On the genus of order difference interval graph of a finite abelian group

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ABSTRACT
The order difference interval graph of a group G, denoted by ΓODI(G), is a graph with $V(\Gamma_{ODI}(G))$=G and two vertices a and b are adjacent in $\Gamma_{ODI}(G)$ if and only if o(b) − o(a)∈ [o(a), o(b)]. Without loss of generality, assume that o(a)≤o(b). In this paper, we try to classify all finite abelian groups whose order difference interval graphs are toroidal and projective.

Keywords: order difference interval graph, finite group, planar, genus, crosscap.

INTRODUCTION
There are different ways to associate to a group a certain graph. In this context, it is interesting to ask for the relation between the structure of the group, given in group theoretical terms, and the structure of the graph, given in the language of graph theory. There are many papers on assigning a graph to a group and investigating algebraic properties of group using the associated graph, for instance, see[1,2,3].
Let G be a finite group. One can associate a graph to G in many different ways. Since the order of an element is one of the most basic concepts of group theory, Balakrishnan and Kala [4] defined the order difference interval graph of a group G denoted by ΓODI(G) as follows: Take V (ΓODI(G)) = G and two vertices a and b are adjacent in ΓODI(G)if and only if o(b)−o(a)∈[o(a),o(b)]. Without loss of generality, assume that o(a)≤o(b). Here o(a) and o(b)denote the orders of a and b, respectively. In this paper, we try to classify all finite abeliean group G whose order difference interval graph has genus at most one.
It is well known that any compact surface is either homeomorphic to a sphere, or to a connected sum of g tori, or to a connected sum of k projective planes (see [8, Theorem 5.1]). We denote by Sg the surface formed by a connected sum of g tori, and by Nk the one formed by a connected sum of k projective planes. The number g is called the genus of the surface Sgand k is called the crosscap of Nk. When considering the orientability, the surfaces Sg and sphere are among the orientable class and the surfaces Nk are among the non-orientable one.
A simple graph which can be embedded in S_g but not in S_{g-1} is called a graph of genus g. Similarly, if it can be embedded in N_k but not in N_{k-1}, then we call it a graph of crosscapk. The notation sγ(G) and γ(G) are denoted for the genus and crosscap of a graph G, respectively. It is easy to see that γ(H)≤γ(G) and γ(H)≤γ(G) for all subgraph H of G. Also a graph G is called planar if γ(G)=0, it is called toroidali fγ(G)=1, and it is called projective if γ(G)=1.
A remarkable characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is planar if and only ifit contains no subdivision of K_5 or $K_{3,3}$. A graph is outerplanar if it can be embedded into the plane so that all its vertices lie on the same face.
Throughout this paper, we assume that G is a finite group. We denote the group of integers addition modulo n by Z_n and the Euler function by φ(n). For basic definitions on groups, one may refer[7].

Genus and Crosscap of $\Gamma_{ODI}(G)$
The main goal of this section is to determine all finite abelian groups G whose order difference interval graph has genus one.
Lemma 2.1. [4]Let a be a generator element in group G of order n. Then a is adjacent to all the non-generator elements of G in the graph $\Gamma_{ODI}(G)$.

In view of preceding lemma, we have the following result.

Lemma 2.2. Let G be a cyclic group of order n. Then $\Gamma_{ODI}(G)$ has a subgraph isomorphic to $K\varphi(n),n-\varphi(n)$. Moreover, if n is prime, then $\Gamma_{ODI}(G)\sim=K_{1,n-1.}$

The following characterization of outer plannar graphs was given by Char-tr and and Harary [6]. Using this characterization, we characterize all finite groups G whose ΓG is outerplanar.

Theorem 2.3.[6]A graph G is outer planar if and only if it contains no subdivision of K_4 or $K_{2,3.}$

Theorem 2.4. Let G be a finite abelian group. Then $\Gamma_{ODI}(G)$ is outer planar if and only if G is isomorphic to Z^n,n≥1 orZ_4, where p is a prime.

Proof. Assume that $\Gamma_{ODI}(G)$ is outerplanar. Since G is finite, $|G| = p_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}$, 1. Suppose G has a cyclic subgroup of order p_1p_2. Then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{2,3}$ as a subgraph, a contradiction. Hence G is a p-group and so $|G| = p^n$.

Suppose G has an element of order p^m,m ≥3. Then G has a cyclic subgroup H of order p^m.Then by Lemma 2.2, $\Gamma_{ODI}(H)$ contains $K_{4,4}$ as a subgraph. Therefore $\Gamma_{ODI}(G)$ has a subgraph which is isomorphic to $K_{4,4,}$ a contradiction. Thus G has elements of order at most p_2.

If order of every element of G is p, then $G\sim=Z^n$.

Suppose G has an element of order p^2.If p ≥ 3, then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{2,3}$ as a subgraph which is a contradiction. Hence p must be 2.

Suppose G is a cyclic 2- group, then $G\sim=Z_4$.

Suppose G is a non-cyclic 2-group. If G has a subrgoup which is isomorphic to $Z_2 \times Z_4$, then $\Gamma_{ODI}(Z_2 \times Z_4)$ contains $K_{2,3}$ as a subgraph. Hence $\Gamma_{ODI}(G)$contains $K_{2,3}$ as a subgraph, a contradiction.

Conversely, if $G\sim=Z_n$, then $\Gamma ODI(G)\sim=K_{1,pn-1.}$ If $G\sim=Z_4$,then $\Gamma_{ODI}(G)\sim=K_{4-e.}$

Theorem2.5. Let G be a finite abelian group. Then $\Gamma_{ODI}(G)$ is planar if and only if G is isomorphic to Zn,n≥1orZ_4, where p is a prime.

Proof. Assume that $\Gamma_{ODI}(G)$ is a planar graph. Since G is finite, $|G| = p_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}$, $n \geq 1$.

Case 1. Suppose G is a p-group. Suppose G has an element of order p^m, m≥ 3. Then G has a cyclic subgroup H of order p^m. Then by Lemma 2.2, ΓODI(H) contains $K_{4,4}$ as a subgraph. Therefore $\Gamma_{ODI}(G)$has a subgraph
which is isomorphic to $K_{4,4,}$ a contradiction. Thus G has elements of order atmost $p_{2.}$

If every element of G is of order p, then G∼=Zn.

Suppose G has an element of order p2. Then G has a cyclic subgroup H of order p2.Suppose p ≥ 3, then by Lemma 2.2, $\Gamma_{ODI}(H)$ contains a subgraph which is isomorphic to $K_{3,6}$ and so is Γ_{ODI} (G). Hence p=2.

Suppose G is a cyclic 2-group, then $G\sim=Z_4$.

Suppose G is a non-cyclic 2-group. If G has a subrgoup which is isomorphic to $Z_2 \times Z_4$, then $\Gamma_{ODI}(Z_2\times Z_4)$ contains $K_{3,3}$ as a subgraph, $\Gamma_{ODI}(Z_2\times Z_4)$ is non-planar and so is $\Gamma_{ODI}(G)$.

Case 2. Suppose G is not a p-group. Then $|G| = p_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}, n \geq 2$. Clearly G has a cyclic subgroup of order $m = p_1p_2\cdots p_n$. Consider $S_1 = \{a_i \in G : o(a_i) = m\} \cup \{e\}$ and $S_2 = \{b_j \neq e \in G : o(b_j)|m$ and $o(b_j) \neq m\}$. It is clear that $o(b_j) \leq \frac{m}{2}$ and $o(a_i) - o(b_j) \geq \frac{m}{2}$. Hence a_i is adjacent to b_j for all i and j. Thus $\Gamma_{ODI}(G)$ contains $K_{3,3}$ as a subgraph, a contradiction.

Conversely, if $G \cong \mathbb{Z}_p^n$, then $\Gamma_{ODI}(G) \cong K_{1,p^n-1}$. If $G \cong \mathbb{Z}_4$, then $\Gamma_{ODI}(G) \cong K_4 - e$. □

For a rational number q, [q] is the first integer number greater or equal than q. In the following lemma we bring some well-known formulas for genus of a graph(see[9]).

Lemma 2.6. *The following statements hold:*

(i) $\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$ *if* $n \geq 3$;

(ii) $\gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil$ *if* $m, n \geq 2$.

In the following theorem we determine all finite groups whose ΓODI (G) has genus one.

Theorem 2.7. Let G be a finite abelian p-group. Then γ $(\Gamma_{ODI}(G))$=1 if and only if G is isomorphic to $Z_2\times Z_4$, Z_8,Z_9.

Proof. Assume that $\gamma(\Gamma_{ODI}(G))$=1. Suppose G is a p-group of order p^n. Then by Theorem 2.5, n≥2 and G≇Zn.

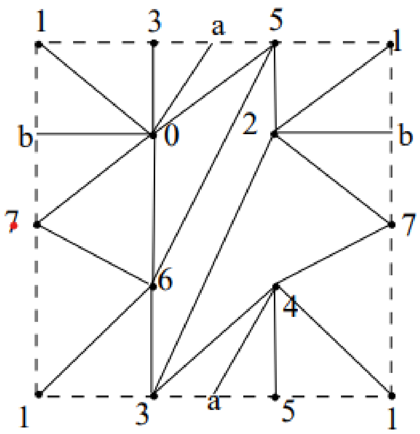
Case 1. Supppose G is cyclic. If n≥4, then by Lemma 2.2, $K_{8,8}$ is a subgraph of $\Gamma_{ODI}(G)$. Therefore by Lemma 2.6, $\gamma(\Gamma_{ODI}(G))$≥9, a contradiction. Thus n≤3.

Suppose n = 3.If p ≥ 3, then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{18,9}$ as a subgraph. Hence by Lemma 2.6, $\gamma(\Gamma_{ODI}(G))$ ≥ 28, which is a contradiction. Hence p=2 and G∼=Z_8.

Suppose n = 2.If p ≥ 5, then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{20,5}$ as a subgraph. Therefore by Lemma 2.6, γ $(\Gamma_{ODI}(G))$≥14, a contradiction. Hence p=2,3. By Theorem 2.5, $\Gamma_{ODI}(Z_4)$is planar and hence G∼=Z_9.

Case 2. Suppose G is non-cyclic. Then G has a subgroup which is isomorphic to Zp×Zpm, m≥2. Consider the sets S1={x∈G:|x|=p},S2={y∈G:|y|=pm}.Clearly|S1|≥p2−1 and |S2|≥pm(p−1).If p≥3,then the subgraph induced by S1∪S2contains K8,18as a subgraph. Therefore γ(ΓODI(G)) ≥ 24, a contradiction. Thus p = 2.If m ≥ 3, then $\Gamma_{ODI}(G)$contains $K_{3,8\ c}$as a subgraph and so $\gamma(\Gamma_{ODI}(G))$≥2, a contradiction.

Suppose G has a subgroup H which is isomorphic to Zp× Zp× Zp2 . Then consider S1={a∈H:|a|=p2} and S2={b∈H:|b|=p}. It is clear that |S1| ≥ 8 and |S2| ≥ 7.Hence the subgraph induced by S1∪S2 is isomorphic to $K_{8,7}$. Therefore by Theorem 2.6, $\gamma(\Gamma_{ODI})(H))$ ≥ 8 and so is $\gamma(\Gamma_{ODI})(H))$ ≥ 8,which is a contradiction.



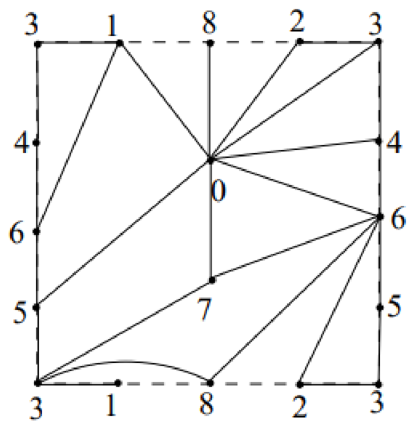


Fig.1 Embedding of $\Gamma_{ODI}(Z_8)$∼=$\Gamma_{ODI}(Z_2\times Z_4)$ **Fig.2** Embedding of $\Gamma_{ODI}(Z_9)$

Suppose G has a subgroup H which is isomorphic to $Z_{p2}\times Z_{p2}$.Then it is easily seen that $K_{4,5}$ as a subgraph of $\Gamma_{ODI}(G)$. Therefore by Lemma 2.6,$\gamma(\Gamma_{ODI}(G))$≥2,a contradiction. Thus G∼=$Z_2\times Z_4$.

Converse follows from Figs. 1,2. □

Theorem 2.8. Let G be a finite abelian non-p-group. Then γ(ΓODI(G))=1 if and only if G is isomorphic to Z6.

Proof. Assume that $\gamma(\Gamma_{ODI}(G)) = 1$. Since G is not a p-group, $|G| = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$

$n > 2$.

Case 1.G is a cyclic group. If n ≥ 3, then by Lemma 2.2, $K_{8,22}$ as a subgraph of $\Gamma_{ODI}(G)$.Therefore by Lemma 2.6, $\gamma(\Gamma_{ODI}(G))$ ≥ 40. Thus n = 2. Suppose k i≥ 2 for some i. Then by Lemma 2.2, $K_{4,8}$ as a subgraph of $\Gamma_{ODI}(G)$. Hence by Lemma $\gamma(\Gamma_{ODI}(G))$≥3, which is a contradiction. Therefore k i=1,i=1,2 and so|G|=p_1p_2. If pi≥5 ,then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{4,6}$ as a subgraph, a contradiction. Therefore |G|=6 and hence G∼=Z_6.

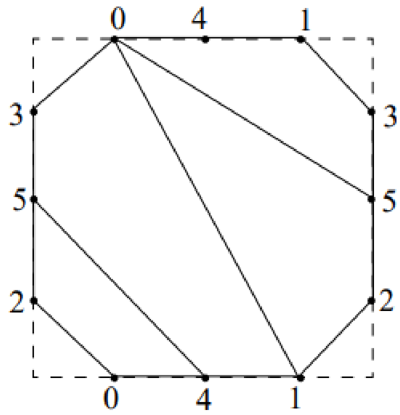


Fig. 3 Embedding of $\Gamma_{ODI}(\mathbb{Z}_6)$

Case 2. Suppose G is not a cyclic group. Then G has a subgroup H which is isomorphic to $Z_{p1}\times Z_{p1}\times Z_{p2}$. Consider $S_1=\{x\in H:|x|=p_1p_2\}$ and $S_2 = \{y \in H : |y| = p_1\} \cup \{e\}$.It is clear that, the subgraph induced by $S_1 \cup S_2$ contains $K_{4,6}$ as a subgraph, a contradiction.

Converse follows from Fig.3. □

In the following lemma we bring some well-known formulas for crosscap of a graph(see[9]).

Lemma 2.9. *The following statements hold:*

$$(i)\ \overline{\gamma}(K_n) = \begin{cases} \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil & if\ \ n \geq 3\ \ and\ \ n \neq 7 \\ 3 & if\ \ n = 7; \end{cases}$$

$$(ii)\ \overline{\gamma}(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil\ if\ m,n \geq 2.$$

By slight modifications in the proof of Theorem 2.7 and Theorem 2.8 with Lemma 2.9, one can prove the following theorem.

Theorem 2.10. Let G be a finite abelian group. Then $\overline{\gamma}(\Gamma ODI(G))=1$ if and only if G is isomorphic to Z_6.

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