to \infty$. So, we conclude that

$$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x) = \mathcal{Q}(x) \tag{2.11}$$

for all $x \in \mathcal{X}$;

(3) $d(f, \mathcal{Q}) \le \frac{1}{1-\alpha} d(f, \mathcal{J}f)$, which gives by (2.9) the inequality

$$d(f, \mathcal{Q}) \le \frac{\alpha}{1-\alpha}.$$

This proves that the inequality (2.8) holds.

To end the proof we show that $\mathcal{Q}$ is an orthogonally quadratic mapping.

By (2.11), (2.7), (2.6) and the fact that $\alpha < 1$,

$$\begin{aligned}
&\Big\|\mathcal{Q}\Big(\frac{x+y+z}{2}\Big) + \mathcal{Q}\Big(\frac{x-y-z}{2}\Big) + \mathcal{Q}\Big(\frac{y-x-z}{2}\Big) + \mathcal{Q}\Big(\frac{z-x-y}{2}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \mathcal{Q}(x) - \mathcal{Q}(y)\Big\| \\
&= \lim_{n\to\infty} \frac{1}{4^n}\Big\|f\big(2^{n-1}(x+y+z)\big) + f\big(2^{n-1}(x-y-z)\big) \\
&\qquad + f\big(2^{n-1}(y-z-x)\big) + f\big(2^{n-1}(z-x-y)\big) - f\big(2^n x\big) - f\big(2^n y\big)\Big\| \\
&\le \Big\|\lim_{n\to\infty} \frac{1}{4^n} f\big(2^n z\big)\Big\| + \lim_{n\to\infty} \frac{1}{4^n}\varphi\big(2^n x, 2^n y, 2^n z\big) \\
&\le \big\|\mathcal{Q}(z)\big\| + \lim_{n\to\infty} \alpha^n \varphi(x, y, z) \\
&= \big\|\mathcal{Q}(z)\big\|
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $x \perp z$ and $y \perp z$. And, now applying Proposition 2.2, we obatin that $\mathcal{Q}$ is an orthogonally quadratic mapping and the proof is complete. $\square$

Theorem 2.4. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function such that $\varphi(0, 0, 0) = 0$ and there exists an $\alpha < 1$ with*

$$\varphi(x, y, z) \le \frac{\alpha}{4}\varphi\big(2x, 2y, 2z\big)$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y, y \perp z$ and $x \perp z$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying (2.7)*. Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\big\|f(x) - \mathcal{Q}(x)\big\| \le \frac{1}{1-\alpha}\varphi(x, 0, 0) \tag{2.12}$$

for all $x \in \mathcal{X}$.

Proof. Let $(\mathcal{S}, d)$ be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ such that

$$\mathcal{J}g(x) := 4g\Big(\frac{x}{2}\Big)$$

for all $x \in \mathcal{X}$.

Similar to the proof of Theorem 2.3, from (2.7) one can get

$$\Big\|4f\left(\frac{x}{2}\right) - f(x)\Big\| \le \varphi(x, 0, 0)$$

for all $x \in \mathcal{X}$, which means $d(f, \mathcal{J}f) \le 1$.

We can also show that $\mathcal{J}$ is a strictly contractive mapping with Lipschitz constant $\alpha < 1$. So by applying Theorem 1.3 again, we have

$$d(f, \mathcal{Q}) \le \frac{1}{1-\alpha} d(f, \mathcal{J}f) \le \frac{1}{1-\alpha}$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of the previous theorem. □

Corollary 2.5. *Let $\mathcal{X}$ be a normed orthogonality space. Let δ be a nonnegative real number and $p \neq 2$ be a positive real number. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned}
&\Big\|f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) + f\Big(\frac{z-x-y}{2}\Big) - f(x) - f(y)\Big\| \\
&\qquad \le \Big\|f(z)\Big\| + \delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $y \perp z$ and $x \perp z$. Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\| \le \frac{2^p}{|2^p - 4|}\delta\|x\|^p$$

for all $x \in \mathcal{X}$.

Proof. Define $\varphi(x, y, z) := \delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$ for all $x, y, z \in \mathcal{X}$.

First assume that $0 < p < 2$.

Take $\alpha := 2^{p-2}$. Since $p < 2$, obviously $\alpha < 1$. Hence there exists an $\alpha < 1$ such that

$$\begin{aligned}
\varphi(x, y, z) &= \delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big) \\
&= 4\alpha 2^{-p}\delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big) \\
&= 4\alpha\delta\Big(\Big\|\frac{x}{2}\Big\|^p + \Big\|\frac{y}{2}\Big\|^p + \Big\|\frac{z}{2}\Big\|^p\Big) \\
&= 4\alpha\varphi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big)
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ (particularly, with $x \perp y, y \perp z$ and $x \perp z$). The recent term allows to use Theorem 2.3. So by applying Theorem 2.3, it follows from (2.8) that

$$\big\|f(x) - \mathcal{Q}(x)\big\| \le \frac{2^p}{4 - 2^p}\delta\|x\|^p$$

for all $x \in \mathcal{X}$.

For the case $p > 2$, taking $\alpha := 2^{2-p}$, and then applying Theorem 2.4, we similarly obtain the desired result. □

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