

# CF Pebbling Number of Path and Path Related Graphs

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## ABSTRACT

Assume  $G$  is a graph with some pebbles distributed over its vertices. A CF pebbling move is when  $x$  pebbles are removed from one vertex,  $\lfloor \frac{x}{2} \rfloor$  pebbles are thrown away and  $\lceil \frac{x}{2} \rceil$  pebbles are moved to an adjacent vertex. The CF pebbling number  $\lambda(G)$ , of a connected graph  $G$  is the least positive integer  $n$  such that any distribution of  $n$  pebbles on  $G$ , allows one pebble to be carried to any arbitrary vertex using a sequence of CF pebbling moves. The CF pebbling number of path and path related graphs are determined in this study.

**Keywords:** CF pebbling move, CF pebbling number, path related graphs.

## 1. INTRODUCTION

Graph theory, an extensive and vibrant area of mathematics, delves into the properties and applications of graphs, which are structures used to model relationships between objects. Among the many intriguing topics within graph theory, pebbling problems have significant attention due to their combinatorial complexity and practical relevance in areas such as network optimization and resource management. This paper examines a particular variant known as the CF (Ceiling Floor) pebbling.

The CF pebbling number of a graph quantifies the minimum number of pebbles needed to guarantee that, regardless of their initial distribution, a pebble can be moved to any target vertex through a series of CF pebbling moves. A CF pebbling move involves removing  $x$  pebbles from a vertex, discarding  $\lfloor \frac{x}{2} \rfloor$  pebbles, and placing  $\lceil \frac{x}{2} \rceil$  pebbles on an adjacent vertex. This variant, incorporating the ceiling and floor functions, introduces additional complexity to the pebbling process, making the determination of the CF pebbling number a challenging and intriguing problem.

In this paper we study the CF pebbling number of path and path-related graphs, analyzing how this number evolves with varying graph configurations. We present precise formulations for the CF pebbling number of path  $P_n$  of length  $n$  and extend our analysis to path related graphs. Here,  $p(v)$  denotes the number of pebbles placed in the vertex  $v$  in a graph  $G$ .

## 2. Preliminaries

**Definition 2.1:** Assume  $G$  is a graph with some pebbles distributed over its vertices. A CF pebbling move is when  $x$  pebbles are removed from one vertex,  $\lfloor \frac{x}{2} \rfloor$  pebbles are thrown away and  $\lceil \frac{x}{2} \rceil$  pebbles are moved to an adjacent vertex. The CF pebbling number  $\lambda(G)$ , of a connected graph  $G$  is the least positive integer  $n$  such that any distribution of  $n$  pebbles on  $G$  allows one pebble to be carried to any arbitrary vertex using a sequence of CF pebbling moves.

**Definition 2.2:** A CF pebbling number  $\lambda(G, v)$ , of a vertex  $v$  of a graph  $G$  is the smallest number  $\lambda(G, v)$  such that at least one pebble may be moved to target vertex  $v$  using a sequence of CF pebbling moves, for any placement of  $\lambda(G, v)$  pebbles on the vertices of  $G$ . The maximum  $\lambda(G, v)$  over all the vertices of  $G$  is the CF pebbling number of a graph denoted as  $\lambda(G)$ .

**Definition 2.3:** A pendant vertex is a vertex that has a degree 1, meaning it is only connected to one edge. Pendant vertices are also known as leaf vertices or end vertices. In trees, pendant vertices are called terminal nodes or simply leaves.

### 3. The CF pebbling number of path and path related graphs

**Theorem 3.1.** For the path  $P_1$ ,  $\lambda(P_1)$  is 2.

**Proof.** Let  $V(P_1) = \{v_1, v_2\}$  and  $E(P_1) = \{v_1v_2\}$ . Without loss of generality, assume  $v_1$  is our target vertex and has zero pebbles. By placing a single pebble on  $v_2$ , a pebble cannot be moved to  $v_1$ . So  $\lambda(P_1) \geq 2$ . If vertex  $v_1$  receives a pebble then there is nothing to prove. Assume that  $v_1$  has zero pebbles, then with two pebbles in  $v_2$ , a pebble can be moved to  $v_1$ . So  $\lambda(P_1) \leq 2$ .

**Theorem 3.2.** For the path  $P_2$ ,  $\lambda(P_2)$  is 3.

**Proof.** Let  $V(P_2) = \{v_1, v_2, v_3\}$  and  $E(P_2) = \{v_1v_2, v_2v_3\}$ . By placing two pebbles on  $v_3$ , a pebble cannot be moved to a target vertex  $v_1$ . So  $\lambda(P_2) \geq 3$ . Without loss of generality, assume that  $v_1$  is our target vertex and distribute three pebbles on vertices of  $P_2$ . If vertex  $v_1$  receives a pebble, then there is nothing to prove, so assume  $v_1$  has zero pebbles. If  $v_2$  receives at least two pebbles then a pebble can be moved from  $v_2$  to  $v_1$ . so assume  $v_2$  receives at most one pebble.

If  $v_2$  has a single pebble then from  $v_3$ , a pebble can be moved to  $v_2$  and from  $v_2$  a pebble can be moved to  $v_1$ . If  $v_2$  has zero pebbles, then from  $v_3$  using CF pebbling move, two pebbles can be moved to  $v_2$  and from  $v_2$  a pebble can be moved to  $v_1$ .

If  $v_2$  is our target vertex, then at least one of  $v_1$  or  $v_3$  receives at least two pebbles, then a pebble can be moved to  $v_2$ . So  $\lambda(P_2) \leq 3$ .

**Theorem 3.3.** For a path  $P_n$  of length  $n$ ,  $\lambda(P_n) = 2^{n-1} + 1, \forall n \geq 2$ .

**Proof:** Let  $V(P_n) = \{v_1, v_2, \dots, v_n, v_{n+1}\}$  and

$E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_{n+1}\}$ . Assume  $v_1$  is our target vertex and it has zero pebbles.

By placing  $2^{n-1}$  pebbles on  $v_{n+1}$ , a pebble cannot be moved to  $v_1$ , hence  $\lambda(P_n) \geq 2^{n-1} + 1$ . Distributing  $2^{n-1} + 1$  pebbles on all vertices of path  $P_n$ . The result is true when  $n = 2$ . Assume the result is true for a path  $P_{k-1}$  (i.e.,  $\lambda(P_{k-1}) = 2^{k-2} + 1$ ).

To prove that the result is true for path  $P_k$ . Any path  $P_k$  can be divided into two paths say  $P_{k_1}$  and  $P_{k_2}$ .

Let  $V(P_{k_1}) = \{v_1\}$  and  $V(P_{k_2}) = \{v_2, \dots, v_n, v_{n+1}\}$ ,  $E(P_{k_2}) = \{v_2v_3, v_3v_4, \dots, v_nv_{n+1}\}$ .

**Case (i):** If  $P_{k_1}$  has no pebble and all pebbles are placed on  $P_{k_2}$ . Let  $v_1$  be our target vertex. Using  $2^{n-2} + 1$  pebbles, two pebbles can be moved to  $v_2$  and a pebble can be moved to  $v_1$ .

**Case (ii):** If  $P_{k_1}$  has all pebbles and  $P_{k_2}$  receives no pebbles. Let any vertex  $v_i, i \neq 1$ , be our target vertex. Then from  $v_1$ ,  $2^{n-2} + 1$  pebbles can be moved to  $v_2$ , using  $2^{n-2} + 1$  pebbles, a pebble can be moved to any target vertex of  $P_{k_2}$  using induction. So  $\lambda(P_n) \leq 2^{n-1} + 1$ .

**Theorem 3.4.** For star graph  $K_{1,n}$ ,  $\lambda(K_{1,n}) = n + 1$  for  $n \geq 1$ .

**Proof:** Let  $V(K_{1,n}) = \{v_0, v_1, \dots, v_n\}$  such that  $\deg(v_0) = n$  and  $\deg(v_i) = 1; 1 \leq i \leq n$ .

Now, put 2 pebbles on  $v_n$  and one pebble on each of the vertices  $v_i, 1 < i < n$ . Then no pebble could be moved to  $v_1$ . Thus  $\lambda(K_{1,n}) \geq n + 1$ .

Now, Consider a distribution  $L$  of  $n+1$  pebbles on the vertices of  $K_{1,n}$ .

**Case (i):** Assume that  $v_1$  be our target vertex and  $p(v_1) = 0$ . If  $n+1$  pebbles are distributed to  $n$  vertices of  $K_{1,n} - \{v_1\}$  such that there exists a vertex with at least 2 pebbles.

Suppose,  $p(v_0) = 2$ . Since  $d(v_0, v_i) = 1; 1 \leq i \leq n$ , one pebble could easily be moved to  $v_1$  by CF pebbling move.

On the other hand if  $p(v_0) \leq 1$  then at least  $n$  pebbles are distributed in  $n-1$  pendant vertices other than  $v_1$ .

Then either any one pendant vertex has at least 3 pebbles or two pendant vertices has at least 2 pebbles. However, by CF pebbling move two pebbles could be moved to  $v_0$  and hence could place a pebble in  $v_1$ .

**Case(ii):** Now, assume  $v_0$  as the target vertex such that  $p(v_0) = 0$ . Then the distribution of  $n+1$  pebbles in the  $n$  pendant vertices of  $K_{1,n}$ , one vertex has at least 2 pebbles. Clearly,  $d(v_0, v_i) = 1; 1 \leq i \leq n$ , a pebble could be moved to  $v_0$ . Hence  $\lambda(K_{1,n}) \leq n + 1$ .

#### Definition 3.1:

The addition of two graphs  $G_1$  and  $G_2$  is a graph with a vertex set that is the union of  $G_1$  and  $G_2$  and an edge set that is the union of  $G_1$  and  $G_2$ .

**Theorem 3.5.** Let  $P_n$  be a path of length  $n$ . Then  $\lambda(P_n + K_1) = n + 2$  for  $n \geq 1$ .

**Proof:** Let  $V(P_n + K_1) = \{v_0, v_1, v_2, \dots, v_{n+1}\}$ . Let  $\deg(v_0) = n + 1, \deg(v_i) = 2,$

$i = 1, n + 1$  and  $\deg(v_j) = 3$  for all  $j = 2, 3, \dots, n$ .

By putting one pebble each on the vertices  $v_1, v_2, \dots, v_{n+1}$ , we cannot move a pebble to  $v_0$ . Thus,  $\lambda(P_n + K_1) \geq n + 2$

**Case (i):** Suppose there are  $n+2$  pebbles, which has been distributed on the vertices of  $P_n + K_1$ .

Let  $v_0$  be the target vertex. If  $p(v_0) = 0$ , then there exists some  $i \in \{1, 2, \dots, n + 1\}$  such that  $p(v_i) \geq 2$ . So, we can move one pebble to  $v_0$  by CF move from such  $v_i$ .

**Case (ii):** Let  $v_k$  be the target vertex such that  $p(v_k) = 0$  and  $1 \leq k \leq n + 1$ .

**Sub case a:** If  $p(v_0) \geq 2$  or  $p(v_i) \geq 3$  for some  $i \neq k$ , then  $\{v_0, v_k\}$  or  $\{v_i, v_0, v_k\}$  forms a transmitting sub graph. Hence we can move a pebble to  $v_k$ .

**Sub case b:** If  $p(v_0) = 1$ , then there exist some  $v_j, j \neq 0, k$  such that  $p(v_j) \geq 2$ . Then  $\{v_j, v_0, v_k\}$  forms a transmitting sub graph and we are done.

**Sub case c:** If  $p(v_0) = 0$  then there should be at least one vertex  $v_s$  such that  $p(v_s) \geq 3$  or there exists atleast two vertices  $v_j$  and  $v_t$  such that  $p(v_j) \geq 2$  and  $p(v_t) \geq 2$ , then two pebbles can be moved to  $v_0$  each one from  $v_j$  and  $v_t$  and hence a pebble could be moved from  $v_0$  to  $v_k$ . Thus  $\lambda(P_n + K_1) \leq n + 2$ .

**Definition 3.2:[2]** A graph which joins the empty graph  $K_m$  on  $m$  nodes and the path graph  $P_n$  on  $n$  nodes is called fan graph. If  $m = 1$  then it is called fan graph and if  $m = 2$  it is called double fan.

**Theorem 3.6.**  $\lambda(P_n + 2K_1) = n + 3$  for  $n \geq 1$ .

**Proof:** Let  $V(P_n + K_1) = \{x_0, y_0, v_1, v_2, \dots, v_{n+1}\}$ . Let  $\deg(x_0)$  and  $\deg(y_0) = n + 1$ ,  $\deg(v_i) = 3$ ,  $i = 1, n + 1$  and  $\deg(v_j) = 4$  for all  $j = 2, 3, \dots, n$ .

Placing  $n+2$  pebbles, one on each vertices  $x_0, y_0, v_1, v_2, \dots, v_n$  leaves the vertex  $v_{n+1}$  unpebbled. Thus  $\lambda(P_n + 2K_1) \geq n + 3$ .

**Case (i):** Suppose there are  $n+3$  pebbles, which has been distributed on the vertices of  $P_n + 2K_1$ .

Let  $x_0$  be the target vertex such that  $p(x_0) = 0$ , then there exists some  $i \in \{1, 2, \dots, n + 1\}$  such that  $p(v_i) \geq 2$ . So, we can move one pebble to  $x_0$  by CF move from such  $v_i$ 's.

Suppose all  $p(v_i) < 2$ , then

i) If  $p(y_0) \geq 3$  then  $x_0$  can be pebbled as  $d(x_0, y_0) = 2$ .

ii) If  $p(y_0) \leq 2$ , then there exists atleast one  $v_i$  such that  $p(v_i) \geq 1$ , thus  $\{y_0, v_i, x_0\}$  forms a transmitting subgraph and hence  $x_0$  can be pebbled.

A similar case holds if  $y_0$  is the target vertex.

**Case (ii):** Let  $v_k$  be the target vertex such that  $p(v_k) = 0$  and  $1 \leq k \leq n + 1$ .

**Sub case (a):** If  $p(x_0) \geq 2$  or  $p(v_i) \geq 3$  for some  $i \neq k$ , then  $\{x_0, v_k\}$  or  $\{v_i, x_0, v_k\}$  forms a transmitting sub graph. Hence we can move a pebble to  $v_k$ . A similar case holds if  $x_0$  is replaced by  $y_0$ .

**Sub case (b):** If  $p(x_0) = 1$ , then there exist some  $v_i$  such that  $p(v_i) \geq 2$ . Thus  $\{v_i, x_0, v_k\}$  forms a transmitting sub graph and we are done. If all  $v_i$  such that  $p(v_i) < 2$ , then

$p(y_0) \geq 2$ . Thus,  $\{y_0, v_j, x_0, v_k\}$  forms a transmitting subgraph such that  $p(v_j) = 1$ ,  $j \neq k$  and hence a pebble can be moved to  $v_k$ .

**Sub case (c):** If  $p(x_0) = 0$  then there exist some  $v_i$  such that  $p(v_i) \geq 2$ . Thus  $\{v_i, x_0, v_k\}$  forms a transmitting sub graph and we are done. If all  $v_i$  such that  $p(v_i) < 2$ , then

$p(y_0) \geq 3$ . Thus,  $\{y_0, v_j, x_0, v_k\}$  forms a transmitting subgraph such that  $p(v_j) = 1$ ,  $j \neq k$  and hence a pebble can be moved to  $v_k$ . Thus  $\lambda(P_n + 2K_1) \leq n + 3$ .

**Theorem 3.7:**

For cycle  $C_n$  with  $n$  vertices,  $n < 8$ ,  $\lambda(C_n) = n$ .

**Proof:**

Let  $V(C_n) = \{v_1, v_2, \dots, v_n\}$  where  $n < 8$ .

Without loss of generality, let us assume that  $v_1$  be our target vertex and  $p(v_1) = 0$ .

Let  $n-1$  pebbles be placed on the vertices of  $C_n$  in such a way that  $p(v_i) = 1$  for all  $i \neq 1$ . Then  $v_1$  cannot be pebbled. Hence  $\lambda(C_n) \geq n$ .

**Case (i) :**  $n$  is even

Let  $n$  pebbles be placed on the vertices of  $C_n$  other than  $v_1$ . That is,  $n$  pebbles are placed on  $n-1$  vertices.

Consider the following distribution.

**Subcase 1:** If all pebbles are placed on  $v_{\frac{n}{2}+1}$ , then by CF pebbling move  $v_1$  can be pebbled.

**Subcase 2:** If  $p(v_{\frac{n}{2}+1}) = 0$ . Consider the paths

$$P_1: v_1, v_2, \dots, v_{\frac{n}{2}}.$$

$$P_2: v_{\frac{n}{2}+2}, \dots, v_n, v_1.$$

Here  $P_1$  and  $P_2$  are paths of lengths  $\frac{n}{2} - 1$ . Since the pebbles are distributed in these paths, either  $P_1$  or  $P_2$  has atleast  $\frac{n}{2}$  pebbles. Since  $\frac{n}{2} > 1 + 2^{\frac{n}{2}-2}$  for  $n = 4, 6$ ,  $v_1$  can be pebbled.

**Subcase 3:** If  $0 < p(v_{\frac{n}{2}+1}) < n$ . Consider the paths

$$P_1: v_1, v_2, \dots, v_{\frac{n}{2}+1}.$$

$$P_2: v_{\frac{n}{2}+1}, \dots, v_n, v_1.$$

Both  $P_1$  and  $P_2$  are paths of length  $\frac{n}{2}$ . If either the path has atleast  $1 + 2^{\frac{n}{2}-1}$  pebbles then  $v_1$  can be pebbled. If both path has less than  $1 + 2^{\frac{n}{2}-1}$  pebbles, then

- 1) For  $n=4$ ,  $v_1$  can be pebbled by CF pebbling move.
- 2) For  $n = 6$ , either  $P_1$  or  $P_2$  has atleast  $\frac{n}{2}$  pebbles, thus  $v_1$  can be pebbled.

**Case (ii):**  $n$  is odd

Let  $n$  pebbles be placed on the vertices of  $C_n$  other than  $v_1$ . Consider the paths

$$P_1: v_1, v_2, \dots, v_{\lfloor \frac{n}{2} \rfloor}.$$

$$P_2: v_{\lfloor \frac{n}{2} \rfloor + 1}, \dots, v_n, v_1.$$

The paths  $P_1$  and  $P_2$  are of length  $\lfloor \frac{n}{2} \rfloor$ . Clearly either  $P_1$  or  $P_2$  has atleast  $\lfloor \frac{n}{2} \rfloor$  pebbles. Thus when  $n=3,5$   $v_1$  can be pebbled.

If  $n = 7$ , placing a pebble in an intermediate vertex of either  $P_1$  or  $P_2$  means  $\lfloor \frac{n}{2} \rfloor$  pebbles are enough to pebble  $v_1$  by CF pebbling moves through the corresponding path.

Without loss of generality, let us assume that  $p(v_{\lfloor \frac{n}{2} \rfloor}) = 4$  and the path  $P_2$  has 3 pebbles. If  $p(v_{\lfloor \frac{n}{2} \rfloor + 1}) \geq 2$ , then by CF pebbling move 2 pebbles can be moved to  $v_{\lfloor \frac{n}{2} \rfloor}$  and thus  $v_1$  can be pebbled.

If  $p(v_{\lfloor \frac{n}{2} \rfloor + 1}) \leq 1$ , then atleast one of the intermediate vertices of  $P_2$  has pebbles. Thus from  $v_{\lfloor \frac{n}{2} \rfloor}$ , by CF pebbling move 2 pebbles can be moved to  $v_{\lfloor \frac{n}{2} \rfloor + 1}$  and hence  $v_1$  can be pebbled.

Thus,  $\lambda(C_n) \leq n$ .

**Lemma 3.1:** For  $n = 1,2$ ,  $\lambda(P_1 \times K_2) = 4$  and  $\lambda(P_2 \times K_2) = 6$ .

**Proof:**

Since  $P_1 \times K_2$  is a cycle  $C_4$  and  $\lambda(C_4) = 4$ .

Now consider  $P_2 \times K_2$ . Let  $V(P_2 \times K_2) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$  and  $E(P_2 \times K_2) = \{\{v_i v_{i+1}\} \cup \{u_i u_{i+1}\} \cup \{v_j u_j\} : 1 \leq i \leq 2, 1 \leq j \leq 3\}$ . Let  $\deg(v_1) = \deg(v_3) = \deg(u_1) = \deg(u_3) = 2$  and  $\deg(v_2) = \deg(u_2) = 3$ .

Let  $u_3$  be the target vertex. If five pebbles are placed in such a way that  $p(v)=1$  for all  $v \neq u_3$ . Then no pebble can reach  $u_3$ . Thus  $\lambda(P_2 \times K_2) \geq 6$ .

**Case (1):** Let  $v$  be the target vertex such that  $\deg(v)=2$  and  $p(v)=0$ . Let 6 pebbles be placed on the graph as follows.

- i) If any vertex  $w$  such  $w \neq v$  and  $w$  is adjacent to  $v$  has atleast 2 pebbles then one pebble can be moved to the target vertex.
- ii) If for all vertex  $w$ ,  $p(w) = 1$ , such that  $w \neq v$  and  $w$  is adjacent to  $v$  and if there exists any  $x$  such that  $p(x) \geq 2$  and  $x$  is adjacent to  $w$ , then one pebble can be moved to  $w$  and  $v$  can be pebbled.
- iii) If for all vertex  $w$ ,  $p(w) = 1$ , such that  $w \neq v$  and  $w$  is adjacent to  $v$  and if  $x$  is adjacent to  $w$  and
  - a) for all  $x$ ,  $p(x) = 0$  then the remaining vertex  $y$  has four pebbles and thus one pebble can be moved to  $w$  and from  $w$ ,  $v$  can be pebbled.
  - b) for all  $x$ ,  $p(x) = 1$  then the remaining vertex  $y$  has two pebbles and thus  $v$  can be pebbled.
  - c) either of the vertex  $x$  adjacent to  $w$  has one pebble then the remaining vertex  $y$  has 3 pebbles, then two pebbles can be moved to  $x$  and from there one pebble can be moved to  $w$  and hence  $v$  can be pebbled.
- iv) if for all vertex  $w$ ,  $p(w) = 0$ , such that  $w \neq v$  and  $w$  is adjacent to  $v$ , then consider the path  $P_1: \{v, w, x, y\}$  and its symmetric path  $P_2: \{v, w, x\}$ . If  $P_1$  has atleast 5 pebbles then  $v$  can be pebbled. If  $P_1$  has less than 5 pebbles then the remaining pebbles must be in  $P_2$ . If  $P_2$  has atleast 3 pebbles then one pebble can be moved to  $v$  through the path  $P_2$ . Otherwise  $P_2$  has 2 pebbles then a pebble can be moved to  $P_1$  and hence the target can be reached.

**Case (2):** Let  $v$  be the target vertex such that  $\deg(v)=3$  and  $p(v)=0$ . Let 6 pebbles be placed on the graph as follows.

- i) If any  $u$  such that  $p(u) \geq 2$  such that  $u$  is adjacent to  $v$  then one pebble can be moved to  $v$ .
- ii) Let  $p(u) < 2$  for all  $u$  such that  $u$  is adjacent to  $v$ . Let the remaining vertices be  $x$  and  $y$  and  $\deg(x)=\deg(y)=2$  and  $d(x,v)=d(y,v)=2$ . Then if  $p(x)=3$  or  $p(y)=3$  then one pebble can be moved to  $v$ . If  $p(x)=0$ , then there exists a  $y-v$  path with atleast 4 pebbles thus  $v$  can be pebbled. If  $p(x)=1$ , then there exists a  $\{y,u,v\}$  path with 3 pebbles such that  $p(y)=2$  and  $p(u)=1$  or  $p(y) \geq 3$ . Thus,  $\{y,u,v\}$  forms a transmitting subgraph and hence  $v$  can be pebbled. A similar case holds if  $x$  is replaced by  $y$ .

Thus  $\lambda(P_2 \times K_2) \leq 6$ .

**Theorem 3.8.** Let  $P_n$  be a path of length  $n$ . Then  $\lambda(P_n \times K_2) = 1 + 2^n$  for  $n \geq 3$ .

**Proof:** Let  $V(P_n \times K_2) = \{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n+2}\}$  and

$$E(P_n \times K_2) = \{ \{v_i, v_{i+1}\} \cup \{v_j, v_{j+1}\} \cup \{v_k, v_{k+n+1}\} : 1 \leq i \leq n, n+1 \leq j \leq 2n+1, 1 \leq k \leq n+1 \}$$

Let  $\deg(v_1) = \deg(v_n) = \deg(v_{n+1}) = \deg(v_{2n+2}) = 2$  and  $\deg(v_i)=3$  for all  $i \neq 1, n, n+1, 2n+2$ .

If the vertex  $v_{2n+2}$  contains  $2^n$  pebbles, then by CF pebbling move, no pebbles could be shifted to  $v_1$ . Thus  $\lambda(P_n \times K_2) \geq 1 + 2^n$ .

Now prove that  $\lambda(P_n \times K_2) \leq 2^n + 1$ .

Consider a distribution  $L$  of  $2^n + 1$  pebbles on  $V(P_n \times K_2)$ .

**Case (i):** Let  $y$  be a target vertex such that  $\deg(y) = 2$ . Without loss of generality, let  $y = v_1$ . Then  $d(v_1, v_t) \leq n+1, 2 \leq t \leq 2n+2$ . And there is a path from  $v_{n+2}$  to  $v_1$ , a pebble could be shifted to the target  $y = v_1$  by the distribution of  $2^n + 1$  pebbles on either path

$$P_1 = \{v_{2n+2}, v_{n+1}, \dots, v_2, v_1\} \text{ or path } P_2 = \{v_{2n+2}, v_{2n+1}, \dots, v_{n+2}, v_1\}.$$

If pebbles are distributed on both the paths, then one of the paths has atleast  $1 + 2^{n-1}$

pebbles. If all  $1 + 2^{n-1}$  are placed on  $v_{2n+2}$  then through the path that contains all  $1 + 2^n$  pebbles, one pebble can be moved to the target vertex. If  $1 + 2^{n-1}$  pebbles are distributed on some vertices of any one of the path say  $P_1$  then clearly atleast  $2^{n-1} - n \geq 1$  pebbles can be moved to  $P_1$  from the path  $P_2$  and as these pebbles are distributed in the intermediate vertices of the path, also one pebble in the intermediate vertex at a distance  $i$  from the initial vertex of the path is equivalent to  $1 + 2^{i-1}$  pebbles placed at the initial vertex of the path, the target vertex can be pebbled.

**Case (ii):** Let  $y$  be any target vertex,  $\deg(y) = 3$  and  $p(y) = 0$ . Consider two paths

$$P_1 = \{v_1, v_2, \dots, v_i, \dots, v_{n+1}\} \text{ and } P_2 = \{v_{n+2}, v_{n+3}, \dots, v_{n+i+1}, \dots, v_{2n+2}\}.$$

Let  $y = v_i$ . Then if  $p(P_1) \geq 1 + 2^{n-1}$  then  $v_i$  can be pebbled as  $d(v, v_i) < n$  for all  $v \in V(P_1)$ . If  $p(P_2) \geq 1 + 2^{n-1}$  then two pebbles can be moved to  $v_{n+i+1}$  as  $d(v, v_{n+i+1}) < n$  for all  $v \in V(P_2)$  and hence one pebble can be moved to  $v_i$ .

A similar proof holds if the target vertex is on the path  $P_2$ .

Thus  $\lambda(P_n \times K_2) \leq 1 + 2^n$  for  $n \geq 3$ .

**Theorem 3.9.**  $\lambda(P_n \odot K_1) = 2^{n+1} + n$  for  $n \geq 1$ .

**Proof:** Let  $V(P_n) = \{v_1, \dots, v_{n+1}\}$  and  $V(P_n \odot K_1) = \{v_1, \dots, v_{n+1}, v'_1, \dots, v'_{n+1}\}$  and

$$E(P_n \odot K_1) = E(P_n) \cup \{v_i v'_i : 1 \leq i \leq n+1\}.$$

Let  $v'_{n+1}$  be the target vertex.

Without loss of generality place  $2^{n+1}$  pebbles on  $v'_1$  and remaining  $n-1$  pebbles on the pendant vertices except  $v'_{n+1}$ .

By the successive CF pebbling moves,  $v_1$  will receive  $\lfloor \frac{2^{n+1}}{2} \rfloor$  pebbles, then  $v_2$  will receive  $\lfloor \frac{2^{n+1}}{4} \rfloor$  pebbles in the next move.

proceeding like this  $v_{n+1}$  will receive exactly one pebble, this will make  $v'_{n+1}$  as unreachable.

Hence  $\lambda(P_n \odot K_1) \geq 2^{n+1} + n$ .

Consider a distribution of  $2^{n+1} + n$  pebbles.

**Case (i):** Choose any vertex  $v_i, 1 \leq i \leq n+1$  such that  $p(v_i) = 0$  as a target vertex. Consider the path  $P' = \{v'_1, v_1, \dots, v_{n+1}, v'_{n+1}\}$  of length  $n+2$ .

If all  $2^{n+1} + n$  pebbles are distributed on the path  $P'$  then our target vertex can be easily pebbled. If all  $2^{n+1} + n$  pebbles are distributed in a way that  $p(P') = 0$  then atleast  $2^n + 1$  pebbles will reach the intermediate vertices  $v_j, 2 \leq j \leq n, i \neq j$ , of  $P'$ . Thus, our target vertex can be pebbled as the distance between the target vertex and the intermediate vertices  $v_j, 2 \leq j \leq n, i \neq j$ , of  $P'$  is atmost  $n-1$ .

If  $2^{n+1} + n$  pebbles are distributed on both  $P_n$  and pendant vertices. Let  $s > 0$  be the number of pebbles distributed on the pendant vertices such that  $p(P_n) = 2^{n+1} + n - s$ . If  $s = 1$ , then  $2^{n+1} + n - 1$  pebbles on  $P_n$  are enough to reach the target vertex with one pebble.

Suppose  $n > s > 1$ . Then the number of pebbles on the path  $P_n$  will be at least  $2^{n+1} + 1$ . Hence one pebble can reach the target vertex. Suppose  $s \geq n$ . Then the number of pebbles the path  $P_n$  can receive from both the path  $P_n$  and the pendant vertices will be at least  $1 + 2^n$ . Since  $P_n$  is a path of length  $n$ , one pebble can be moved to the target vertex.

**Case (ii):** Let  $v_i'$  be the target vertex such that  $p(v_i') = 0$  and  $1 \leq i \leq n + 1$ .

If all the pebbles are placed either on path  $P_n$  or on the pendant vertices except the target vertex then the target vertex can be easily pebbled as the distance between the target vertex and any other vertex in the graph is at most  $n+2$ .

If  $2^{n+1} + n$  pebbles are distributed on both  $P_n$  and pendant vertices. Let  $s > 0$  be the number of pebbles distributed on the pendant vertices such that  $p(P_n) = 2^{n+1} + n - s$ .

Now, proceeding as above in case (i), for any values of  $s$ , the target vertex can be easily pebbled as each pendant vertex is non-adjacent, an intermediate vertex of path  $P_n$  at a distance of  $i$  from the initial vertex of path  $P_n$  has one pebble is equivalent to placing  $1 + 2^{i-1}$  pebbles on the initial vertex of path  $P_n$  and the distance between the target vertex and any other vertex in the graph is at most  $n+2$ .

Thus  $\lambda(P_n \odot K_1) \leq 2^{n+1} + n$ .

Hence  $\lambda(P_n \odot K_1) = 2^{n+1} + n$ .

#### 4. CONCLUSION

In this paper we find the CF pebbling number of path and path related graphs. The CF pebbling number of other standard graphs is an open problem.

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