On the stability of 3-Lie homomorphisms and 3-Lie derivations

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Abstract. In this paper, we prove the Hyers-Ulam stability of 3-Lie homomorphisms in 3-Lie algebras for Cauchy-Jensen functional equation. We also prove the Hyers-Ulam stability of 3-Lie derivations on 3-Lie algebras for Cauchy-Jensen functional equation.

1. Introduction and preliminaries

The stability problem of functional equations had been first raised by Ulam [21]. In 1941, Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. The generalizations of this result have been published by Aoki [2] for $(0 < p < 1)$, Rassias [19] for $(p < 0)$ and Gajda [8] for $(p > 1)$ for additive mappings and linear mappings by a general control function $\theta(\|x\|^p + \|y\|^p)$, respectively. In 1994, Găvruta [9] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions, i.e., who replaced $\theta(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Several stability problems for various functional equations have been investigated in [1, 4, 6, 7, 12, 14, 15, 16, 17, 18, 20].

A Lie algebra is a Banach algebra endowed with the Lie product

$$[x, y] := \frac{(xy - yx)}{2}.$$

Similarly, a 3-Lie algebra is a Banach algebra endowed with the product

$$\Big[[x, y], z\Big] := \frac{[x, y]z - z[x, y]}{2}.$$

Let A and B be two 3-Lie algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a 3-Lie homomorphism if

$$H([[x, y], z]) = [[H(x), H(y)], H(z)]$$

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for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a 3-Lie derivation if

$$D\Big([[x,y],z]\Big) = [[D(x),y],z] + [[x,D(y)],z] + [[x,y,],D(z)]$$

for all $x, y, z \in A$ (see [22]).

Throughout this paper, we suppose that A and B are two 3-Lie algebras. For convenience, we use the following abbreviation for a given mapping $f : A \to B$

$$D_\mu f(x,y,z) := f(\frac{\mu x + \mu y}{2} + \mu z) + f(\frac{\mu x + \mu z}{2} + \mu y) + f(\frac{\mu y + \mu z}{2} + \mu x)$$
$$- 2\mu f(x) - 2\mu f(y) - 2\mu f(z)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in A$.

Throughout this paper, assume that A is a 3-Lie algebra with norm $\|\cdot\|$ and that B is a 3-Lie algebra with norm $\|\cdot\|$.

2. Stability of 3-Lie homomorphisms in 3-Lie algebras

We need the following lemmas which have been given in for proving the main results.

Lemma 2.1. ([11]) *Let X be a uniquely 2-divisible abelian group and Y be linear space. A mapping $f : X \to Y$ satisfies*

$$f(\frac{x+y}{2} + z) + f(\frac{x+z}{2} + y) + f(\frac{y+z}{2} + x) = 2[f(x) + f(y) + f(z)] \tag{2.1}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is additive.

Lemma 2.2. *Let X and Y be linear spaces and let $f : X \to Y$ be a mapping such that*

$$D_\mu f(x,y,z) = 0 \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then the mapping $f : X \to Y$ is $\mathbb{C}$-linear.

Proof. By Lemma 2.2, f is additive. Letting $y = z = 0$ in (2.1), we get $2f\left(\mu\frac{x}{2}\right) = \mu f(x)$ and so $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of [13, Theorem 2.1], the mapping $f : X \to Y$ is $\mathbb{C}$-linear. □

In the following, we investigate the Hyers-Ulam stability of (2.1).

Theorem 2.3. *Let $\varphi : A^3 \to [0,\infty)$ be a function such that*

$$\widetilde{\varphi}(x,y,z) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) < \infty \tag{2.3}$$

for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping satisfying

$$\|D_\mu f(x,y,z)\| \le \varphi(x,y,z), \tag{2.4}$$

$$\|f([[x,y],z]) - [[f(x),f(y)],f(z)]\| \le \varphi(x,y,z) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{1}{6}\widetilde{\varphi}(x, x, x) \tag{2.6}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x = y = z$ in (2.4), we get

$$\|3f(2x) - 6f(x)\| \le \varphi(x, x, x) \tag{2.7}$$

for all $x \in A$. If we replace x by $2^n x$ in (2.7) and divide both sides by $3 \cdot 2^{n+1}$. then we get

$$\|\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^n}f(2^n x)\| \le \frac{1}{3 \cdot 2^{n+1}}\varphi(2^n x, 2^n x, 2^n x)$$

for all $x \in A$ and all nonnegative integers n. Hence

$$\begin{aligned}\|\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^m}f(2^m x)\| &= \|\sum_{k=m}^{n}[\frac{1}{2^{k+1}}f(2^{k+1}x) - \frac{1}{2^k}f(2^k x)]\| \\ &\le \sum_{k=m}^{n}\|\frac{1}{2^{k+1}}f(2^{k+1}x) - \frac{1}{2^k}f(2^k x)\| \\ &\le \frac{1}{6}\sum_{k=m}^{n}\frac{1}{2^k}\varphi(2^k x, 2^k x, 2^k x)\end{aligned} \tag{2.8}$$

for all $x \in A$ and all nonnegative integers $n \ge m \ge 0$. It follows from (2.3) and (2.8) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence in B for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges for all $x \in A$. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.8), we get (2.6). It follows from (2.3) that

$$\begin{aligned}\|D_\mu H(x, y, z)\| &= \lim_{n\to\infty}\frac{1}{2^n}\|D_\mu f(2^n x, 2^n y, 2^n z)\| \\ &\le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0\end{aligned}$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$. So $D_\mu H(x, y, z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By Lemma 2.2, the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.5) that

$$\begin{aligned}&\|H([[x, y], z]) - [[H(x), H(y)], H(z)]\| \\ &= \lim_{n\to\infty}\frac{1}{8^n}\|f([[2^n x, 2^n y], 2^n z]) - [[f(2^n x), f(2^n y)], f(2^n z)]\| \\ &\le \lim_{n\to\infty}\frac{1}{8^n}\varphi(2^n x, 2^n y, 2^n z) \le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0\end{aligned}$$

for all $x, y, z \in A$. Thus

$$H([[x, y], z]) = [[H(x), H(y)], H(z)]$$

for all $x, y, z \in A$.

Therefore, the mapping $H : A \to B$ is a 3-Lie homomorphism. □

Corollary 2.4. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be positive real numbers such that $p_1, p_2, p_3 < 1$ and $q_1, q_2, q_3 < 3$. Suppose that $f : A \to B$ is a mapping such that*

$$\|D_\mu f(x, y, z)\| \leq \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3}), \tag{2.9}$$

$$\|f([[x, y], z]) - [[f(y), f(z)], f(x)]\| \leq \varepsilon(\|x\|^{q_1} + \|y\|^{q_2} + \|z\|^{q_3}) \tag{2.10}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{\theta}{3}\{\frac{1}{2 - 2^{p_1}}\|x\|^{p_1} + \frac{1}{2 - 2^{p_2}}\|x\|^{p_2} + \frac{1}{2 - 2^{p_3}}\|x\|^{p_3}\}$$

for all $x \in A$.

Theorem 2.5. *Let $\Phi : A^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=1}^{\infty} 8^n \psi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) < \infty \tag{2.11}$$

for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping such that

$$\|D_\mu f(x, y, z)\|_B \leq \psi(x, y, z),$$

$$\|f([[x, y], z]) - [[f(x), f(y)], f(z)]\| \leq \psi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{6}\widetilde{\psi}(x, x, x) \tag{2.12}$$

for all $x \in A$, where $\widetilde{\psi}(x, y, z) := \sum_{n=1}^{\infty} 2^n \psi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})$ for all $x, y, z \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.3, there exists a unique 3-Lie homomorphism $H : A \to B$ satisfying (2.12). The mapping $H : A \times A \to B$ is given by

$$H(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

The rest of the proof is similar to the proof of Theorem 2.3 □

Corollary 2.6. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and q_3 be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : A \to B$ is a mapping satisfying (2.9) and (2.10). Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\| \leq \frac{\theta}{3}\{\frac{1}{2^{p_1} - 2}\|x\|^{p_1} + \frac{1}{2^{p_2} - 2}\|x\|^{p_2} + \frac{1}{2^{p_3} - 2}\|x\|^{p_3}\}$$

for all $x \in A$.

3. Stability of 3-Lie derivations on 3-Lie algebras

In this section, we prove the Hyers-Ulam stability of 3-Lie derivations on 3-Lie algebras for the functional equation $D_\mu f(x,y,z)=0$.

Theorem 3.1. *Let $\varphi : A^3 \to [0,\infty)$ be a function satisfying (2.3). Suppose that $f : A \to A$ is a mapping satisfying*

$$\|D_\mu f(x,y,z)\| \le \varphi(x,y,z),$$

$$\|f([[x,y],z]) - [[f(x),y],z] - [[x,f(y)],z] - [[x,y],f(z)]\| \le \varphi(x,y,z) \tag{3.1}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique 3-Lie derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \frac{1}{6}\widetilde{\varphi}(x,x,x) \tag{3.2}$$

for all $x \in A$, where $\widetilde{\varphi}$ is given in Theorem 2.3.

Proof. By the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (3.2) and

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. It follows from (3.1) that

$$\begin{aligned}
&\|D([[x,y],z]) - [[D(x),y],z] - [[x,D(y)],z] - [[x,y],D(z)]\| \\
&= \lim_{n\to\infty} \frac{1}{8^n}\|f([[2^n x, 2^n y], 2^n z]) - [[f(2^n x), 2^n y], 2^n z] - [[2^n x, f(2^n y)], 2^n z] - [[2^n x, 2^n x], f(2^n z)]\| \\
&\le \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n x, 2^n y, 2^n z) = 0
\end{aligned}$$

for all $x,y,z \in A$. So

$$D([[x,y],z]) = [[D(x),y],z] + [[x,G(y)],z] + [[x,y],D(z)]$$

for all $x,y,z \in A$. Therefore, the mapping $D : A \to A$ is a 3-Lie derivation. □

Corollary 3.2. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be positive real numbers such that $p_1, p_2, p_3 < 1$ and $q_1, q_2, q_3 <$ 3. Suppose that $f : A \to A$ is a mapping such that*

$$\|D_\mu f(x,y,z)\| \le \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3}), \tag{3.3}$$

$$\|f([[x,y],z]) - [[f(x),y],z] - [[x,f(y)],z] - [[x,y],f(z)]\| \le \varepsilon(\|x\|^{q_1} + \|y\|^{q_2} + \|z\|^{q_3}) \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique 3-Lie derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \frac{\theta}{3}\{\frac{1}{2-2^{p_1}}\|x\|^{p_1} + \frac{1}{2-2^{p_2}}\|x\|^{p_2} + \frac{1}{2-2^{p_3}}\|x\|^{p_3}\}$$

for all $x \in A$.

Theorem 3.3. *Let $\psi : A^3 \to [0,\infty)$ be a function satisfying* (2.11)*. Suppose that $f : A \to A$ is a mapping satisfying*

$$\|D_\mu f(x,y,z)\| \le \psi(x,y,z),$$

$$\|f([[x,y],z]) - [[f(x),y],z] - [[x,f(y)],z] - [[x,y],f(z)]\| \le \psi(x,y,z)$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique 3-Lie derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \frac{1}{6}\widetilde{\psi}(x,x,x) \tag{3.5}$$

for all $x \in A$, where $\widetilde{\psi}$ is given in Theorem 2.5.

Proof. By the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (3.5) and

$$D(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in A$.

The rest of proof is similar to the proof Theorem 3.1. □

Corollary 3.4. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and q_3 be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : A \to B$ is a mapping satisfying (3.3) and (3.4). Then there exists a unique 3-Lie derivation $D : A \to A$ such that*

$$\|f(x) - H(x)\| \le \frac{\theta}{3}\{\frac{1}{2^{p_1}-2}\|x\|^{p_1} + \frac{1}{2^{p_2}-2}\|x\|^{p_2} + \frac{1}{2^{p_3}-2}\|x\|^{p_3}\}$$

for all $x \in A$.

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