NEUTROSOPHIC EXTENDED TRIPLET GROUPS AND HOMOMORPHISMS IN C^*-ALGEBRAS

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ABSTRACT. Çelik, Shalla and Olgun [2] defined neutro-homomorphisms in neutrosophic extended triplet groups and Zhang et al. [8] investigated neutro-homomorphisms in neutrosophic extended triplet groups.

In this note, we apply the results on neutro-homomorphisms in neutrosophic extended triplet groups to investigate C^*-algebra homomorphisms in unital C^*-algebras.

1. Introduction and preliminaries

As an extension of fuzzy sets and intuitionistic fuzzy sets, Smarandache [4] proposed the new concept of neutrosophic sets.

Definition 1.1. ([5, 6]) Let N be a nonempty set together with a binary operation $*$. Then N is called a neutrosophic extended triplet set if, for any $a \in N$, there exist a neutral of a (denoted by neut(a)) and an opposite of a (denoted by anti(a)) such that neut(a) $\in N$, anti(a) $\in N$ and

$$\begin{aligned} a * \text{neut}(a) &= \text{neut}(a) * a = a, \\ a * \text{anti}(a) &= \text{anti}(a) * a = \text{neut}(a). \end{aligned}$$

The triplet (a, neut(a), anti(a)) is called a neutrosophic extended triplet.

Note that, for a neutrosophic triplet set $(N, *)$ and $a \in N$, neut(a) and anti(a) may not be unique.

Definition 1.2. ([5, 6]) Let $(N, *)$ be a neutrosophic extended triplet set.

Then N is called a neutrosophic extended triplet group (NETG) if the following conditions hold:

(1) $(N, *)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$;

(2) $(N, *)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

N is called a commutative neutrosophic extended triplet group if, for all $a, b \in N$, $a * b = b * a$.

Let A be a unital C^*-algebra with multiplication operation $\bullet$, unit e and unitary group $U(A) := \{u \in A \mid u^* \bullet u = u \bullet u^* = e\}$. Then $u \bullet v \in U(A)$ and $(u \bullet v) \bullet w = u \bullet (v \bullet w)$ for all $u, v, w \in U(A)$ (see [3]). So $(U(A), \bullet)$ is an NETG.

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Proposition 1.3. ([7]) *Let $(N, *)$ be an NETG. Then*
(1) *neut(a) is unique for each $a \in N$;*
(2) *neut(a) $*$ neut(a)= neut(a) for each $a \in N$.*

Note that $u \bullet e = e \bullet u = u$ for any $u \in (U(A), \bullet)$. By Proposition 1.3, $\text{neut}(u) = e$ for each $u \in U(A)$.

Definition 1.4. ([7]) Let $(N, *)$ be an NETG. Then N is called a weak commutative neutrosophic extended triplet group (briefly, WCNETG) if $a * \text{neut}(b) = \text{neut}(b) * a$ for all $a, b \in N$.

Since $\text{neut}(v) = e$ for all $v \in U(A)$, $u \bullet \text{neut}(v) = \text{neut}(v) \bullet u$ for all $u, v \in U(A)$. So $(U(A), \bullet)$ is a WCNETG.

2. Neutrosophic extended triplet groups and C^*-algebra homomorphisms in unital C^*-algebras

Definition 2.1. ([8]) Let $(N, *)$ be a WCNETG. Then N is called a perfect NETG if anti(neut(a))= neut(a) for all $a \in N$.

Since $\text{anti}(e) = e$ and $\text{neut}(u) = e$ for all $u \in U(A)$, $\text{anti}(\text{neut}(u)) = \text{neut}(u) = e$ for all $u \in U(A)$. Thus $(U(A), \bullet)$ is a perfect NETG.

Definition 2.2. ([1, 2]) Let $(N_1, *)$ and $(N_2, *)$ be neutrosophic extended triplet groups. A mapping $f : N_1 \to N_2$ is called a neutro-homomorphism if

$$f(x * y) = f(x) * f(y)$$

for all $x, y \in N_1$.

From now on, assume that A is a unital C^*-algebra with multiplication operation $\bullet$, unit e and unitary group $U(A)$ and that B is a unital C^*-algebra with multiplication operation $\bullet$ and unitary group $U(B)$.

Definition 2.3. Let $(U(A), \bullet)$ and $(U(B), \bullet)$ be unitary groups of unital C^*-algebras A and B, respectively. A mapping $h : U(A) \to U(B)$ is called a neutro-$*$-homomorphism if

$$\begin{aligned} h(u \bullet v) &= h(u) \bullet h(v), \\ h(u^*) &= h(u)^* \end{aligned}$$

for all $u, v \in U(A)$.

Theorem 2.4. *Let A and B be unital C^*-algebras. Let $H : A \to B$ be a $\mathbb{C}$-linear mapping and let $h : (U(A), \bullet) \to (U(B), \bullet)$ be a neutro-$*$-homomorphism. If $H|_{U(A)} = h$, then $H : A \to B$ is a C^*-algebra homomorphism.*

Proof. Since $H : A \to B$ is $\mathbb{C}$-linear and $x, y \in A$ are finite linear combinations of unitary elements (see [3]), i.e., $x = \sum_{j=1}^m \lambda_j u_j,\ y = \sum_{i=1}^n \mu_i v_i$ $(\lambda_j, \mu_i \in \mathbb{C},\ u_j, v_i \in U(A))$,

$$\begin{aligned} H(x \bullet y) &= H((\sum_{j=1}^m \lambda_j u_j) \bullet (\sum_{i=1}^n \mu_i v_i)) = H(\sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i (u_j \bullet v_i)) \\ &= \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i H(u_j \bullet v_i) = \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i h(u_j \bullet v_i) \\ &= \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i h(u_j) \bullet h(v_i) = \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i H(u_j) \bullet H(v_i) \\ &= H(\sum_{j=1}^m \lambda_j u_j) \bullet H(\sum_{i=1}^n \mu_i v_i) = H(x) \bullet H(y) \end{aligned}$$

for all $x, y \in A$.

Since $H : A \to B$ is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [3]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C},\ u_j \in U(A))$,

$$\begin{aligned} H(x^*) &= H((\sum_{j=1}^m \lambda_j u_j)^*) = H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} h(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} h(u_j)^* \\ &= \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in A$. Thus the $\mathbb{C}$-linear mapping $H : A \to B$ is a C^*-algebra homomorphism. □

3. Conclusions

In this note, we have studied unitary groups of unital C^*-algbras as neutrosophic extended triplet groups and have extended neutro-homomorphisms in neutrosophic extended triplet groups to neutro-$*$-homomorphisms in unitary groups of unital C^*-algebras. We have obtained C^*-algebra homomorphisms in unital C^*-algebras by using neutro-$*$-homomorphisms in unitary groups of unital C^*-algebras.

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Competing interests

The authors declare that they have no competing interests.

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