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ON TOPOLOGICAL ROUGH GROUPS

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Abstract. In this paper, we give an introduction for rough groups and rough homomorphisms. Then we present some properties related to topological rough subgroups and rough subsets. Finally we construct the product of topological rough groups and give an illustrated example.

1. Introduction

In [2], Bagirmaz *et al.* introduced the concept of topological rough groups. They extended the notion of a topological group to include algebraic structures of rough groups. In addition, they presented some examples and properties.

The main purpose of this paper is to introduce some basic definitions and results about topological rough groups and topological rough subgroups. We also introduce the Cartesian product of topological rough groups.

This paper is as follows: Section 2 gives basic results and definitions on rough groups and rough homomorphisms. In Section 3, following results and definitions of [2], we give some more interesting and nice results about topological rough groups. Finally, in Section 4 we prove that the product of topological rough groups is a topological rough group. Further, an example is provided.

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2. Rough groups and rough homomorphisms

First, we give the definition of rough groups introduced by Biswas and Nanda in 1994 [3].

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Let (U, R) be an approximation space. For a subset $X \subseteq U$,

$$\overline{X} = \{[x]_R : [x]_R \cap X \neq \emptyset\}$$

and

$$\underline{X} = \{[x]_R : [x]_R \subseteq X\}.$$

Suppose that $*$ is a binary operation defined on U. We will use xy instead of $x * y$ for each composition of elements $x, y \in U$ as well as for composition of subsets XY, where $X, Y \subseteq U$.

Definition 2.1. [2] *Let $G = (\underline{G}, \overline{G})$ be a rough set in the approximation space (U, R). Then $G = (\underline{G}, \overline{G})$ is called a rough group if the following conditions are satisfied:*

(1) *for all $x, y \in G$, $xy \in \overline{G}$ (closed);*
(2) *for all $x, y, z \in \overline{G}$, $(xy)z = x(yz)$ (associative law);*
(3) *for all $x \in G$, there exists $e \in \overline{G}$ such that $xe = ex = x$ (e is the rough identity element);*
(4) *for all $x \in G$, there exists $y \in G$ such that $xy = yx = e$ (y is the rough inverse element of x. It is denoted as x^{-1}).*

Definition 2.2. [2] *A nonempty rough subset $H = (\underline{H}, \overline{H})$ of a rough group $G = (\underline{G}, \overline{G})$ is called a rough subgroup if it is a rough group itself.*

A rough set $G = (\underline{G}, \overline{G})$ is a trivial rough subgroup of itself. Also the rough set $e = (\underline{e}, \overline{e})$ is a trivial rough subgroup of the rough group G if $e \in G$.

Theorem 2.1. [2] *A rough subset H is a rough subgroup of the rough group G if the two conditions are satisfied:*

(1) *for all $x, y \in H, xy \in \overline{H}$;*
(2) *for all $y \in H, y^{-1} \in H$.*

Also, a rough normal subgroup can be defined. Let N be a rough subgroup of the rough group G. Then N is called a rough normal subgroup of G if for all $x \in G, xN = Nx$.

Definition 2.3. [4] *Let (U_1, R_1) and (U_2, R_2) be two approximation spaces and $*, *'$ be two binary operations on U_1 and U_2, respectively. Suppose that $G_1 \subseteq U_1$, $G_2 \subseteq U_2$ are rough groups. If the mapping $\varphi : \overline{G_1} \to \overline{G_2}$ satisfies $\varphi(x * y) = \varphi(x) *' \varphi(y)$ for all $x, y \in \overline{G_1}$, then φ is called* **a rough homomorphism**.

Definition 2.4. [4] *A rough homomorphism φ from a rough group G_1 to a rough group G_2 is called:*

(1) *a rough epimorphism (or surjective) if $\varphi : \overline{G_1} \to \overline{G_2}$ is onto.*
(2) *a rough embedding (or monomorphism) if $\varphi : \overline{G_1} \to \overline{G_2}$ is one-to -one.*
(3) *a rough isomorphism if $\varphi : \overline{G_1} \to \overline{G_2}$ is both onto and one-to-one.*

3. Topological rough groups

We study a topological rough group, which has an ordinary topology on a rough group, i.e., a topology τ on $\overline{G}$ induced a subspace topology τ_G on G. Suppose that (U, R) is an approximation space with a binary operation $*$ on U. Let G be a rough group in U.

Definition 3.1. [2] *A topological rough group is a rough group G with a topology τ_G on $\overline{G}$ satisfying the following conditions:*

(1) *the product mapping $f : G \times G \to \overline{G}$ defined by $f(x, y) = xy$ is continuous with respect to a product topology on $G \times G$ and the topology τ on G induced by τ_G;*
(2) *the inverse mapping $\iota : G \to G$ defined by $\iota(x) = x^{-1}$ is continuous with respect to the topology τ on G induced by τ_G.*

Elements in the topological rough group G are elements in the original rough set G with ignoring elements in approximations.

Example 3.1. *Let $U = \{\overline{0}, \overline{1}, \overline{2}\}$ be any group with 3 elements. Let $U/\mathcal{R} = \{\{\overline{0}, \overline{2}\}, \{\overline{1}\}\}$ be a classification of equivalent relation. Let $G = \{\overline{1}, \overline{2}\}$. Then $\underline{G} = \{\overline{1}\}$ and $\overline{G} = \{\overline{0}, \overline{1}, \overline{2}\} = U$. A topology on $\overline{G}$ is $\tau_G = \{\varphi, \overline{G}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1}, \overline{2}\}\}$ and the relative topology is $\tau = \{\varphi, G, \{\overline{1}\}, \{\overline{2}\}\}$. The conditions in Definition 3.1 are satisfied and hence G is a topological rough group.*

Example 3.2. *Let $U = \mathbb{R}$ and $U/\mathcal{R} = \{\{x : x \geqslant 0\}, \{x : x < 0\}\}$ be a partition of $\mathbb{R}$. Consider $G = \mathbb{R}^* = \mathbb{R} - 0$. Then G is a rough group with addition. It is also a topological rough group with the standard topology on $\mathbb{R}$.*

Example 3.3. *Consider $U = S_4$ the set of all permutations of four objects. Let $(*)$ be the multiplication operation of permutations. Let*

$$U/\mathcal{R} = \{E_1, E_2, E_3, E_4\}$$

be a classification of U, where

$$E_1 = \{1, (12), (13), (14), (23), (24), (34)\},$$

$$E_2 = \{(123), (132), (142), (124), (134), (143), (234), (243)\},$$

$$E_3 = \{(1234), (1243), (1342), (1324), (1423), (1432)\},$$

$$E_4 = \{(12)(34), (13)(24), (14)(23)\}.$$

Let $G = \{(12), (123), (132)\}$. Then $\overline{G} = E_1 \cup E_2$. Clearly, G is a rough group. Consider a topology on $\overline{G}$ as $\tau_G = \{\varphi, \overline{G}, \{(12)\}, \{1, (123), (132)\}, \{1, (12), (123), (132)\}\}$. Then the relative topology on G is $\tau = \{\varphi, G, \{(12)\}, \{(123), (132)\}\}$. The conditions in Definition 3.1 are satisfied and hence G is a topological rough group.

Proposition 3.1. [2] *Let G be a topological rough group and fix $a \in G$. Then*

(1) *the mapping* $L_a : G \to \overline{G}$ *defined by* $L_a(x) = ax$*, is one-to-one and continuous for all* $x \in G$.
(2) *the mapping* $R_a : G \to \overline{G}$ *defined by* $R_a(x) = xa$*, is one-to-one and continuous for all* $x \in G$.
(3) *the inverse mapping* $\iota : G \to G$ *is a homeomorphism for all* $x \in G$.

Proposition 3.2. [2] *Let* G *be a topological rough group. Then* $G = G^{-1}$.

Proposition 3.3. [2] *Let* G *be a topological rough group and* $V \subseteq G$. *Then* V *is open (resp. closed) if and only if* V^{-1} *is open (resp. closed).*

Proposition 3.4. [2] *Let* G *be a topological rough group and* W *be an open set in* $\overline{G}$ *with* $e \in W$. *Then there exists an open set* V *with* $e \in V$ *such that* $V = V^{-1}$ *and* $VV \subseteq W$.

Proposition 3.5. [2] *Let* G *be a rough group. If* $G = \overline{G}$, *then* G *is a topological group.*

Definition 3.2. *Let* G *be a topological rough group. Then a subset* U *of* G *is called rough symmetric if* $U = U^{-1}$.

From the definition of rough subgroups, we obtain the following result.

Corollary 3.1. *Every rough subgroup of a topological rough group is rough symmetric.*

Theorem 3.1. *Let* G *be a topological rough group. Then the closure of any rough symmetric subset* A *of* G *is again rough symmetric.*

Proof. Since the inverse mapping $\iota : G \to G$ is a homeomorphism, $cl(A) = (cl(A))^{-1}$. □

Theorem 3.2. *Let* G *be a topological rough group and* H *be a rough subgroup. Then* $cl(H)$ *is a rough group in* $\overline{G}$.

Proof. (1) Identity element: $H \subseteq cl(H)$ implies that $\overline{H} \subseteq \overline{cl(H)}$ and so $e \in \overline{cl(H)}$. Since $cl(H) \subseteq G$, we have $ex = xe = x$ for all $x \in cl(H)$.
(2) Inverse element: $cl(H)^{-1} \subseteq cl(H^{-1}) = cl(H)$.
(3) Closed under product: Let $x, y \in cl(H)$. Then $xy \in \overline{G}$, which implies that there exists an open set $U \in \overline{G}$ such that $xy \in U$. We will prove that $U \wedge H \neq \varphi$. Consider the multiplication mapping $\mu : G \times G \to \overline{G}$. This implies that there exist open sets W, V of G such that $x \in W, y \in V$, $W \wedge H \neq \varphi, V \wedge H \neq \varphi$. Since the topology on G is a relative topology on $\overline{G}$, there exist open sets W', V' of $\overline{G}$ such that $W \subseteq W', V \subseteq V'$. Hence $W' \wedge H \neq \varphi, V' \wedge H \neq \varphi$. Then $\mu(W \times V) \wedge H \neq \varphi$, but we have $\mu(W \times V) \subseteq U$, which implies $H \wedge U \neq \varphi$. So $xy \in cl(H) \subseteq \overline{cl(H)}$. This implies that $cl(H)$ is a rough group of $\overline{G}$.

Thus $cl(H)$ is a rough group in $\overline{G}$. □

Definition 3.3. *Let (X,τ) be a topological rough space of approximation space $(U,\mathcal{R})$, and let $\mathcal{B} \subseteq \tau$ be a base for τ. For $x \in X$, the family*

$$\mathcal{B}_x = \{O \in \mathcal{B} : x \in O\} \subseteq \mathcal{B}$$

is called a base at x.

Theorem 3.3. *Let G be a topological rough space with $\overline{G}$ group. For $g \in \overline{G}$, the base at g is equal to*

$$\mathcal{B}_g = \{gO : O \in \mathcal{B}_e\},$$

where e is the identity element of a rough group G.

4. Product of topological rough groups

Let $(U,\mathcal{R}_1)$ and $(V,\mathcal{R}_2)$ be approximation spaces with binary operations $*_1$ and $*_2$, respectively. Consider the Cartesian product of U and V: let $x, x^{'} \in U$ and $y, y^{'} \in V$. Then $(x,y),(x^{'},y^{'}) \in U \times V$. Define $*$ as $(x,y)*(x^{'},y^{'}) = (x *_1 x^{'}, y *_2 y^{'})$. Then $*$ is a binary operation on $U \times V$. In [1], Alharbi *et al.* proved that the product of equivalence relations is also an equivalence relation on $U \times V$.

Theorem 4.1. [1] *Let $G_1 \subseteq U$ and $G_2 \subseteq V$ be two rough groups. Then the Cartesian product $G_1 \times G_2$ is a rough group.*

The following conditions are satisfied:

(1) For all $(x,y),(x^{'},y^{'}) \in G_1 \times G_2$, $(x_1,y_1^{'})*(x_2,y_2^{'}) = (x_1 *_1 x_2, y_1^{'} *_2 y_2^{'}) \in \overline{G_1} \times \overline{G_2}$.
(2) Associative law is satisfied over all elements in $\overline{G_1} \times \overline{G_2}$.
(3) There exists an identity element $(e,e^{'}) \in \overline{G_1} \times \overline{G_2}$ such that $\forall (x,x^{'}) \in G_1 \times G_2, (x,x^{'}) \times (e,e^{'}) = (e,e^{'}) \times (x,x^{'}) = (ex, e'x^{'}) = (x.x')$.
(4) For all $(x,x^{'}) \in G_1 \times G_2$, there exists an element $(y,y^{'}) \in G_1 \times G_2$ such that $(x,x^{'})*(y,y^{'}) = (y,y^{'})*(x,x^{'}) = (e,e^{'})$.

Example 4.1. *Consider Example 3.1 where $U = \{\overline{0},\overline{1},\overline{2}\}$ and $U/R = \{\{\overline{0},\overline{2}\},\{\overline{1}\}\}$. Then the Cartesian product $U \times U$ is as follows:*

$$U \times U = \{(\overline{0},\overline{0}),(\overline{0},\overline{2}),(\overline{0},\overline{1}),(\overline{2},\overline{0}),(\overline{2},\overline{2}),(\overline{2},\overline{1}),(\overline{1},\overline{0}),(\overline{1},\overline{2}),(\overline{1},\overline{1})\}.$$

Then the new classification is

$$\{\{\overline{0},\overline{0}),(\overline{0},\overline{2}),(\overline{2},\overline{0}),(\overline{2},\overline{2})\},\{(\overline{0},\overline{1}),(\overline{2},\overline{1})\},\{(\overline{1},\overline{0}),(\overline{1},\overline{2})\},\{(\overline{1},\overline{1})\}\}.$$

Consider the rough group $G = \{\overline{1},\overline{2}\}$. Then the Cartesian product $G \times G$ is

$$G \times G = \{(\overline{2},\overline{2}),(\overline{2},\overline{1}),(\overline{1},\overline{2}),(\overline{1},\overline{1})\},$$

where $\overline{G \times G} = \overline{G} \times \overline{G} = U \times U$. From the definition of a rough group, we have that

(1) *the multiplication of elements in* $G \times G$ *is closed under* $\overline{G} \times \overline{G}$, *i.e.* $(\overline{2},\overline{2})(\overline{2},\overline{2}) = (\overline{1},\overline{1}), (\overline{2},\overline{2})(\overline{2},\overline{1}) = (\overline{1},\overline{0}), (\overline{2},\overline{2})(\overline{1},\overline{1}) = (\overline{0},\overline{0}), (\overline{2},\overline{2})(\overline{1},\overline{2}) = (\overline{0},\overline{1}), (\overline{2},\overline{1})(\overline{2},\overline{1}) = (\overline{1},\overline{2})$,
$(\overline{2},\overline{1})(\overline{1},\overline{1}) = (\overline{0},\overline{2}), (\overline{2},\overline{1})(\overline{1},\overline{2}) = (\overline{0},\overline{0}), (\overline{1},\overline{1})(\overline{1},\overline{1}) = (\overline{2},\overline{2}), (\overline{1},\overline{1})(\overline{1},\overline{2}) = (\overline{2},\overline{0})$;

(2) *there exists* $(\overline{0},\overline{0}) \in \overline{G}\times\overline{G}$ *such that for every* $(g,g^{'}) \in G\times G$, *we have* $(\overline{0},\overline{0})(g,g^{'}) = (g,g^{'})$;

(3) *for every element of* $G \times G$, *there exists an inverse element in* $G \times G$, *where* $(\overline{1},\overline{1})^{-1} = (\overline{2},\overline{2}) \in G \times G$, $(\overline{2},\overline{1})^{-1} = (\overline{1},\overline{2}) \in G \times G$;

(4) *the associative law is satisfied.*

Hence $G \times G$ *is a rough group.*

From Example 3.1, *we have* $\tau_G = \{\varphi, \overline{G}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1},\overline{2}\}\}$ *as a topology on* $\overline{G}$. *Then* $\tau_G \times \tau_G$ *is the product topology of* $\overline{G} \times \overline{G}$. *Also we have* $\tau = \{\varphi, G, \{\overline{1}\}, \{\overline{2}\}\}$ *as a relative topology on* G. *So* $\tau \times \tau$ *is a topology on* $G \times G$ *induced by* $\tau_G \times \tau_G$.

Consider the multiplication mapping $\mu : (G \times G) \times (G \times G) \to \overline{G} \times \overline{G}$. *This mapping is continuous with respect to topology* $\tau \times \tau$ *and the product topology on* $(G \times G) \times (G \times G)$. *Also, we can show that the inverse mapping* $\iota : G \times G \to G \times G$ *is continuous. Hence* $G \times G$ *is a topological rough group.*

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