Binary Nature of a Conjecture on Nonelementary Integrals in Context of Elementary Functions

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Abstract

Binary nature has been observed in the characters of the indefinite integrals of some elementary functions having inverse hyperbolic functions as a component in the integrand. It doesn't lie between one (elementary) and zero (nonelementary) but it is exactly either one or zero. In this case the two characters behave like a member of a classical set (or crisp set) and in the language of mathematical logic it is either elementary or nonelementary functions. In this paper we have proffered a conjecture on the antiderivative containing the elementary functions made of inverse hyperbolic functions and polynomials, where the elementary function written in the numerator is made by taking the composition of two functions: the inverse hyperbolic functions and the polynomial functions. It contains two polynomials, which may or may not be equal, where one polynomial occurs as an argument in the inverse hyperbolic function and another one as a denominator of the integrand, which means that the integrand is always a fraction. The computer software mathematica has played an important role in integrating the assumed functions originating as a particular case of the proffered conjecture. It has been found that the nature of the elementary functions written as integrands varies as the degree of the polynomials increases. Two interesting antiderivatives have been observed in the study, which are always elementary containing inverse hyperbolic tangent and inverse hyperbolic cotangent functions. We have ended the paper with conclusion, its limitations and the further scope of research.

Key-words: Conjecture, elementary and nonelementary functions, polylogarithm function, dilogarithm function, elliptic function, hypergeometric function.

Introduction

Binary nature of functions originated from the antiderivatives is not only a study of numbers (types of functions) lying between one (elementary) and zero (nonelementary) but also a study of the characteristics of mathematical operations and their outputs for some character of results of some particular functions composed of following some particular patterns (Elementary function-Wikipedia contributors, 2024). The study of types of indefinite integrals is such an interesting topic in calculus, where we can easily apply the concept of binary nature.

The study of outputs originating from the antiderivatives has been a subject of research of many efforts in the past (Chaudhary et al, 2024a, 2024b; Elementary function-Wikipedia contributors, 2024; Nonelementary integrals-Wikipedia contributors, 2024; Hardy, 2018; Yadav et al., 2012; Yadav, 2023). The first example beyond the elementary functions was the study of elliptic integrals and the first known study of such integral was due to John Wallis in 1655, when he tried to find the arc length of an ellipse (Elementary function-Wikipedia contributors, 2024; Nonelementary integrals-Wikipedia contributors, 2024; Hardy, 2018; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972).

The study of nature of the indefinite integrals of some particular elementary functions as conjectures has attracted the mathematics researchers in context of elementary and nonelementary functions. In mathematics by a conjecture we mean to guess something mathematically without having a proof. It proffers a mathematical statement that is based on sufficient examples and facts without a solid proof (Conjecture-Wikipedia contributors, 2024). In this paper we have discussed such particular functions and their antiderivatives, as conjecture, because we have not proved the cases for general polynomial expressions of arbitrary degree, which is perhaps not possible. We have also tried to find some interesting nature of integrals of some particular types of integrands.

Preliminary Ideas

To study the properties of a proffered conjecture, we should know some basic concepts of the elementary and nonelementary functions in context of antiderivatives and about some special functions beyond elementary functions region. In context of available literatures, elementary and nonelementary functions were first propounded by Joseph Liouville during 1833 to 1841 and its algebraic treatment was started by Joseph Fels Ritt in 1930s (Chaudhary et al, 2024a, 2024b; Elementary function-Wikipedia contributors, 2024; Nonelementary integrals-Wikipedia contributors, 2024; Hardy, 2018; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav & Sen, 2012; Yadav, 2023).

Elementary Function: An elementary function is a single variable (real or complex valued) function, which can be written as the sums, differences, products, divisions, roots and composition of finitely many polynomials, rational, logarithms, trigonometric, hyperbolic, exponential, and their inverses. For example, $2x^2+5$, $\sqrt{3x^2+4x+1}$, e^{3x}, $\log(2x+1)$, $sin3x+2x^2+3x$, $\int 3x\, sinx\, dx$, $\int 13x\, e^{2x^2}\, dx$, π, e, 5, $sinh5x$, $arcsin6x$, $|5x|$, etc. are elementary functions. But every function cannot be an elementary function. For example, the antiderivative $\int e^{-t^2} dt$ is not an elementary function (Chaudhary et al, 2024a, 2024b;

Elementary function-Wikipedia contributors, 2024; Hardy, 2018; Kasper, 1980; Lutfi, 2016; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav & Sen, 2012; Yadav, 2023).

Nonelementary Function: If a function is written in closed form, its derivative can also be written in closed form. But its antiderivative may or may not be written in closed form. One such example for an integrand (elementary function) whose integral does not have a closed form expression is e^{-x^2}, whose one of the antiderivative is the well known error function

$$\operatorname{erf}(x) = \frac{1}{\pi}\int_0^x e^{-t^2}\,dt.$$

Such antiderivative is called a nonelementary integral or nonelementary function (Chaudhary et al, 2024a, 2024b; Elementary function-Wikipedia contributors, 2024; Nonelementary integrals-Wikipedia contributors, 2024; Hardy, 2018; Marchisotto et al., 1994; Risch, 1969; 1970, 2022; Ritt, 2022; Rosenlicht, 1972; Yadav & Sen, 2012; Yadav, 2023). For this study, a nonelementary function of a given elementary function is an indefinite integral that is not an elementary function and that cannot be expressed in closed form expression also (Chaudhary et al, 2024a, 2024b; Nonelementary integrals-Wikipedia contributors, 2024; Hardy, 2018; Marchisotto et al., 1994; Yadav et al., 2012; Yadav, 2023).

Methodology: When we use the software computing mathematica code to find the integral of a function f(x) with respect to x using the code '*In[i]:Integrate[f[x], x]*', we get its integral in terms of elementary functions, where In[i] stands for input[i] i.e. ith input to find the ith output. When the output comes in terms of some special types of functions like hypergeometric, dilogarithm, polylogarithm, elliptic, error, etc., it means the indefinite integral is not an elementary function. This concept will be throughout used in the present study (Chaudhary et al, 2024a, 2024b; Elliptic function-Wikipedia contributors, 2024; Hypergeometric function -Wikipedia contributors, 2024; Nonelementary integrals-Wikipedia contributors, 2024; Polylogarithm function-Wikipedia contributors, 2024).

Discussion

Yadav & Sen (2008, 2012) proffered six types of conjectures on nonelementary integrals and proved them for some particular cases using strong Liouville's theorem and its corollaries. Chaudhary & Yadav (2024a, 2024b), Yadav & Chaudhary (2024a), and Yadav & Yadav (2024a, 2024b) proposed another three conjectures based on those functions, which were left earlier. They proved it using strong Liouville's theorem, Laplace's theorem and computing software mathematica. In continuation, we are here studying the characteristics of a special type of integrand in an antiderivative.

Before proffering a conjecture, let us consider the antiderivatives of the indefinite integral

$$\int \frac{fog(x)}{h(x)}\,dx \qquad (1)$$

where $fog(x) = f[g(x)]$ is an elementary function made by the composition of two functions: inverse hyperbolic function f(x) and a polynomial function g(x), where both g(x) and h(x) are polynomials of degree greater than or equal to one. The two polynomials g(x) and h(x) are arbitrary and are independent of each other i.e. they may or may not be related to each other.

To find the above antiderivative, we first denote the above integral by I. We know that there are six types of inverse hyperbolic functions, so there will be six cases for different inverse hyperbolic functions. Let us consider each of them one by one:

Case-I: When $f(x) = \sinh^{-1}$x. The antiderivative (1) becomes

$$\mathrm{I} = \int \frac{\sinh^{-1}[\mathrm{g(x)}]}{\mathrm{h(x)}}\mathrm{dx} \qquad (1.1)$$

When g(x) = h(x) = x. Then from (1.1), we have

$$\mathrm{I} = \int \frac{\sinh^{-1}\mathrm{x}}{\mathrm{x}}\mathrm{dx}$$

Using mathematica, we get

$$\mathrm{In}[1]\text{: Integrate}[\frac{\mathrm{ArcSinh}[x]}{x}, x]$$

$$\mathrm{Out}[1]\text{: }\frac{1}{2}(\mathrm{ArcSinh}[x](\mathrm{ArcSinh}[x] + 2\mathrm{Log}[1 - e^{-2\mathrm{ArcSinh}[x]}]) - \mathrm{PolyLog}[2, e^{-2\mathrm{ArcSinh}[x]}])$$

where PolyLog$[2, e^{-2ArcSinh[x]}]$ is a polylogarithm function $\mathrm{Li}_2(\mathrm{e}^{-2\mathrm{ArcSinh}[x]})$. We know that the polylogarithm function $\mathrm{Li}_\mathrm{n}(\mathrm{z})$ is elementary for only special values of n and the special case is that n = 1 involves the ordinary natural logarithm and the special cases n = 2 and n = 3 are called dilogarithm (Polylogarithm - Wikipedia contributors, 2024). Thus the above antiderivative is not an elementary integral but it is a nonelementary integral.

When h(x) is the derivative of g(x) like g(x) = x and h(x) = 1, although it is beyond our assumption since we have taken both g and h of degree at least equal to one. Then from (1.1), we have

$$\mathrm{I} = \int \frac{\sinh^{-1}[\mathrm{x}]}{1}\mathrm{dx} = \int \sinh^{-1}[\mathrm{x}]\mathrm{dx}$$

Using mathematica, we get

$$\mathrm{In}[2]\text{: Integrate}[\mathrm{ArcSinh}[x], x]$$

$$\mathrm{Out}[2]\text{: } -\sqrt{1 + x^2} + x\mathrm{ArcSinh}[x]$$

which is elementary. This is why we have taken the degree of polynomials greater than or equal to one. Let us take another case when g(x) = x^2 and h(x) = 2x, here h(x) is the first derivative of g(x). Then from (1.1), we get

$$\mathrm{I} = \int \frac{\sinh^{-1}[\mathrm{x}^2]}{2\mathrm{x}}\mathrm{dx}$$

Using mathematica, we get

$$\mathrm{In}[3]\text{: Integrate}[\frac{\mathrm{ArcSinh}[x^2]}{2 * x}, x]$$

$$\mathrm{Out}[3]\text{:}\frac{1}{8}(\mathrm{ArcSinh}[x^2](\mathrm{ArcSinh}[x^2] + 2\mathrm{Log}[1 - e^{-2\mathrm{ArcSinh}[x^2]}]) - \mathrm{PolyLog}[2, e^{-2\mathrm{ArcSinh}[x^2]}])$$

which is nonelementary due to PolyLog function, as discussed earlier. So let us repeat it for more degree polynomials as

$$\mathrm{I} = \int \frac{\sinh^{-1}[\mathrm{x}^3]}{\mathrm{x}^2}\mathrm{dx}$$

$$\mathrm{In}[4]\text{: Integrate}[\frac{\mathrm{ArcSinh}[x^3]}{x^2}, x]$$

$$\mathrm{Out}[4]\text{:} -\frac{\mathrm{ArcSinh}[x^3]}{x} + \frac{1}{\sqrt{1+x^6}}(-1)^{1/6}3^{3/4}\sqrt{-(-1)^{1/6}((-1)^{2/3}+x^2)}\sqrt{1+(-1)^{1/3}x^2+(-1)^{2/3}x^4}.$$

$$\mathrm{EllipticF}[\mathrm{ArcSin}[\frac{\sqrt{-(-1)^{5/6}(1+x^2)}}{3^{1/4}}], (-1)^{1/3}]$$

which is elliptic functions form in mathematical nature, and so it is nonelementary. We know that the elliptic functions are special cases of meromorphic functions and they have been derived from elliptic integrals (Elliptic function - Wikipedia contributors, 2024), which are themselves nonelementary. Similarly taking more degree for polynomials for g(x) and h(x), we get

$$\mathrm{In}[5]\text{: Integrate}[\frac{\mathrm{ArcSinh}[x^4]}{x^3}, x]$$

$$Out[5]\text{:} -\frac{\mathrm{ArcSinh}[x^4]}{2x^2} + x^2\mathrm{Hypergeometric2F1}[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -x^8]$$

$$\mathrm{In}[6]\text{: Integrate}[\frac{\mathrm{ArcSinh}[x^5]}{x^5}, x]$$

$$Out[6]\text{:} -\frac{\mathrm{ArcSinh}[x^5]}{4x^4} + \frac{5}{4}x\mathrm{Hypergeometric2F1}\left[\frac{1}{10}, \frac{1}{2}, \frac{11}{10}, -x^{10}\right]$$

$$\mathrm{In}[7]\text{: Integrate}[\frac{\mathrm{ArcSinh}[x^5]}{x^4}, x]$$

$$Out[7]: -\frac{\text{ArcSinh}[x^5]}{3x^3} + \frac{5}{6}x^2\text{Hypergeometric2F1}[\frac{1}{5},\frac{1}{2},\frac{6}{5},-x^{10}]$$

In above results we see that when we increase the degree, the integrals come out in nonelementary functions form. So we get nonelementary integrals. Thus we find that for all polynomials of degree greater than or equal to one, the integral (1) is nonelementary i.e. for all m and n greater than or equal to 1, the integral

$$\mathrm{I} = \int \frac{\sinh^{-1}[\mathrm{x^m}]}{\mathrm{x^n}}\,\mathrm{dx}$$

is nonelementary.

In above integrands, we have taken only one term of the polynomials. What will happen, when we take the polynomials in mixed forms? Let us take some arbitrary polynomials of degree greater than one like $g(x) = x^2 + 2x$ and $h(x) = 2x^2+1$. From (1.1), we get

$$\mathrm{I} = \int \frac{\sinh^{-1}[\mathrm{x^2 + 2x}]}{\mathrm{2x^2 + 1}}\,\mathrm{dx}$$

In this case the computing software mathematica doesn't produce any output although for another polynomials in mixed terms, it gives output both in elementary and nonelementary forms. Thus we can state that for $f(x) = \sinh^{-1}x$ and for any polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is always nonelementary, where the polynomials have only one term with the leading coefficient only.

Case-II: When $\mathrm{f(x)} = \cosh^{-1}\mathrm{x}$. The antiderivative (1) becomes

$$\mathrm{I} = \int \frac{\cosh^{-1}[\mathrm{g(x)}]}{\mathrm{h(x)}}\,\mathrm{dx} \qquad (1.2)$$

When g(x) = h(x) = x, then from (1.2), we have

$$\mathrm{I} = \int \frac{\cosh^{-1}\mathrm{x}}{\mathrm{x}}\,\mathrm{dx}$$

Using mathematica, we get

$$\text{In}[8]: \text{Integrate}[\frac{\text{ArcCosh}[x]}{x}, x]$$

$$\text{Out}[8]: \frac{1}{2}\left(\text{ArcCosh}[x]\left(\text{ArcCosh}[x] + 2\text{Log}\left[1 + e^{-2\text{ArcCosh}[x]}\right]\right) - \text{PolyLog}\left[2, -e^{-2\text{ArcCosh}[x]}\right]\right)$$

where $\text{PolyLog}[2, e^{-2ArcCosh[x]}]$ is a polylogarithm function $\mathrm{Li_2}(\mathrm{e}^{-2\text{ArcCosh}[x]})$. Thus the above integral is not an elementary integral but it is a nonelementary integral.

When g(x) = x and h(x) = 1, then from (1.2), we have

$$I = \int \frac{\cosh^{-1}[x]}{1} dx = \int \cosh^{-1}[x] dx$$

Using mathematica, we get

$$\text{In}[9]: \text{Integrate}[\text{ArcCosh}[x], x]$$

$$\text{Out}[9]: -\sqrt{-1+x}\sqrt{1+x} + x\text{ArcCosh}[x]$$

which is elementary. When g(x) = x^2 and h(x) = 2x, then from (1.2), we get

$$I = \int \frac{\cosh^{-1}[x^2]}{2x} dx$$

Using mathematica, we get

$$\text{In}[10]: \text{Integrate}[\frac{\text{ArcCosh}[x^2]}{2*x}, x]$$

$$\text{Out}[10]: \frac{1}{8}(\text{ArcCosh}[x^2](\text{ArcCosh}[x^2] + 2\text{Log}[1 + e^{-2\text{ArcCosh}[x^2]}]) - \text{PolyLog}[2, -e^{-2\text{ArcCosh}[x^2]}])$$

which is nonelementary due to PolyLog function, as discussed earlier. Let us repeat it for more degree polynomials as

$$I = \int \frac{\cosh^{-1}[x^3]}{x^2} dx$$

$$\text{In}[11]: \text{Integrate}[\frac{\text{ArcCosh}[x^3]}{x^2}, x]$$

$$\text{Out}[11]: \frac{1}{x(-1+x^6)}(-(-1+x^6)\text{ArcCosh}[x^3] + i3^{3/4}x\sqrt{(-1)^{5/6}(-1+x^2)}\sqrt{-1+x^3}\sqrt{1+x^3}.$$

$$\sqrt{1+x^2+x^4}\text{EllipticF}[\text{ArcSin}[\frac{\sqrt{-(-1)^{5/6}-ix^2}}{3^{1/4}}], (-1)^{1/3}])$$

which is in terms of elliptic functions in mathematical notation, and so it is nonelementary as discussed in Case-I. Taking more degree for polynomials for g(x) and h(x), we get

$$\text{In}[12]: \text{Integrate}[\frac{\text{ArcCosh}[x^4]}{x^3}, x]$$

$$\text{Out[12]:} \frac{1}{2x^2(-1+x^4)}\left(-(-1+x^4)\text{ArcCosh}[x^4] + 2x^4\sqrt{\frac{-1+x^4}{1+x^4}}\sqrt{1-x^8}\text{Hypergeometric2F1}\left[\frac{1}{4},\frac{1}{2},\frac{5}{4},x^8\right]\right)$$

$$\text{In[13]: Integrate}[\frac{\text{ArcCosh}[x^5]}{x^5},x]$$

$$\text{Out[13]:} \frac{1}{4x^4(-1+x^{10})}\left(-(-1+x^{10})\text{ArcCosh}[x^5] + 5x^5\sqrt{-1+x^5}\sqrt{1+x^5}\sqrt{1-x^{10}}\text{Hypergeometric2F1}\left[\frac{1}{10},\frac{1}{2},\frac{11}{10},x^{10}\right]\right)$$

$$\text{In[14]: Integrate}[\frac{\text{ArcCosh}[x^5]}{x^4},x]$$

$$\text{Out[14]:} \frac{1}{6x^3(-1+x^{10})}\left(-2(-1+x^{10})\text{ArcCosh}[x^5] + 5x^5\sqrt{-1+x^5}\sqrt{1+x^5}\sqrt{1-x^{10}}\text{Hypergeometric2F1}\left[\frac{1}{5},\frac{1}{2},\frac{6}{5},x^{10}\right]\right)$$

In above outputs we can see that when we increase the degree of polynomials, the integrals come out in nonelementary functions form. So we get nonelementary integrals. Thus we find that for all polynomials of degree greater than or equal to one and f(x) = $\cosh^{-1}$x, the integral (1) is nonelementary i.e. for all m and n greater than or equal to 1, the integral

$$\mathrm{I} = \int \frac{\cosh^{-1}[\mathrm{x^m}]}{\mathrm{x^n}}\mathrm{dx}$$

is nonelementary.

Again let us consider what will happen, when we take the polynomials having mixed terms? Let us take some arbitrary polynomials of degree greater than one like g(x) = x^2 + 2x and h(x) = $2x^2$+1, then from (1.2), we get

$$\mathrm{I} = \int \frac{\cosh^{-1}[\mathrm{x^2+2x}]}{\mathrm{2x^2+1}}\mathrm{dx}$$

In this case the computing software mathematica doesn't produce any output although for another polynomials in mixed terms, it gives output both in elementary and nonelementary forms. Thus we can state that for f(x) = $\cosh^{-1}$x and for any polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is always nonelementary, where the polynomials have only one term with the leading coefficient only.

Case-III: When we take $\mathrm{f(x)} = \tanh^{-1}\mathrm{x}$, the integral (1) becomes

$$\mathrm{I} = \int \frac{\tanh^{-1}[\mathrm{g(x)}]}{\mathrm{h(x)}}\mathrm{dx} \qquad (1.3)$$

For g(x) = h(x) = x, we get from (1.3)

$$I = \int \frac{\tanh^{-1}x}{x}dx$$

Applying mathematica, we get

$$\mathrm{In}[15]\text{: Integrate}[\frac{\mathrm{ArcTanh}[x]}{x}, x]$$

$$\mathrm{Out}[15]\text{:} \frac{1}{2}(-\mathrm{PolyLog}[2, -x] + \mathrm{PolyLog}[2, x])$$

which is nonelementary.

Taking g(x) = x and h(x) = 1, then from (1.3), we have

$$\mathrm{I} = \int \frac{\tanh^{-1}[\mathrm{x}]}{1}\mathrm{dx} = \int \tanh^{-1}[\mathrm{x}]\mathrm{dx}$$

Integrating using mathematica, we get

$$\mathrm{In}[16]\text{: Integrate}[\mathrm{ArcTanh}[x], x]$$

$$\mathrm{Out}[16]\text{:} x\mathrm{ArcTanh}[x] + \frac{1}{2}\mathrm{Log}[1 - x^2]$$

which is elementary. Let us take g(x) = x^2 and h(x) = 2x, then from (1.3), we get

$$\mathrm{I} = \int \frac{\tanh^{-1}[\mathrm{x}^2]}{2\mathrm{x}}\mathrm{dx}$$

Using mathematica for integration, we get

$$\mathrm{In}[17]\text{: Integrate}[\frac{\mathrm{ArcTanh}[x^2]}{2 * x}, x]$$

$$\mathrm{Out}[17]\text{:} \frac{1}{8}(-\mathrm{PolyLog}[2, -x^2] + \mathrm{PolyLog}[2, x^2])$$

obviously it is nonelementary, as discussed earlier. Let us take one more example

$$\mathrm{I} = \int \frac{\tanh^{-1}[\mathrm{x}^2]}{\mathrm{x}^2}\mathrm{dx}$$

and using mathematica, we get

$$\text{In}[18]\text{: Integrate}[\frac{\text{ArcTanh}[x^2]}{x^2}, x]$$

$$\text{Out}[18]\text{: ArcTan}[x] - \frac{\text{ArcTanh}[x^2]}{x} - \frac{1}{2}\text{Log}[-1+x] + \frac{1}{2}\text{Log}[1+x]$$

which is elementary. Let us take some arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = $2x^2+1$. Then from (1.3), we get

$$\mathrm{I} = \int \frac{\tanh^{-1}[\mathrm{x}^2 + 2\mathrm{x}]}{2\mathrm{x}^2 + 1}\mathrm{dx}$$

In this case the mathematica doesn't produce any output. After trying on many functions it was found that no satisfactory result comes out for mixed terms polynomials. So let us simplify it for polynomials having only term with leading coefficient, like as

$$\mathrm{I} = \int \frac{\tanh^{-1}[\mathrm{x}^3]}{\mathrm{x}^2}\mathrm{dx}$$

$$\text{In}[19]\text{: Integrate}[\frac{\text{ArcTanh}[x^3]}{x^2}, x]$$

$$\text{Out}[19]\text{: } \frac{1}{4x}\Big(-4\text{ArcTanh}[x^3] + x\Big(2\sqrt{3}\text{ArcTan}\Big[\frac{-1+2x}{\sqrt{3}}\Big] - 2\sqrt{3}\text{ArcTan}\Big[\frac{1+2x}{\sqrt{3}}\Big] - 2\text{Log}[-1+x] - 2\text{Log}[1+x] + \text{Log}[1-x+x^2] + \text{Log}[1+x+x^2]\Big)\Big)$$

which is elementary. Similarly let us verify it for more example as

$$\mathrm{I} = \int \frac{\tanh^{-1}[\mathrm{x}^3]}{\mathrm{x}^3}\mathrm{dx}$$

We get from mathematica

$$\text{In}[20]\text{: Integrate}[\frac{\text{ArcTanh}[x^3]}{x^3}, x]$$

$$\text{Out}[20]\text{: } \frac{1}{8x^2}\Big(-4\text{ArcTanh}[x^3] + x^2\Big(2\sqrt{3}\text{ArcTan}\Big[\frac{-1+2x}{\sqrt{3}}\Big] + 2\sqrt{3}\text{ArcTan}\Big[\frac{1+2x}{\sqrt{3}}\Big] - 2\text{Log}[-1+x] + 2\text{Log}[1+x] - \text{Log}[1-x+x^2] + \text{Log}[1+x+x^2]\Big)\Big)$$

which is elementary. For

$$I = \int \frac{\tanh^{-1}[x^4]}{x^4} dx$$

We get using mathematica,

$$\text{In[21]: Integrate}[\frac{\text{ArcTanh}[x^4]}{x^4}, x]$$

$$\text{Out[21]:} \frac{1}{12x^3}\Big(-4\text{ArcTanh}[x^4] + x^3\big(4\text{ArcTan}[x] - 2\sqrt{2}\text{ArcTan}\big[1-\sqrt{2}x\big] + 2\sqrt{2}\text{ArcTan}\big[1+\sqrt{2}x\big] - 2\text{Log}[-1+x] + 2\text{Log}[1+x] - \sqrt{2}\text{Log}\big[1-\sqrt{2}x+x^2\big] + \sqrt{2}\text{Log}\big[1+\sqrt{2}x+x^2\big]\big)\Big)$$

which is elementary. For

$$I = \int \frac{\tanh^{-1}[x^4]}{x^3} dx$$

and using mathematica, we get

$$\text{In[22]: Integrate}\left[\frac{\text{ArcTanh}[x^4]}{x^3}, x\right]$$

$$\text{Out[22]:} -\frac{1}{2}\text{ArcTan}[\frac{1}{x^2}] - \frac{\text{ArcTanh}[x^4]}{2x^2} - \frac{1}{4}\text{Log}[-1+x^2] + \frac{1}{4}\text{Log}[1+x^2]$$

which is elementary.

Thus we can state that for f(x) = $\tanh^{-1}$x and for polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is elementary for some cases and nonelementary for some cases. It cannot be unified in a single statement. This is the binary nature of the integral discussed in (1).

But a particular chain is seen in the integral of the form

$$I = \int \frac{\tanh^{-1}[x^m]}{x^n} dx$$

that it is elementary for all values of n and m except for m = 1, n = 1 and m = 2, n = 1, for which it is nonelementary.

Case-IV: When $f(x) = \coth^{-1}x$, the integral (1) becomes

$$I = \int \frac{\coth^{-1}[g(x)]}{h(x)} dx \qquad (1.4)$$

For g(x) = h(x) = x, we get from (1.4)

$$I = \int \frac{\coth^{-1}x}{x}\,dx$$

Applying mathematica, we get

$$\text{In[23]: Integrate}[\frac{\text{ArcCoth}[x]}{x}, x]$$

$$\text{Out[23]:}\frac{1}{2}\left(\text{PolyLog}\left[2,-\frac{1}{x}\right]-\text{PolyLog}\left[2,\frac{1}{x}\right]\right)$$

where $\text{PolyLog}\left[2,-\frac{1}{x}\right]$ and $\text{PolyLog}\left[2,\frac{1}{x}\right]$ are polylogarithm function, which are not elementary.

Taking g(x) = x and h(x) = 1, from (1.4), we have

$$\text{I} = \int \frac{\coth^{-1}[\text{x}]}{1}\text{dx} = \int \coth^{-1}[\text{x}]\text{dx}$$

Integrating using mathematica, we get

$$\text{In[24]: Integrate[ArcCoth}[x], x]$$

$$\text{Out[24]: } x\text{ArcCoth}[x] + \frac{1}{2}\text{Log}[1-x^2]$$

which is elementary. Let us take another case when g(x) = x^2 and h(x) = 2x, then from (1.4), we get

$$\text{I} = \int \frac{\coth^{-1}[\text{x}^2]}{2\text{x}}\text{dx}$$

Using mathematica for integration, we get

$$\text{In[25]: Integrate}[\frac{\text{ArcCoth}[x^2]}{2*x}, x]$$

$$\text{Out[25]:}\frac{1}{8}\left(\text{PolyLog}\left[2,-\frac{1}{x^2}\right]-\text{PolyLog}\left[2,\frac{1}{x^2}\right]\right)$$

obviously it is nonelementary, as discussed earlier. Let us take another example

$$\text{I} = \int \frac{\coth^{-1}[\text{x}^2]}{\text{x}^2}\text{dx}$$

and using mathematica, we get

$$\text{In}[26]\text{: Integrate}[\frac{\text{ArcCoth}[x^2]}{x^2}, x]$$

$$\text{Out}[26]\text{: } -\frac{\text{ArcCoth}[x^2]}{x} + \text{ArcTan}[x] - \frac{1}{2}\text{Log}[-1+x] + \frac{1}{2}\text{Log}[1+x]$$

which is elementary. Let us take some arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = 2x^2+1. Then from (1.4), we get

$$\mathrm{I} = \int \frac{\coth^{-1}[\mathrm{x}^2 + 2\mathrm{x}]}{2\mathrm{x}^2 + 1}\,\mathrm{dx}$$

In this case the mathematica doesn't produce any output. For any such mixed polynomials, we get such problem in mathematica and due to the complexity of polynomials in mixed terms, we cannot generalize it for the result. So let us simplify it as

$$\mathrm{I} = \int \frac{\coth^{-1}[\mathrm{x}^3]}{\mathrm{x}^2}\,\mathrm{dx}$$

$$\text{In}[27]\text{: Integrate}[\frac{\text{ArcCoth}[x^3]}{x^2}, x]$$

$$\text{Out}[27]\text{: } \frac{1}{4x}\Bigg(-4\text{ArcCoth}[x^3] + x\left(2\sqrt{3}\text{ArcTan}\left[\frac{-1+2x}{\sqrt{3}}\right] - 2\sqrt{3}\text{ArcTan}\left[\frac{1+2x}{\sqrt{3}}\right] - 2\text{Log}[-1+x] - 2\text{Log}[1+x] + \text{Log}[1-x+x^2] + \text{Log}[1+x+x^2]\right)\Bigg)$$

which is elementary. Let us verify it for one more example as

$$\mathrm{I} = \int \frac{\coth^{-1}[\mathrm{x}^3]}{\mathrm{x}^3}\,\mathrm{dx}$$

We get that

$$\text{In}[28]\text{: Integrate}[\frac{\text{ArcCoth}[x^3]}{x^3}, x]$$

$$\text{Out}[28]\text{: } \frac{1}{8x^2}\Bigg(-4\text{ArcCoth}[x^3] + x^2\left(2\sqrt{3}\text{ArcTan}\left[\frac{-1+2x}{\sqrt{3}}\right] + 2\sqrt{3}\text{ArcTan}\left[\frac{1+2x}{\sqrt{3}}\right] - 2\text{Log}[-1+x] + 2\text{Log}[1+x] - \text{Log}[1-x+x^2] + \text{Log}[1+x+x^2]\right)\Bigg)$$

which is elementary. Again taking one more example

$$I = \int \frac{\coth^{-1}[x^4]}{x^4} dx$$

We get that

$$\text{In}[29]\text{: Integrate}[\frac{\text{ArcCoth}[x^4]}{x^4}, x]$$

$$\begin{aligned} Out[29]: \frac{1}{12x^3}\Big(&-4\text{ArcCoth}[x^4] \\ &+ x^3\big(4\text{ArcTan}[x] - 2\sqrt{2}\text{ArcTan}\big[1-\sqrt{2}x\big] + 2\sqrt{2}\text{ArcTan}\big[1+\sqrt{2}x\big] \\ &- 2\text{Log}[-1+x] + 2\text{Log}[1+x] - \sqrt{2}\text{Log}\big[1-\sqrt{2}x+x^2\big] \\ &+ \sqrt{2}\text{Log}\big[1+\sqrt{2}x+x^2\big]\big)\Big) \end{aligned}$$

which is elementary. For another example

$$I = \int \frac{\coth^{-1}[x^4]}{x^3} dx$$

and using mathematica, we get

$$\text{In}[30]\text{: Integrate}\left[\frac{\text{ArcCoth}[x^4]}{x^3}, x\right]$$

$$Out[30]: -\frac{\text{ArcCoth}[x^4]}{2x^2} - \frac{1}{2}\text{ArcTan}[\frac{1}{x^2}] - \frac{1}{4}\text{Log}[-1+x^2] + \frac{1}{4}\text{Log}[1+x^2]$$

which is elementary.

Thus we can state that for f(x) = $\coth^{-1}x$ and for polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is elementary for some cases and nonelementary for some cases. It cannot be unified in a single statement. This is the binary nature of the integral discussed in (1).

But again a particular chain is seen in the integral of the form

$$I = \int \frac{\coth^{-1}[x^m]}{x^n} dx$$

that it is elementary for all values of n and m except for m = 1, n = 1 and m = 2, n = 1, for which it is nonelementary.

Case-V: When we take $f(x) = \text{cosech}^{-1}x$, the antiderivative (1) becomes

$$I = \int \frac{\text{cosech}^{-1}[g(x)]}{h(x)} dx \qquad (1.5)$$

Taking g(x) = h(x) = x, we get from (1.5)

$$I = \int \frac{\text{cosech}^{-1}\text{x}}{\text{x}}\,\text{dx}$$

Applying mathematica, we get

$$\text{In[31]: Integrate}[\frac{\text{ArcCsch}[x]}{x}, x]$$

$$\text{Out[31]:}\frac{1}{2}\left(-\text{ArcCsch}[x]\left(\text{ArcCsch}[x] + 2\text{Log}\left[1 - e^{-2\text{ArcCsch}[x]}\right]\right) + \text{PolyLog}\left[2, e^{-2\text{ArcCsch}[x]}\right]\right)$$

which is in the form of polylogarithm and thus nonelementary.

If we consider another case g(x) = x and h(x) = 1, then from (1.5), we have

$$I = \int \frac{\text{cosech}^{-1}[\text{x}]}{1}\,\text{dx} = \int \text{cosech}^{-1}[\text{x}]\text{dx}$$

Integrating using mathematica, we get

$$\text{In[32]: Integrate[ArcCsch}[x], x]$$

$$\text{Out[32]:}\, x\left(\text{ArcCsch}[x] + \frac{\sqrt{1 + \frac{1}{x^2}}\,\text{ArcSinh}[x]}{\sqrt{1 + x^2}}\right)$$

which is elementary. Taking g(x) = x^2 and h(x) = 2x, then from (1.5), we get

$$I = \int \frac{\text{cosech}^{-1}[\text{x}^2]}{2\text{x}}\,\text{dx}$$

Using mathematica for integration, we get

$$\text{In[33]: Integrate}[\frac{\text{ArcCsch}[x^2]}{2 * x}, x]$$

$$\text{Out[33]:}\frac{1}{8}\left(-\text{ArcCsch}[x^2]\left(\text{ArcCsch}[x^2] + 2\text{Log}\left[1 - e^{-2\text{ArcCsch}[x^2]}\right]\right) + \text{PolyLog}\left[2, e^{-2\text{ArcCsch}[x^2]}\right]\right)$$

obviously it is nonelementary, as discussed earlier. Let us take one more example

$$I = \int \frac{\text{cosech}^{-1}[\text{x}^2]}{\text{x}^2}\,\text{dx}$$

and using mathematica, we get

$$\text{In[34]: Integrate}[\frac{\text{ArcCsch}[x^2]}{x^2}, x]$$

$$\text{Out[34]:} -\frac{1}{x^3}(x^2\text{ArcCsch}[x^2] + \frac{1}{\sqrt{1+\frac{1}{x^4}}}2(-1-x^4 + (-1)^{3/4}\sqrt{x^2}\sqrt{1+x^4}(-\text{EllipticE}[i\text{ArcSinh}[(-1)^{1/4}\sqrt{x^2}], -1] + \text{EllipticF}[i\text{ArcSinh}[(-1)^{1/4}\sqrt{x^2}], -1])))$$

which is nonelementary. Let us take arbitrary polynomials of degree greater than one like g(x) = x^2+2x and h(x) = $2x^2$+1. Then from (1.5), we get

$$\mathrm{I} = \int \frac{\mathrm{cosech}^{-1}[\mathrm{x}^2 + 2\mathrm{x}]}{2\mathrm{x}^2 + 1}\,\mathrm{dx}$$

In this case mathematica doesn't produce any output. In fact when we take polynomials having mixed terms, we get such problems. So let us simplify it in simple form as

$$\mathrm{I} = \int \frac{\mathrm{cosech}^{-1}[\mathrm{x}^3]}{\mathrm{x}^2}\,\mathrm{dx}$$

$$\text{In[35]: Integrate}[\frac{\text{ArcCsch}[x^3]}{x^2}, x]$$

$$\text{Out[35]:}\frac{1}{5\sqrt{1+\frac{1}{x^6}}x^4}\left(15 + 15x^6 - 5\sqrt{1+\frac{1}{x^6}}x^3\text{ArcCsch}[x^3] - 6x^6\sqrt{1+x^6}\text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{5}{6}, \frac{11}{6}, -x^6\right]\right)$$

which gives integral in hypergeometric function, which is obviously nonelementary. Let us verify it for more example as

$$\mathrm{I} = \int \frac{\mathrm{cosech}^{-1}[\mathrm{x}^3]}{\mathrm{x}^3}\,\mathrm{dx}$$

We get that

$$\text{In[36]: Integrate}[\frac{\text{ArcCsch}[x^3]}{x^3}, x]$$

$$\text{Out[36]:}\frac{1}{4x^5}(-2x^3\text{ArcCsch}[x^3] + \frac{1}{\sqrt{1+\frac{1}{x^6}}}(3(1+x^6) + 3^{3/4}(x^3)^{2/3}\sqrt{-(-1)^{1/6}((-1)^{2/3}+(x^3)^{2/3})}\sqrt{1+(-1)^{1/3}(x^3)^{2/3}+(-1)^{2/3}(x^3)^{4/3}}$$

$$(\sqrt{3}\text{EllipticE}[\text{ArcSin}[\frac{\sqrt{-(-1)^{5/6}(1+(x^3)^{2/3})}}{3^{1/4}}],(-1)^{1/3}]$$
$$+(-1)^{5/6}\text{EllipticF}[\text{ArcSin}[\frac{\sqrt{-(-1)^{5/6}(1+(x^3)^{2/3})}}{3^{1/4}}],(-1)^{1/3}])))$$

which is in elliptic function form, and so it is nonelementary. Again for one more example

$$\mathrm{I}=\int\frac{\mathrm{cosech}^{-1}[\mathrm{x}^4]}{\mathrm{x}^4}\mathrm{dx}$$

We get that

$$\text{In}[37]\text{: Integrate}[\frac{\text{ArcCsch}[x^4]}{x^4},x]$$

$$\text{Out}[37]\text{:}\frac{1}{45\sqrt{1+\frac{1}{x^8}}x^7}(5(4+4x^8-3\sqrt{1+\frac{1}{x^8}}x^4\text{ArcCsch}[x^4])$$
$$-4x^8\sqrt{1+x^8}\text{Hypergeometric2F1}[\frac{1}{2},\frac{5}{8},\frac{13}{8},-x^8])$$

which is obviously nonelementary. Let us take one more example

$$\mathrm{I}=\int\frac{\mathrm{cosech}^{-1}[\mathrm{x}^4]}{\mathrm{x}^3}\mathrm{dx}$$

and using mathematica, we get

$$\text{In}[38]\text{: Integrate}\left[\frac{\text{ArcCsch}[x^4]}{x^3},x\right]$$

$$Out[38]\text{:}-\frac{1}{2x^6}(x^4\text{ArcCsch}[x^4]+\frac{1}{\sqrt{1+\frac{1}{x^8}}}2(-1-x^8$$
$$+(-1)^{3/4}\sqrt{x^4}\sqrt{1+x^8}(-\text{EllipticE}[i\text{ArcSinh}[(-1)^{1/4}\sqrt{x^4}],-1]$$
$$+\text{EllipticF}[i\text{ArcSinh}[(-1)^{1/4}\sqrt{x^4}],-1])))$$

which is nonelementary.

Thus we can state that for f(x) = cosech^{-1}x and for polynomial h(x) of degree greater than or equal to one, the integral (1) is elementary for m = 1, n = 0 and nonelementary for other cases i.e., the integral of the form

$$\mathrm{I}=\int\frac{\mathrm{cosech}^{-1}[\mathrm{x}^{\mathrm{m}}]}{\mathrm{x}^{\mathrm{n}}}\mathrm{dx}$$

is nonelementary except for m = 1, n = 0, whereas no result comes out always from computing software mathematica for complex polynomial having more than one term in it.

Case-VI: When we take $f(x) = \text{sech}^{-1}x$, the integral (1) becomes

$$\text{I} = \int \frac{\text{sech}^{-1}[\text{g}(\text{x})]}{\text{h}(\text{x})}\text{dx} \qquad (1.6)$$

Taking g(x) = h(x) = x, we get from (1.6)

$$I = \int \frac{\text{sech}^{-1}x}{x}dx$$

Applying mathematica, we get

$$\text{In}[38]\text{: Integrate}[\frac{\text{ArcSech}[x]}{x}, x]$$

$$\text{Out}[38]\text{:}\frac{1}{2}(-\text{ArcSech}[x](\text{ArcSech}[x] + 2\text{Log}[1 + e^{-2\text{ArcSech}[x]}]) + \text{PolyLog}[2, -e^{-2\text{ArcSech}[x]}])$$

which is in the form of polylogarithm and thus nonelementary.

Taking g(x) = x and h(x) = 1, then from (1.6), we have

$$\text{I} = \int \frac{\text{sech}^{-1}[\text{x}]}{1}\text{dx} = \int \text{sech}^{-1}[\text{x}]\text{dx}$$

Integrating using mathematica, we get

$$\text{In}[39]\text{: Integrate}[\text{ArcSech}[x], x]$$

$$\text{Out}[39]\text{: }x\text{ArcSech}[x] - \frac{2\sqrt{\frac{1-x}{1+x}}\sqrt{1-x^2}\text{ArcSin}[\frac{\sqrt{1+x}}{\sqrt{2}}]}{-1+x}$$

which is elementary. Let us take g(x) = x^2 and h(x) = 2x, then from (1.6), we get

$$\text{I} = \int \frac{\text{sech}^{-1}[\text{x}^2]}{2\text{x}}\text{dx}$$

Using mathematica for integration, we get

$$\text{In}[40]\text{: Integrate}[\frac{\text{ArcSech}[x^2]}{2 * x}, x]$$

$$\text{Out}[40]\text{:}\frac{1}{8}(-\text{ArcSech}[x^2](\text{ArcSech}[x^2] + 2\text{Log}[1 + e^{-2\text{ArcSech}[x^2]}]) + \text{PolyLog}[2, -e^{-2\text{ArcSech}[x^2]}])$$

obviously it is nonelementary, as discussed earlier. Let us take one more example

$$\mathrm{I} = \int \frac{\mathrm{sech}^{-1}[\mathrm{x}^2]}{\mathrm{x}^2}\,\mathrm{dx}$$

and using mathematica, we get

$$\mathrm{In}[41]\colon \mathrm{Integrate}[\frac{\mathrm{ArcSech}[x^2]}{x^2}, x]$$

$$\mathrm{Out}[41]\colon \frac{2\sqrt{\frac{1-x^2}{1+x^2}}(1+x^2)}{x} - \frac{\mathrm{ArcSech}[x^2]}{x} - \frac{2\sqrt{\frac{1-x^2}{1+x^2}}\sqrt{1-x^4}(\mathrm{EllipticE}[\mathrm{ArcSin}[x],-1] - \mathrm{EllipticF}[\mathrm{ArcSin}[x],-1])}{-1+x^2}$$

which is nonelementary. Let us take g(x) = x^2+2x and h(x) = $2x^2+1$. Then from (1.6), we get

$$\mathrm{I} = \int \frac{\mathrm{sech}^{-1}[\mathrm{x}^2 + 2\mathrm{x}]}{2\mathrm{x}^2 + 1}\,\mathrm{dx}$$

In this case mathematica doesn't produce any output. In fact using mathematica, we cannot always find the integrals of integrands having polynomials in mixed terms in the antiderivative (1). So let us simplify it as

$$\mathrm{I} = \int \frac{\mathrm{sech}^{-1}[\mathrm{x}^3]}{\mathrm{x}^2}\,\mathrm{dx}$$

$$\mathrm{In}[42]\colon \mathrm{Integrate}[\frac{\mathrm{ArcSech}[x^3]}{x^2}, x]$$

$$\mathrm{Out}[42]\colon \frac{1}{15x}(\frac{1}{(\frac{x^3}{-1+x^3})^{2/3}\sqrt{\frac{1+x^3}{-1+x^3}}}\sqrt{\frac{1-x^3}{1+x^3}}(-9(x^3 + x^6)\mathrm{AppellF1}[-\frac{5}{3},-\frac{2}{3},-\frac{1}{2},-\frac{2}{3},\frac{1}{1-x^3},-\frac{2}{-1+x^3}] + 5(\frac{x^3}{-1+x^3})^{2/3}\sqrt{\frac{1+x^3}{-1+x^3}}(9(1+x^3) + \sqrt{2}(x^3)^{1/3}(-1+x^3)\sqrt{1+x^3}\mathrm{AppellF1}[\frac{3}{2},-\frac{2}{3},\frac{1}{2},\frac{5}{2},1 - x^3,\frac{1}{2}(1-x^3)]))) - 15\mathrm{ArcSech}[x^3])$$

which gives integral in hypergeometric function, which is obviously nonelementary. Let us verify it for more example as

$$\mathrm{I}=\int \frac{\mathrm{sech}^{-1}[\mathrm{x}^3]}{\mathrm{x}^3}\,\mathrm{dx}$$

We get that

$$\mathrm{In}[43]\text{: Integrate}[\frac{\mathrm{ArcSech}[x^3]}{x^3},x]$$

$$\mathrm{Out}[43]\text{: }\frac{1}{96x^2}(\frac{1}{(\frac{x^3}{-1+x^3})^{1/3}\sqrt{\frac{1+x^3}{-1+x^3}}}\sqrt{\frac{1-x^3}{1+x^3}}(-9(x^3+x^6)\mathrm{AppellF1}[-\frac{4}{3},-\frac{1}{3},-\frac{1}{2},-\frac{1}{3},\frac{1}{1-x^3},-\frac{2}{-1+x^3}]+4(\frac{x^3}{-1+x^3})^{1/3}\sqrt{\frac{1+x^3}{-1+x^3}}(18(1+x^3)+\sqrt{2}(x^3)^{2/3}(-1+x^3)\sqrt{1+x^3}\mathrm{AppellF1}[\frac{3}{2},-\frac{1}{3},\frac{1}{2},\frac{5}{2},1-x^3,\frac{1}{2}(1-x^3)]))-48\mathrm{ArcSech}[x^3])$$

which is in hypergeometric function form, and so it is nonelementary. Again for the integral

$$\mathrm{I}=\int \frac{\mathrm{sech}^{-1}[\mathrm{x}^4]}{\mathrm{x}^4}\,\mathrm{dx}$$

We get that

$$\mathrm{In}[44]\text{: Integrate}[\frac{\mathrm{ArcSech}[x^4]}{x^4},x]$$

$$Out[44]\text{: }\frac{1}{270x^3}(\frac{1}{(\frac{x^4}{-1+x^4})^{1/4}\sqrt{\frac{1+x^4}{-1+x^4}}}\sqrt{\frac{1-x^4}{1+x^4}}(-12(x^4+x^8)\mathrm{AppellF1}[-\frac{5}{4},-\frac{1}{4},-\frac{1}{2},-\frac{1}{4},\frac{1}{1-x^4},-\frac{2}{-1+x^4}]+5(\frac{x^4}{-1+x^4})^{1/4}\sqrt{\frac{1+x^4}{-1+x^4}}(24(1+x^4)+\sqrt{2}(x^4)^{3/4}(-1+x^4)\sqrt{1+x^4}\mathrm{AppellF1}[\frac{3}{2},-\frac{1}{4},\frac{1}{2},\frac{5}{2},1-x^4,\frac{1}{2}(1-x^4)]))-90\mathrm{ArcSech}[x^4])$$

which is obviously nonelementary. Let us take one more example

$$\mathrm{I}=\int \frac{\mathrm{sech}^{-1}[\mathrm{x}^4]}{\mathrm{x}^3}\,\mathrm{dx}$$

and using mathematica, we get

$$\text{In}[45]\text{: Integrate}\left[\frac{\text{ArcSech}[x^4]}{x^3}, x\right]$$

$$Out[45]\text{:} -\frac{\text{ArcSech}[x^4] - \frac{2\sqrt{\frac{1-x^4}{1+x^4}}\left(-1+x^8+\sqrt{1+x^4}\sqrt{x^4-x^8}\,\text{EllipticE}[\text{ArcSin}[\frac{\sqrt{1-x^4}}{\sqrt{2}}],2]\right)}{-1+x^4}}{2x^2}$$

which is nonelementary.

Thus we can state that for f(x) = sech^{-1}x and for polynomial h(x), g(x) of degree greater than or equal to one, the integral (1) is elementary for m = 1, n = 0 and nonelementary for other cases i.e., the integral of the form

$$\text{I} = \int \frac{\text{sech}^{-1}[\text{x}^{\text{m}}]}{\text{x}^{\text{n}}}\text{dx}$$

is nonelementary except for m = 1, n = 0.

From above discussion, we can conclude that the antiderivative (1) has the same nature of integral for the pair wise inverse hyperbolic functions ($\sinh^{-1}$x, $\cosh^{-1}$x), ($\tanh^{-1}$x, $\coth^{-1}$x) and (cosech^{-1}x, sech^{-1}x). The common nature for all inverse hyperbolic functions is that the integral (1) is nonelementary for g(x) = h(x) = x and h(x) = 2x, g(x) = x^2, and elementary for g(x) = x, h(x) = 1. For g(x) = x^3, h(x) = x^2 and f(x) = $\sinh^{-1}$x, $\cosh^{-1}$x, cosech^{-1}x, sech^{-1}x, the integral (1) is nonelementary but for f(x) = $\tanh^{-1}$x and $\coth^{-1}$x, it is elementary.

The integral (1) is always elementary for f(x) = $\tanh^{-1}$x, $\coth^{-1}$x for all polynomials g(x), h(x) of degree greater than or equal to two. Whereas for f(x) = $\sinh^{-1}$x, $\cosh^{-1}$x, cosech^{-1}x, sech^{-1}x, the integral (1) is always nonelementary for higher degree polynomials g(x), h(x) of degree greater than or equal to two. In all cases each polynomial has only one term of highest degree. For mixed polynomial like g(x) = x^2+2x, h(x) = 2x + 1 etc. the mathematica7 code doesn't produce any result. Obviously for complex polynomials, we cannot decide its nature using computer software.

Conclusion

We can conclude and proffer the conjecture on the antiderivative (1) that it is always elementary for f(x) = $\tanh^{-1}$x, $\coth^{-1}$x and nonelementary for f(x) = $\sinh^{-1}$x, $\cosh^{-1}$x, cosech^{-1}x, sech^{-1}x for all polynomials g(x), h(x) of degree greater than or equal to two. The only condition is imposed on all polynomials is that it should have only one term with highest degree.

Limitation

The polynomial has only one term with the highest degree. No coefficient has been considered to make the calculation easy. When we take the polynomials in complex form containing more than one term, the mathematica fails to give output, which is the biggest limitation of the study.

Future Scope of Research

The article has been presented for the polynomials having lower degree with only one term. A lot of scope is available for research for standard forms of polynomials of higher degrees.

Acknowledgement

This article is based on the third conjecture proffered by Yadav & Sen (2008, 2012) in their doctoral thesis. A paper based on this conjecture with different f(x) has already been published, which has been proffered on the base of inverse trigonometric functions and has been listed in the references.

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