Identity Intersection graph of a group

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ABSTRACT
Let G be a group with identity e. The Identity Intersection graph $\Gamma_{II}(G)$ of G is a graph with $V(\Gamma_{II}(G))$=G and two distinct vertices x and y are adjacent in $\Gamma_{II}(G)$ if and only if $< x > \cap < y > = \{e\}$, where $< x >$ is the cyclic subgroup of G generated by $x \in G$. In this paper, we want to explore how the group theoretical properties of G can effect on the graph theoretical properties of $\Gamma_{II}(G)$. Some characterizations for fundamental properties of $\Gamma_{II}(G)$ have also been obtained.

Keywords: Identity Intersection graph, Star graph, finite group, p-Sylow subgroup.

1. INTRODUCTION

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and thereby investigating algebraic properties of the ring or group using the associated graph, for instance, see [1, 2]. In the present article, to any group G, we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts. Before starting, let us introduce some necessary notation and definitions.

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph Γ = (V, E), V denote the set of all vertices and E denote the set of all edges in Γ. The degree degΓ(v) of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote degΓ(v) simply by deg(v). The order of Γ is defined |V (Γ)| and its maximum and its minimum degrees will be denoted, respectively, by Δ(Γ) and δ(Γ). A graph Γ is regular if the degrees of all vertices of Γ are the same. A vertex of degree 0 is known as an isolated vertex of Γ. A graph Ω is called a subgraph of Γ if V (Ω) ⊆ V (Γ), E(Ω) ⊆ E(Γ). Let Γ = (V, E) be a graph and let S ⊆ V . A subgraph Ω of Γ is said to be an induced subgraph of Γ induced by S, if V (Ω) = S and each edge of Γ having its ends in S is also an edge of Ω. A simple graph Γ is said to be complete if every pair of distinct vertices of Γ are adjacent in Γ. A graph Γ is said to be connected if every pair of distinct vertices of Γ are connected by a path in Γ. An Eulerian graph has an Eulerian trail, a closed trail containing all vertices and edges. The Union of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a graph Γ = (V, E) with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The join of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a graph denoted by $\Gamma_1 + \Gamma_2 = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup$ {Edges joining every vertex of V_1 with every vertex of V_2}.

Let G be a group with identity e. The order of the group G is the number of elements in G and is denoted by O(G). The order of an element a in a group G is the smallest positive integer k such that ak = e. If no such integer exists, we say a has infinite order. The order of an element a is denoted O(a). Let p be a prime number. A group G with $O(G) = p^k$ for some $k \in Z^+$, is called a p-group.

2. Preparation of Manuscript

In this section, we observe certain basic properties of Identity Intersection graphs.

Proposition 2.1. Let G be a finite group of order n. Then $\Delta(\Gamma_{II}(G))$ is n-1.

Proof. Clearly $< x > \cap < e > = \{e\}$, for all $x \in G$, where e is the identity of the group G. Hence the proof as follows.

Proposition 2.2. If any finite group G and $x \in G$, then x and x^{-1} are non adjacent in $\Gamma_{II}(G)$.

Proof. Always $< x > = < x-1 >$, $< x > \cap < x-1 > /= \{e\}$, where e is an identity element of G. Hence the proof as follows.

Remark 2.3. The converse of the Proposition 2.2 is not true. For, let G = (Z_6, $+_6$). Here 1 and 3 are non adjacent but they are not inverse to each other.

Proposition 2.4. Let G be a finite group and x, y be any two elements of G such that O(x) = p and O(y) = q, where p and q are distinct prime. Then x and y are adjacent in $\Gamma_{II}(G)$.
Proof. Let G be a finite group and x, y be any two elements of G such that O(x) = p and O(y) = q, where p and q are distinct prime. Clearly < x > ∩ < y >= {e} and hence the proof as follows.

Theorem 2.5. Let G be a finite group. Then $\Gamma_{II}(G)$ is a complete graph if and only if every element of G − e is a self inverse element.
Proof. Assume that $\Gamma_{II}(G)$ is a complete graph. Suppose there exist an element a ∈ G − e, which is non self inverse element. Then by Proposition 2.2, a and a^{-1} are non adjacent in $\Gamma_{II}(G)$, which is a contradiction. Therefore every element of G − e is a self inverse element. Conversely, assume that every element of G − e is a self inverse element. Clearly < a > ∩ < b >= {e} for all a, b ∈ G − e and e is adjacent to all other elements. Hence $\Gamma_{II}(G)$ is complete.

Theorem 2.6. For any finite group G, $\Gamma_{II}(G)$ is a star graph if and only if o(G) = p^n, where p is an odd prime number and G has unique subgroup of order p.
Proof. Assume that $\Gamma_{II}(G)$ is a star graph. Suppose not, o(G) /= p^n, where p is an odd prime number. Then there exist two prime p_1 and p_2 such that $p_1|o(G)$ and $p_2|o(G)$. Therefore by cauchy's theorem G has two elements say a and b such that o(a) = p_1 and o(b) = p_2. Clearly < a > ∩ < b >= {e}. Therefore e, a, b form a cycle in $\Gamma_{II}(G)$, which is a contradiction.
Conversely, assume that o(G) = p^n, where p is an odd prime number and G has unique subgroup of order p. Clearly e is adjacent to all other elements in G and every non identity element of G is of order p^k for some 1 ≤ k ≤ n. Since G has unique subgroup H of order p, the intersection of any two subgroup generated by non identity elements must contain H. Therefore any two non identity elements are not adjacent to each other. Hence $\Gamma_{II}(G)$ is a star graph.

Corollary 2.7. For any finite group G, $\Gamma_{II}(G)$ is a tree if and only if o(G) = p^n, where p is an odd prime number and G has unique subgroup of order p.
Proof. Assume that $\Gamma_{II}(G)$ is a tree. Suppose not, o(G) /= p^n, where p is an odd prime number. Then there exist two prime p_1 and p_2 such that $p_1|o(G)$ and $p_2|o(G)$. Therefore by cauchy's theorem G has two elements say a and b such that o(a) = p_1 and o(b) = p_2. Clearly < a > ∩ < b >= {e}. Therefore e, a, b form a cycle in $\Gamma_{II}(G)$, which is a contradiction.
Conversely, assume that o(G) = p^n, where p is an odd prime number and G has unique subgroup of order p. Therefore by Theorem 2.6, $\Gamma_{II}(G)$ is star graph and hence $\Gamma_{II}(G)$ is a tree.

3. Identity Intersection Graph of the Groups has orders pq and p^2.
In this section, we study about the Identity Intersection graph of groups of orders pq and p2 for some prime numbers p and q.

Theorem 3.1. Let G be a group of order pq, where p < q and p, q are two distinct primes. Then
$$\Gamma_{II}(G) \cong K_1 + [K_{p-1,q-1} \cup \overline{K_{\varphi(pq)}},$$
if G is cyclic, where φ is the Euler function and $\Gamma_{II}(G)$ ∼= K_1 + [K_{q-1} + $K_{p-1,p-1,...,p-1}$], if G is non cyclic.
Proof. Case(i). Let G be a cyclic group. In this case G has an unique p-Sylow subgroup and an unique q-Sylow subgroup. G can be partition into 4 sets namely A, B, C, D. Let A = {e}. Let B be the set of elements of order p. Let C be the set of elements of order q and D be the set of elements of order pq. Clearly |A| = 1, |B| = p − 1, |C| = q − 1 and |D| = φ(pq). The element of the set A is adjacent to all other elements. No elements of B is adjacent to itself and every element of B is adjacent to all the element of C. No elements of C is adjacent to itself and every element of C is adjacent to all the element of B. The elements of the set D is adjacent to only the element of the set A. Hence $\Gamma_{II}(G)$ ∼= K_1 + [$K_{p-1,q-1}$ ∪ $K_{\varphi(pq)}$
Case(ii). Let G be a non-cyclic group. In this case G has q p-Sylow subgroups and a unique q-Sylow subgroup. Note that e is adajcent to all other elements. Every non identity elements of q-Sylow subgroup is adajcent to all the elements of p-Sylow subgroups and e. Every non identity element of one p-Sylow subgroup is adjacent to all elements of other p-Sylow subgroup. Hence $\Gamma_{II}(G)$ ∼= K_1 +[K_{q-1} +$K_{p-1,p-1,...,p-1}$].

Theorem 3.2. Let G be a group of order p^2, where p is a prime number.

$$\Gamma_{II}(G) \cong \begin{cases} K_{1,p^2-1} & if\ G\ is\ cyclic \\ K_1 + [K_{p-1,p-1,\ldots,p-1}] & if\ G\ is\ non\ cyclic \end{cases}$$

Proof. Let G be a group of order p^2, where p is a prime number. Cleary G is abelian and also G ∼= Z^{p2} or $Z_p \times Z_p$.
Case(i): If G ∼= Zp_2 , then G is cyclic and has unique subgroup of order p. By Thereome 2.6, $\Gamma_{II}(G)$ must be a star graph and hence the result follows.
Case(ii): If G ∼= Zp × Zp, then G has p + 1 distinct p-sylow subgroups. Each element of one p-Sylow subgroup is adjacent to all the elements of other p-sylow subgroup also e is adjacent to all other elements and hence the result follows.

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