A New Class of $\hat{\delta}$I-Closed Sets With Respect To Ideal Topological Spaces

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ABSTRACT
The purpose of this paper are to introduce a new class of sets namely $\hat{\delta}$I-closed sets in ideal topological spaces. We discuss about the class lies between the class of ⋆-closed sets and the class of $\hat{\delta}$I-closed sets.

Keywords: $\hat{\delta}$I-closed, $\hat{\delta}I_\alpha$-closed, $\hat{\delta}$I-open, $\hat{\delta}$I-int, $\hat{\delta}$I-cl.

1. INTRODUCTION

In 1966, K. Kuratowski [6], introduced topology. S. Jafari and N. Rajesh [7], introduced the generalized closed sets with respect to an ideal. A. Acikgoz and et al. [1], introduced the on α-I-continuous and α-I-open functions. N. Levine [8], introduced the generalized closed sets in topology.
The purpose of this paper are to introduce a new class of sets namely $\hat{\delta}$I-closed sets in ideal topological spaces. We discuss about the class lies between the class of ⋆-closed sets and the class of $\hat{\delta}$I-closed sets.

2. Preliminaries

An ideal I on a topological space (briefly, TPS) (X, τ) is an on empty collection of subsets of X which satisfies
(1)A∈I and B⊆A⇒B∈I and
(2)A∈I and B∈I⇒A∪B∈I.
Given a topological space (X, τ) with an ideal Ion X if P(X) is the set of all subsets of X, a set operator (•)⋆: P(X)→ P(X), called a local function [4] of A with respect to τ and I is defined as follows: for A ⊆X, A⋆(I, τ)={ x ∈ X:U∩A ∉ I for every U ∈ τ(x)} where τ(x) ={U ∈ τ:x ∈ U}. A Kuratowski closure operator cl⋆(•) for a topology τ⋆(I, τ), called the ⋆-topology and finer than τ, is defined by cl⋆(A) = A ∪A⋆(I, τ) [4]. We will simply write A⋆ for A⋆(I, τ) and τ⋆ for τ⋆(I, τ). If I is an ideal on X, then (X, τ, I) is called an ideal topological space (briefly, ITPS). A subset T of an ideal topological space (X, τ, I) is ⋆-closed (briefly, ⋆-closed) [4] if T⋆ ⊆ T. The interior of a subset T in (X, τ⋆(I)) is denoted by int⋆(A).

Definition 2.1
A subset T of a TPS X is called:
(i) semi-open set [2] if T ⊆ cl(int(T));
(ii) α-open set [9] if T ⊆ int(cl(int(T)));
(iii) β-open set (Semi-pre-open) [2] if T ⊆ cl(int(cl(T)));
The complements of the above-mentioned open sets are called their respective closed sets.

Definition 2.2
A subset T of a TPS X is called
(i) αgs -closedset [9] if αcl(T) ⊆ Q whenever T ⊆ Q and Q is semi-open.
(ii) semi-generalized closed [2] if scl(T) ⊆ Q whenever T ⊆ Q and Q is semi-open.
(iii) ψ -closed [11] if scl(T) ⊆ Q whenever T ⊆ Q and Q is sg-open.
(iv) generalized semi-closed [10] if scl(T) ⊆ Q whenever T ⊆ Q and Q is open.
(v) α -generalized closed [9] if α cl(T) ⊆ Q whenever T ⊆ Q and Q is open.
(vi) generalized semi-pre-closed [10] if spcl(T) ⊆ Q whenever T ⊆ Q and Q is open .
The complements of the above-mentioned closed sets are called their respective open sets.

Definition 2.3
A subset T of a ITPS X is called
(i) I_g-closed (briefly, I_g-cld) set [3] if $T^*\subseteq Q$ whenever $T\subseteq Q$ and Q is open.
(ii) I-$\hat{g}$-closed (briefly, I-ω-cld) [5] if $T^*\subseteq Q$ whenever $T\subseteq Q$ and Q is semi-open.
The complements of the above-mentioned closed sets are called their respective open sets.

3. $\hat{\delta}$I-CLOSED SETS
We introduce the following definition.

Definition 3.1
A subset T of X is called
(i) $\hat{\delta}$I-closed if $T^* \subseteq Q$ whenever $T \subseteq Q$ and Q is sg-open.
The complement of $\hat{\delta}$I-closed is called $\hat{\delta}$I-open.
The family of all $\hat{\delta}$I-closed in X is denoted by $\hat{\delta}$IC(X).
(ii) $\hat{\delta}I_\alpha$-closed if $\alpha\, cl(T^*) \subseteq Q$ whenever $T \subseteq Q$ and Q is sg-open.
The complement of $\hat{\delta}I_\alpha$-closed is called $\hat{\delta}I_\alpha$-open.

Proposition 3.2
Every ⋆-closed is $\hat{\delta}$I-closed.
Proof
If T is any ⋆-closed in X and H is any sg-open set containing T, then $H\supseteq T = T^*$. Hence T is $\hat{\delta}$I-closed.
The converse of Proposition 3.2 need not be true as seen from the following example.

Example 3.3
Let X = {p_1, q_1, r_1},τ = {ϕ, {p_1, q_1}, X} and $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$I-C(X) = {ϕ, {r_1}, {p_1, r_1}, {q_1, r_1}, X}. Here, T = {p_1, r_1} is $\hat{\delta}$I-closed set but not ⋆-closed.

Proposition 3.4
Every $\hat{\delta}$I-closed is $\hat{\delta}I_\alpha$-closed.
Proof
If T is a $\hat{\delta}$I-closed subset of X and H is any sg-open set containing T, then $H\supseteq T^*\supseteq \alpha\, cl(T^*)$. Hence T is $\hat{\delta}I_\alpha$-closed.
The converse of Proposition 3.4 need not be true as seen from the following example.

Example 3.5
Let X = {p_1, q_1, r_1}, τ = {ϕ, {q_1}, X} with $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$IC(X) = {ϕ, {p_1, r_1}, X} and $\hat{\delta}I_\alpha$C(X) = {ϕ, {p_1}, {r_1}, {p_1, r_1}, X}. Here, T = {p_1} is $\hat{\delta}I_\alpha$-closed but not $\hat{\delta}$I-closed.

Proposition 3.6
Every $\hat{\delta}$I-closed is ψ-closed.
Proof
If T is a $\hat{\delta}$I-closed subset of X and H is any sg-open set containing T, then $H\supseteq T^*\supseteq scl(T)$. Hence T is ψ-closed.
The converse of Proposition 3.6 need not be true as seen from the following example.

Example 3.7
Let X = {p_1, q_1, r_1}, τ = {ϕ, {p_1}, X} and $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$IC(X) = {ϕ, {q_1, r_1}, X} and ψ C(X) = {ϕ, {q_1}, {r_1}, {q_1, r_1}, X}. Here, T = {q_1} is ψ-closed but not $\hat{\delta}$I-closed.

Proposition 3.8
Every $\hat{\delta}$I-closed set is $\mathcal{J}$-ω-closed.
Proof
Suppose that $T\subseteq H$ and H is semi-open in X. Since every semi-open set is sg-open and T is $\hat{\delta}$I-closed, therefore $T^*\subseteq H$. Hence T is $\mathcal{J}$-ω-closed.
The converse of Proposition 3.8 need not be true as seen from the following example.

Example 3.9

Let X = {p_1, q_1, r_1, s_1},τ = {ϕ, {s_1}, {q_1, r_1}, {q_1, r_1, s_1}, X} and $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$IC(X) = {ϕ, {p_1}, {p_1, s_1}, {p_1, q_1, r_1}, X} and $\mathcal{J}$- ω C(X) = {ϕ, {p_1}, {p_1, q_1}, {p_1, r_1}, {p_1, s_1}, {p_1, q_1, r_1}, {p_1, q_1, s_1}, {p_1, r_1, s_1}, X}. Here, T = {p_1, r1, s_1} is $\mathcal{J}$- ω -closed but not $\hat{\delta}$I-closed.

Proposition 3.10

Every $\hat{\delta}$I-closed is I_g-closed.

Proof

If T is a $\hat{\delta}$I-closed subset of X and H is any open set containing T, since every open set is sg-open, we have H⊇T*. Hence T is I_g-closed.

The converse of Proposition 3.10 need not be true as seen from the following example.

Example 3.11

Let X = {p_1, q_1, r_1},τ = {ϕ, {p_1}, {q_1, r_1}, X} and $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$IC(X) = {ϕ, {p_1}, {q_1, r_1}, X} and I_g-C(X) = P(X). Here, T = {p_1, q_1} is I_g-closed but not $\hat{\delta}$I-closed.

Proposition 3.12

Every $\hat{\delta}$I-closed is αgs -closed.

Proof

If T is a $\hat{\delta}$I-closed subset of X and H is any semi-open set containing T, since every semi-open set is sg-open, we have H⊇T*⊇ α cl(T*). Hence T is αgs -closed.

The converse of Proposition 3.12 need not be true as seen from the following example.

Example 3.13

Let X = {p_1, q_1, r_1},τ = {ϕ, {p_1}, {q_1, r_1}, X} and $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$IC(X) = {ϕ, {p_1}, {q_1, r_1}, X} and αGS C(X) = P(X). Here, T = {p_1, r_1} is αgs -closed but not $\hat{\delta}$I-closed.

Proposition 3.14

Every $\hat{\delta}$I-closed is α g-closed.

Proof

If T is a $\hat{\delta}$I-closed subset of X and H is any open set containing T, since every open set is sg-open, we have H⊇T*⊇ α cl(T). Hence T is α g-closed.

The converse of Proposition 3.14 need not be true as seen from the following example.

Example 3.15

Let X = {p_1, q_1, r_1},τ = {ϕ, {r_1}, {p_1, q_1}, X} and $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$IC(X) = {ϕ, {r_1}, {p_1, q_1}, X} and αG C(X) = P(X). Here, T = {p_1, r_1} is α g-closed but not $\hat{\delta}$I-closed.

Proposition 1.3.16

Every α -closed is $\hat{\delta}I_\alpha$-closed.

Proof

If T is an α -closed subset of X and H is any sg-open set containing T, we have α cl(T*) = T⊆H. Hence T is $\hat{\delta}I_\alpha$-closed.

The converse of Proposition 3.16 need not be true as seen from the following example.

Example 3.17

Let X = {p_1, q_1, r_1}, τ = {ϕ, X, {p_1, q_1}} and $\mathcal{J}$={ϕ}. Then α C(X) = {ϕ, {r_1}, X} and $\hat{\delta}I_\alpha$C(X) = {ϕ, {r_1}, {p_1, r_1}, {q_1, r_1}, X}. Clearly, the set {p_1, r_1} is an $\hat{\delta}I_\alpha$ closed but not an α -closed.

Proposition 3.18

Every $\hat{\delta}$I-closed is gs-closed.

Proof

If T is a $\hat{\delta}$I-closed subset of X and H is any open set containing T, since every open set is sg-open, we have H⊇T*⊇scl(T). Hence T is gs-closed.

The converse of Proposition 3.18 need not be true as seen from the following example.

Example 3.19

Let X = {p_1, q_1, r_1},τ = {ϕ, {p_1}, X} and $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$IC(X) = {ϕ, {q_1, r_1}, X} and GS C(X) = {ϕ, {q_1}, {r_1}, {p_1, q_1}, {p_1, r_1}, {q_1, r_1}, X}. Here, T = {r_1} is gs-closed but not $\hat{\delta}$I-closed.

Proposition 3.20

Every $\hat{\delta}$I-closed is gsp-closed.

Proof

If T is a $\hat{\delta}$I-closed subset of X and H is any open set containing T, every open set is sg-open, we have H⊇T*⊇spcl(T). Hence T is gsp-closed.

The converse of Proposition 3.20 need not be true as seen from the following example.

Example 3.21

Let X = {p_1, q_1, r_1},τ = {ϕ, {q_1}, X} and $\mathcal{J}$= {ϕ}. Then $\breve{\theta}$-$\mathcal{J}$C(X) = {ϕ, {p_1, r_1}, X} and GSP C(X) = {ϕ, {p_1}, {r_1}, {p_1, q_1}, {p_1, r_1}, {q_1, r_1}, X}. Here, T = {r_1} is gsp-closed but not $\hat{\delta}$I-closed.

Remark 3.22

The following example shows that $\hat{\delta}$I-closed sets are independent of α -closed sets and semi-closed sets.

Example 3.23

Let X = {p_1, q_1, r_1},τ = {ϕ, {p_1, q_1}, X} and $\mathcal{J}$= {ϕ}. Then $\hat{\delta}$IC(X) = {ϕ, {r_1}, {p_1, r_1}, {q_1, r_1}, X} and α C(X) = S C(X) = {ϕ, {r_1}, X}. Here, T = {p_1, r_1} is $\hat{\delta}$I-closed but it is neither α -closed nor semi-closed.

Example 3.24

Let X = {p_1, q_1, r_1},τ = {ϕ, {p_1}, X} and $\mathcal{J}$= {ϕ}. Then $\breve{\theta}$-$\mathcal{J}$C(X) = {ϕ, {q_1, r_1}, X} and α C(X) = S C(X) = {ϕ, {q_1}, {r_1}, {q_1, r_1}, X}. Here, T = {q_1} is α -closed as well as semi-closed but it is not $\hat{\delta}$I-closed.

Theorem 3.25

A set T is $\hat{\delta}$I-closed if and only if T*–T contains no nonempty sg-closed set.

Proof

Necessity. Suppose that T is $\hat{\delta}$I-closed. Let T be a sg-closed subset of T*–T. Then T⊆T^c. Since A is $\hat{\delta}$I-closed, we have T*⊆T^c. Consequently, T⊆ $(T^*)^c$. Hence, T⊆T*∩ $(T^*)^c$ = ϕ. Therefore,T is empty.

Sufficiency. Suppose that T*–T contains no nonempty sg-closed set. Let T⊆H and H be ⋆-closed and sg-open. If T*⊄H, then T*∩H^c≠ϕ. Since T* is a ⋆-closed set and H^c is a sg-closed set, T*∩H^c is a nonempty sg-closed subset of T*–T. This is a contradiction. Therefore, T*⊆H and hence T is $\hat{\delta}$I-closed.

Theorem 3.26

A set T of X is $\hat{\delta}$I-open if and only if F ⊆ int(T) whenever F is sg-closed and F ⊆T.

Proof

Suppose that F ⊆ int(T) such that F is sg-closed and F ⊆T. Let T^c⊆G where G is sg-open. Then G^c ⊆T and G^c is sg-closed. Therefore G^c ⊆ int(T) by hypothesis. Since G^c ⊆ int(T), we have $(int(T))^c$⊆G. i.e., $(T^c)^*$⊆G, since $(T^c)^*$ = $(int(T))^c$. Thus,T^c is $\breve{\theta}$-$\mathcal{J}$-closed. i.e., T is $\hat{\delta}$I-open.

Conversely, suppose that T is $\hat{\delta}$I-open such that F ⊆T and F is sg-closed. Then F^c is sg-open and T^c⊆ F^c. Therefore, $(T^c)^*$⊆ F^c by definition of $\hat{\delta}$I-closed and so F⊆ int(T), since $(T^c)^*$ = $(int(T))^c$.

Lemma 3.27

For an x ∈X, x ∈ $\hat{\delta}$I-cl(T) if and only if Q ∩T≠ϕ for every $\hat{\delta}$I-open set Q containing x.

Proof

Let x ∈$\hat{\delta}$I-cl(T) for any x ∈X. To prove Q ∩T≠ϕ for every $\hat{\delta}$I-open set Q containing x. Prove the result by contradiction. Suppose there exists a $\hat{\delta}$I-open set Q containing x such that Q ∩T = ϕ. Then T⊆Q^c and Q^c is $\hat{\delta}$I-closed. We have $\hat{\delta}$I-cl(T) ⊆Q^c. This shows that x ∉ $\hat{\delta}$I-cl(T) which is a contradiction. Hence Q ∩ T ≠ ϕ for every $\hat{\delta}$I-open set Q containing x.

Conversely, let Q ∩ T≠ϕ for every $\hat{\delta}$I-open set Q containing x. To prove x ∈ $\hat{\delta}$I-cl(T). We prove the result by contradiction. Suppose x ∉ $\hat{\delta}$I-cl(T). Then there exists a $\hat{\delta}$I-closed set F containing T such that x ∉ F. Then x ∈ F^c and F^c is $\hat{\delta}$I-open. Also, F^c∩T = ϕ, which is a contradiction to the hypothesis. Hence x ∈ $\hat{\delta}$I-cl(T).

CONCLUSION

The notions of sets and functions in ideal topological spaces and fuzzy topological spaces are extensively developed and used in many engineering problems, information systems, particle physics, computational topology and mathematical sciences.

By researching generalizations of closed sets, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, all topological functions defined in this thesis will have many possibilities of applications in digital topology and computer graphics.

REFERENCES

[1] A. Acikgoz, T. Noiri and S. Yuksel, "On α-I-continuous and α-I-open functions", Acta Math.Hungar., 105(1-2)(2004), 27-37.

[2] A. I. El-Maghrabi, and A. A. Nasef, "Between semi-closed and GS-closed sets", Journal of Taibah University for Science, 2(2009), 78-87.

[3] T. R. Hamlett and D. Jonkovic, "Ideals in General Topology", Lecture notes in pure and Appl. Math., 123(1990), 115-125.

[4] D. Jankovic and T. R. Hamlett, "New topologies from old via ideals", Amer. Math. Monthly, 97(4)(1990), 295-310.

[5] K. Kavitha, P. Revathi, C. Shanmugavadivu and A. Pandi, "I_Ω-Closed Sets with Respect to an Ideal Topological Spaces", GIS Science Journal, 8(6) (2021), 1276-1280.

[6] K. Kuratowski, "Topology", Vol. I. New York, Academic Press (1966).

[7] S. Jafari and N. Rajesh, "Generalized closed sets with respect to an ideal", European J. Pure Appl. Math., 4(2)(2011), 147-151.

[8] N. Levine, "Generalized closed sets in topology", Rend. Circ Mat. Palermo (2), 19(1970), 89-96.

[9] H. Maki, R. Devi and K. Balachandran, "Associated topologies of generalized α-closed sets and α-generalized closed sets", Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 15(1994), 51-63.

[10] K. Nono, R. Devi, M. Devipriya, K. Muthukumarasamy and H. Maki, "On g#α-closed sets and the digital plane", Bull. Fukuoka Univ. Ed. part III, 53(2004), 15-24.

[11] N. Palaniappan and K. C. Rao, "Regular generalized closed sets", Kyungpook Math. J., 33(1993),211-219.