On Essentially Semismall Quasi-Dedekind modules with nonsingular modules

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ABSTRACT
Let L be unitary left module over ring with 1 R. In this paper we training the connection among Essentially Semismall Quasi-Dedekind modules and nonsingular modules. As well, we offer round about examples which explain the relations between them.

Keywords: Essentially semismall Quasi-Dedekind module, nonsingular module.

INTRODUCTION
A submodule U of an R-module L is said to be small in L denoted by (U≪ L) if L = U + V for every submodule V of L then V = L[1].A submodule U of an R-module L is said to be semismall of L denoted by ($U\ll_S$ L) if U = 0 or U/V≪ L/V∀ non zero submodule V of U[2].A submodule U of R-module L is essentially semismall denoted by ($U\ll_{es}$L), if for each nonzero semismall submodule V of L, U∩V≠ 0[3]. An R-moduleL is essentially semismall quasi-Dedekind denoted by (ESSQD) if Hom(L/V, L) = 0 ∀$V\ll_{es}$L[3]. A ring R is ESSQD if R is ESSQDR-module[3].Let L be R-module, put $Z(L)=\{l\in L: ann_R(l) \leq_e R\}$. Z(L) is the singular submodule of L. L is said to be singular if Z(L) = L and L is said to be nonsingular if Z(L) = 0[4].In this paper we give the relationship between ESSQD modules and nonsingular modules.
An R-module L remains semismall quasi-Dedekind (SSQD), if for each submodule 0 ≠ V of L be semismall quasi-invertible; that is Hom(L/V, L) = 0, ∀ 0 ≠$V \ll_s$L[5].

Proposition 1 Let L be nonsingular module, thus each essential semismall submodule of L is semismall quasi-invertible submodule of L.
Proof: Let Uremain essential semismall submodule of Las well asa homomorphism f: L/U → L, f ≠ 0. Then∃ l ∈ L s.t f(l + U) = l ≠ 0 . Let r ∈ R and r ∉ ann(l). Thus rl ≠ 0; rl ∉ U. ButU is essential semismall in L, ∃ s ≠ 0, s ∈ R s.t 0 ≠ srl ∈ U. Then 0 = f(srl + U) = srf(l + U) = srlimpliessr ∈ ann(l). Therefore ann(l) is essential semismall ideal of R. Thusl = 0,implies f = 0. Thus Hom(L/U , L) = 0.
From prop.1, we get the following proposition:

Proposition 2 Every nonsingular module is ESSQD module.
The next example shows the opposite of prop. 2 is not correct.

Example 3 $Z_p$as Z-module, where p be prime number is an ESSQD which is not nonsingular since $Z(Z_P)=\{l\in Z_P : ann_Z(l) \leq_e Z\}= Z_P \neq 0$.
A regular ring remains a ring R with identity in which each element r ∈ R is regular, that is rsr = r for some element s ∈ R[6].
A Rickart ring is a ring R with identity in which the left (right) annihilator of each element be principal left (right) ideal generated by an idempotent[6].
An R-module L is prime if ann(L) = ann(W) for any 0 ≠W ≤ L[7].
An element l ∈ L is torsion element when lc = 0, wherever c beregular element of a ring R. The set of torsion elements of L be submodule Z(L) as well as the module L - Z(L) has no nonzero torsion elements[8],
The set Z(L) of these elements remains a submodule but L - Z(L) is not torsion-free[8].

Remarks 4

1) Every Rickart ring remains nonsingular ring, as a result of[4,prop 1.27, p.35],and hence ESSQD ring.
2) Every regular ring be nonsingular ring, as a result of[4,p.36], and hence ESSQD ring.
3) If W be prime R-module, then $End_R(\overline{W})$ be ESSQD.
Proof: From[9, prop 3.7, p.36] and (Rem. 4(2)).
4) Any direct product of integral domains benonsingular ring, as a result of[4, p.36], and hence ESSQD ring.
5) For any ring R. R/Z(R) be nonsingular ring; that is Z(R/Z (R)) = 0, as a result of[4, Ex.5, p.36], thus R/Z(R) be ESSQD ring.
6) Let $V \leq L$. If V and L/Vare both nonsingular, then L be non singular, by[10 ,Ex.5,p.269], thus L be ESSQD.
7) Let L be R-module with $V \leq L$. If L remains non singular, then V remains non singular. So V be ESSQD R-module, in addition to the opposite holds if V $\ll_{es}$Las a result of [10 ,Ex.7.6, p.247], therefore L be ESSQD R-module.
8) An R-module L over integral domain R be non singular R-module iff L be torsion free R-module [10,p.247]. Thus every torsion free above integral domain be ESSQD.
9) A nonsingular module need not be SSQD module as the next example illustrations:

Example 5

1) Let $L = Z \oplus Z$ as Z-module is nonsingular, since for $(d, f) \in L, (d, f) \neq 0$, $ann_Z(d, f) = 0 \nleq_e Z$; that is Z (L) = 0 which is not SSQD [7, Ex1.5,p.7].
2) For each of the Z-modules $Q \oplus Z$ and $Q \oplus Q$ are nonsingular Z-modules which are not SSQD, by [9 ,Ex 1.10,p.27],[9, Ex 3.21,p.40].

Theorem 6 Let R is nonsingular ring thus every faithful multiplication R-module is ESSQD R-module.
Proof: Let L be faithful multiplication R-module, thus from[11,Coro. 2.14], Z (L) = Z (R).L. Since R remains nonsingular ring, then Z (R) = 0, thus Z (L) = 0. Then L remains non singular R-module. Thus from prop.2, L be ESSQDR-module.

Proposition 7 Let L be essentially semismall prime faithful R-module. Thus R is ESSQD ring.
Proof: Let Y be an ideal of R s.t $Y^2 = (0)$. Assume $Y \neq (0)$. Claim $YL \neq (0)$,if YL = (0) the $Y \subseteq ann_R(L) = (0)$;that is Y = (0), a contradiction. But YL $\not\ll_{es}$ L, since if YL $\ll_{es}$L, But L is an essentially semismall prime faithful R-module thus $ann_R(YL) = ann_R(L) = (0)$. However, it is clear that $Y \subseteq ann_R(YL) = (0)$, thus Y = (0), a contradiction, therefore YL$\not\ll_{es}$L. Let E be a relative complement for YL, thus $YL \oplus E \ll_{es}$L. So $ann_R(YL \oplus E) = ann_R(L) = (0)$, since $YN \subseteq YL$ and $YE \subseteq E$, then $YE \subseteq YL \cap E = (0)$ and hence $Y(YL \oplus E) = Y^2L + YE = (0)$. Therefore $Y \subseteq ann_R(YL \oplus E) = (0)$ and Y = (0) , a contradiction. Then our assumption remainsuntrue. Then Y = (0),thus R remains semiprime ring. Therefore by[3,Proposition9], R is ESSQD ring.

Proposition 8 Let Lbe faithful multiplication module overself-injective ring R. R is nonsingular (ESSQD) ring iff L is nonsingular R-Module (ESSQD ring).
Proof: Assume R is nonsingular ring. ButL is faithful multiplication R-module, thus from[11,Coro 2.14] , Z (L) = Z (R). L, since Z (R) = 0 thenZ (L) = 0, therefore L remains nonsingular R-module. Assume L is nonsingular ring. Thus Z (L) = 0. Now, for all $b \in Z(R)$, bL ⊆ Z(R)L = Z(L) = 0, so $b \in ann_R(L) = 0$, then b = 0. Then R is nonsingular ring.

Proposition 9 If E is semismall quasi-invertible R-submodule of L, then ann(L) = ann(E).
Proof: Clearly ann(L) ⊆ ann(E). let r ∈ ann(E). Define f: $L/_E$ → Lby f(l+E) = rl, ∀l ∈ L. Clearly f is well-defined homomorphism. Thusf = 0. Thereforer ∈ ann(L).

Proposition 10 If L is SSQDR-module, then L be semismall prime R-module.
Proof: Since L be SSQD module, thus each semismall submodule 0 ≠ Y of L remains semismall quasi-invertible submodule of L. Then from prop.9, ann (L) = ann (Y), hence L is semismall prime module.

Proposition 11 If L is prime faithful R-module, thus L be nonsingular R-module, and hence L be ESSQD R-module.
Proof: Since Lbe prime R-module, ann(L) be prime ideal of R. But L be prime R-module, thus from [12, Prop 1.3, ch.1], Lbe torsion-free $\overline{R} = R/ann_R(L) \cong R$. Thus L be torsion-free over integral domain R . Then by (Rem 4(8)), L is nonsingular R-module. Therefore L is ESSQD R-module.

Corollary 12 If L is faithful SSQD R-module, thus L is non singular R-module, and hence L is ESSQD R-module.
Proof: From Prop.10 and Prop.11.
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Proposition 13 Let L be faithful module over integral domain R, If L be nonsingular R-module, thus L be ESSQD R-module.
Proof: By prop.2
The next example shows the converse of proposition13 is not correct

Example 14 The Z-module $L = Q \oplus Z_2$ is faithful module over an integral domain Z and hencESSQD, It is easy to see that L is not nonsingular.

Proposition 15 Let L , N be modules over ring R. Let $f : L \longrightarrow N$ be R-monomorphism. If N be non singular R-module, then L be nonsingular R-module and hence L be ESSQD R-module.
Proof: Since $f : L \longrightarrow N$ remains R-homomorphism, thusfrom[10,Lemma 7.2, p.246], $f(Z(L)) \subseteq Z(N)$. But Z(N) = 0. Since N be nonsingular, thus $f(Z(L)) = 0 = f(0)$, since f remains monomorphism, thus Z(L) = 0. Then L remains nonsingular R-module. Therefore L remains ESSQD R-module.

Proposition16 Let L be R-module. If z^k (L) = 0 then L is ESSQD.
Proof: Suppose that z^k(L). Let f ∈ Hom_R(L) and Kerf$\ll_{es}$ L, $\operatorname{Im} f \subseteq Z^K(L) = 0$, so $\operatorname{Im} f = 0$, hence f = 0. Then L is an ESSQD R-module.

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