Fourth Hankel Determinants for star- Like Functions related with the Analytic functions

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Received: 14.07.2024 Revised: 26.08.2024 Accepted: 14.09.2024

ABSTRACT
In this paper, A newanalytic function Λ_{σ}° subclass is introduced by this paper and deriving the fourth Hankel determinant$H_4(1)$bound for a new class.

Keywords: analytic functions ,univalent functions, fourth-order Hankel determinant, subordination.

1. INTRODUCTION

Suppose we have a functions f with class $\mathcal{B}$where f is analytic function and attributed to open unit disc$U = \{z \in \mathbb{C} : |z| < 1\}$, which take the following form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, z \in U. \qquad (1.1)$$

Suppose we have a univalent function of $\mathcal{B}$,assume$\sigma$is subclass of$\mathcal{B}$, let $\mathfrak{C}$is a analytic functions class where its normalization by:

$$\mathrm{M}(z) = 1 + v_1 z + v_2 z^2 + v_3 z^3 + \cdots, \qquad (1.2)$$

and satisfy the inequality below

$$Re\big(\mathrm{M}(z)\big) > 0, z \in U.$$

Suppose we have two functions f and g where both are analytic in U.
Therfore, $g$is subordinate offunction f, which is given by
$f(z) \prec g(z), z \in U$, if the Schwarz function $w(z)$exists, with $|w(z)| < 1$ and $w(0) = 0$ [seemore[21]], $g(z) = f\big(w(z)\big), z \in U$.
In 2018 Cho et al. [9] had presented function class $\mathcal{S}_s^*$ as follow:

$$\mathcal{S}_s^* = \left\{\frac{zf'(z)}{f(z)} \prec 1 + \sin z \ , f \in \mathcal{A}, z \in U\right\},$$

We introduced the new class.

Definition (1.1) : Suppose that the function $f \in \mathcal{B}$ and be taking (1.1),which take the form of convex function classΛ_{σ}° :

$$\Lambda_{\sigma}^{\circ} = \left\{\frac{zf'(z)}{f(z)} \prec (1 + \sin z)e^{w(z)} , f \in \mathcal{B} z \in U\right\}. \qquad (1.3)$$

The $\mathbb{q}^{th}$ Hankel determinant had been stated by Noonan and Thomas [23] in 1976 where $\mathbb{q} \geq 1$ and $n \geq 1$of functions f as given below:

$$H_{\mathbb{q}}(n) = \begin{vmatrix} \alpha_n & \alpha_{n+1} & \cdots & \alpha_{n+\mathbb{q}-1} \\ \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{n+\mathbb{q}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n+\mathbb{q}-1} & \alpha_{n+\mathbb{q}} & \cdots & \alpha_{n+2\mathbb{q}-2} \end{vmatrix}, a_1 = 1.$$

In particular, we have
For $\mathbb{q} = 2 , n = 1$ and$a_1 = 1$, $\mathrm{H}_2(1) = \alpha_3 - \alpha_2^2$is the well − knownFekete − Szegö functional. $H_2(2)$ known as second Hankel which is given by$H_2(2) = \alpha_2\alpha_4 - \alpha_3^2$ for $\mathbb{q} = 2 , n = 2$, that had been stadied as classes of bi- convex and bi-star like functions(see more[1,2,4,5,10,12,15,16,25,31]).The third Hankel determinant is represented by:

$$H_3(1) = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_3 & \alpha_4 & \alpha_5 \end{vmatrix}, \mathbb{q}_1 = 3, n = 1,$$

where elements of the previous determinant are different classes of analytic functions which equivelent [6,7,8,11,12,13,17,18,20,22,27,28,29,30,32,34]. The bound of third Hankle determinant for the univalent starlike functions had been introduced by Islam et. Zaprawa et al. [33] .Fourth Hankel determinant

$H_4(1)$was studied by Arif et al. [3] . Many classes of functions with bounded functions turning are connected to the sine functions and obtained upper bounds for the third and fourth order were presented by Khan et al. [14] .
Suppose$f \in \sigma, \alpha_1 = 1$therefore

$$H_4(1) = \{(\alpha_2\alpha_4 - \alpha_3^2)\alpha_3 - (\alpha_4 - \alpha_2\alpha_3)\alpha_4 + (\alpha_3 - a_2^2)\alpha_5\}\alpha_7 - \{(\alpha_2\alpha_5 - \alpha_3\alpha_4)\alpha_3 - (\alpha_5 - \alpha_2\alpha_4)\alpha_4 + (\alpha_3 - \alpha_2^2)\alpha_6\}\alpha_6 + \{(\alpha_3\alpha_5 - \alpha_4^2)\alpha_3 - (\alpha_5 - \alpha_2\alpha_4)\alpha_5 + \alpha_4 - \alpha_2\alpha_3\}\alpha_6 + \{(\alpha_3\alpha_5 - \alpha_4^2)\alpha_3 - (\alpha_5 - \alpha_2\alpha_4)\alpha_5 + (\alpha_4 - \alpha_2\alpha_3)\alpha_6\}\alpha_5 - \{(\alpha_3\alpha_5 - \alpha_4^2)\alpha_4 - (\alpha_2\alpha_5 - \alpha_3\alpha_4)\alpha_5 + (\alpha_4 - \alpha_2\alpha_3)\alpha_6\}\alpha_4 . \quad (1.4)$$

The new concept on fourth Hankel determinant is represent the first original result .

Definition (1.2): Theexpression below represent the fourth Hankel determinant of a function f of the form (1.1):

$$H_4(1) = \begin{vmatrix} 1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \end{vmatrix} = \Xi_1\alpha_7 + \Xi_2\alpha_6 + \Xi_3\alpha_5 + \Xi_4\alpha_4,$$

such that

$|\Xi_1| = |\alpha_2\alpha_4 - \alpha_3^2||\alpha_3| + |\alpha_4 - \alpha_2\alpha_3||\alpha_4| + |\alpha_3 - \alpha_2^2||\alpha_5|$.
$|\Xi_2| = |\alpha_2\alpha_5 - \alpha_3\alpha_4||\alpha_3| + |\alpha_5 - \alpha_2\alpha_4||\alpha_4| + |\alpha_3 - \alpha_2^2||\alpha_6|$.
$|\Xi_3| = |\alpha_3\alpha_5 - \alpha_4^2||\alpha_3| + |\alpha_5 - \alpha_2\alpha_4||\alpha_5| + |\alpha_4 - \alpha_2\alpha_3||\alpha_6|$.
$|\Xi_4| = |\alpha_3\alpha_5 - \alpha_4^2||\alpha_4| + |\alpha_2\alpha_5 - \alpha_3\alpha_4||\alpha_5| + |\alpha_4 - \alpha_2\alpha_3||\alpha_6|$.

2. Perliminaries

To prove the important results, we must follow Lemmas are needed.

Lemma(2.1)[19]: SupposeM$(z) \in \mathscr{B}$, thus thereexists some $z, \mathcal{X}$together with$|z| \le 1$, $|x| \le 1$,such that

$2v_2 = v_1^2 + \mathcal{X}(4 - v_1^2)$.
$4v_3 = v_1^3 + 2v_1\mathcal{X}(4 - v_1^2) - (4 - v_1^2)v_1\mathcal{X}^2 + 2(4 - v_1^2)(1 - |\mathcal{X}|)z$.

Lemma(2.2) [26]: SupposeM$(z) \in \mathscr{B}$Then

$|v_1^4 + v_2^2 + 2v_1v_3 - 3v_1^2v_2 - v_4| \le 2$,
$|v_1^5 + 3v_1v_2^2 + 3v_1^2v_3 - 4v_1^3v_2 - 2v_1v_4 - 2v_2v_3 + v_5| \le 2$,
$|v_1^6 + 6v_1^2v_2^2 + 4v_1^3v_3 + 2v_1v_5 + 2v_2v_4 + v_3^2 - v_2^3 - 5v_1^4v_2 - 3v_1^2v_4 - 6v_1v_2v_3 - v_6| \le 2$,
$|v_n| \le 2$, $n = 1,2,3, \dots$.

Lemma (2.3) [24]: SupposeM$(z) \in \mathscr{B}$, therefore

$\left|v_2 - \frac{v_1^2}{2}\right| \le 2 - \frac{|v_1|^2}{2}$,
$|v_{n+k} - jv_nv_k| \le 2$, $0 \le j \le 1$,
$|v_{n+2k} - jv_nv_k^2| \le 2(1 + 2j)$.

3. Main Results

Now we present the prove and statement to our theorems as a part of our work in this paper.

Theorem (3.1): Suppose that$f \in \Lambda_\sigma^\circ$ and be taking(1.1),thus

$|\alpha_2| \le 2$, $|\alpha_3| \le 1$, $|\alpha_4| \le 0.8o2288343$, $|\alpha_5| \le 0.968639472$, $|\alpha_6| \le \frac{38}{75}$, $|\alpha_7| \le 0.311130508$.

Proof : Suppose$f \in \Lambda_\sigma^\circ$ and$\frac{zf'(z)}{f(z)} = (1 + \sin w(z))e^{w(z)}$, such that,

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + (4a_5 - 2a_3^2 - 4a_2a_4 + 4a_2^2a_3 - a_2^4)z^4 + (5a_6 - 2a_3^2 - 5a_2a_5 + 6a_2^2a_4 - 5a_3a_4 + 5a_2a_3^2 - 5a_2^3a_3 + a_2^5)z^5 + (6a_7 - 6a_2a_6 - 2a_5a_3 + \cdots)z^6 = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots, \quad (3.2)$$

Suppose $\mathfrak{p}(z) \in \mathbb{p}(z)$,in the some conditions for Schwarz function $w(z)$,

$$\mathfrak{p}(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + v_1z + v_2z^2 + v_3z^3 + \cdots,$$

consider that$\mathbb{p}(z) \in \mathfrak{C}$ and

$$w(z) = \frac{\mathbb{p}(z) - 1}{1 + \mathbb{p}(z)} = \frac{v_1z + v_2z^2 + v_3z^3 + \cdots}{2 + 1 + v_1z + v_2z^2 + v_3z^3 + \cdots}.$$

On the other side,

$$(1+\sin w(z))e^{w(z)}$$
$$= v_1 z + \left(v_2 - \frac{v_1^2}{8}\right)z^2 + \left(\frac{v_1^3}{16} + v_3 - \frac{v_1 v_2}{2}\right)z^3 + \left(v_4 - \frac{v_1 v_3}{4} + \frac{3v_1^2 v_2}{16} - \frac{v_2^2}{8} - \frac{25v_1^4}{384}\right)z^4$$
$$+ \left(v_5 - \frac{v_2 v_3}{4} - \frac{v_1 v_4}{4} - \frac{3v_1^2 v_3}{16} - \frac{3v_2 v_1^2}{16} - \frac{25v_1^3 v_2}{96} + \frac{143v_1^5}{3840}\right)z^5$$
$$+ \left(v_6 - \frac{-v_1 v_5}{4} - \frac{v_2 v_4}{4} + \frac{3v_1 v_2 v_3}{8} + \frac{v_2^3}{8} - \frac{v_3^2}{4} + \frac{743v_3 v_1^6}{46080} - \right.$$
$$\left. \frac{143v_1^4 v_2}{768} - \frac{25v_1^3 v_3}{16} + \frac{25v_1^2 v_2^2}{16} - \frac{v_2^3 v_3}{16}\right)z^6 + \cdots. \quad (3.3)$$

Comparing the coefficients of $z, z^2, \dots, z^6$ between equations (3.2) and (3.3), we have

$\alpha_2 = \frac{1}{2}v_1,$

$\alpha_3 = \frac{v_2}{2} - \frac{7v_1^2}{16},$ (3,4)

$\alpha_4 = \frac{7v_1^3}{12} + \frac{v_3}{3} - \frac{5v_1 v_2}{12},$

$$\alpha_5 = \frac{v_4}{4} + \frac{17v_1^2 v_2}{192} + \frac{13v_1 v_3}{48} + \frac{3v_2^2}{32} - \frac{61v_1^4}{384}$$

$\alpha_6 = \frac{v_5}{5} + \frac{2v_2 v_3}{60} + \frac{3v_1 v_4}{10} + \frac{49v_1^3 v_2}{24} + \frac{11v_1^2 v_2}{80} + \frac{779v_1^5}{2400} - \frac{v_1^2 v_3}{48} + \frac{v_2^2}{10} + \frac{49v_1^4}{640} + \frac{49v_1 v_2^2}{480}.$ (3.5)

$$\alpha_7 = \frac{v_6}{6} + \frac{13v_1^2 v_4}{320} - \frac{v_2^3 v_3}{96} + \frac{29v_1 v_5}{120} + \frac{7v_1^3 v_3}{768} + \frac{1313v_1^2 v_2^2}{3840} + \frac{7843v_1^4 v_2}{3072} - \frac{743v_1^6 v_3}{276480} + \frac{23v_2^2}{576} + \frac{v_2 v_4}{12}$$
$$+ \frac{353v_1 v_2 v_3}{1440} + \frac{11v_1^3 v_2}{80} - \frac{10039v_1^6}{921600} + \frac{3v_1 v_2^2}{30} + \frac{147v_1^5}{1920} - \frac{v_2^2 v_3}{3} - \frac{5v_1 v_2^3}{12} +$$
$$\frac{7v_1^3 v_2^2}{32} - \frac{7v_1^2 v_2 v_3}{24} + \frac{49v_1^5 v_2}{96}. \quad (3.6)$$

Applying Lemma (2.2), we obtain

$|\alpha_2| \le 1,$

$|\alpha_3| \le \left|\frac{v_2}{2} - \frac{7v_1^2}{16}\right|,$

$|\alpha_3| = \left|\frac{v_1^2 + \chi(4-v_1^2)}{4} - \frac{7v_1^2}{16}\right| = \left|\frac{7v_1^2}{8} + \frac{\chi(4-v_1^2)}{4}\right|.$

Let $v_1 = v \in [0,2]$ and $|\chi| = \varrho \in [0,1]$ to get

$|\alpha_3| \le \frac{7v^2}{8} + \frac{\varrho(4-v^2)}{4}.$

Suppose that$(v,\varrho) = \frac{7v^2}{8} + \frac{\varrho(4-v^2)}{4}.$

Now, we have$\frac{\partial \xi(v,\varrho)}{\partial \varrho} = \frac{(4-v^2)}{4} \ge 0$.

On the interval [0,1]the function $\xi(v,\varrho)$be increasing and $\xi(v,\varrho)$ has the maximum value at $\varrho = 1$,

$max\xi(v,\varrho) = \xi(v,\varrho) = \frac{7v^2}{8} + \frac{(4-v^2)}{4}.$

Suppose that$\rho(v) = 1 + \frac{7v^2}{16},\ \rho'(v) = \frac{7v}{8} \ge 0.$

Now, $\rho(v)$has a maximum value at $v = 0$,we obtain

$|\alpha_3| \le \rho(0) = 1$.

$|\alpha_4| \le \left|\frac{7v_1^3}{12} + \frac{v_3}{3} - \frac{5v_1 v_2}{12}\right| = \left|\frac{1}{3}\left[v_3 - \frac{5v_1 v_2}{4}\right] + \frac{v_1}{6}\left[v_2 - \frac{7v_1^2}{2}\right]\right|,$

Taking $v_1 = v, v \in [0,2]$ and taking Lemma (2.3) , to get

$|\alpha_4| \le \left|\frac{1}{3}\left[v_3 - \frac{5v_1 v_2}{4}\right] + \frac{v_1}{6}\left[v_2 - \frac{7v_1^2}{2}\right]\right| \le \frac{2}{3} + \frac{v\left(2 - \frac{7v_1^2}{2}\right)}{6},$

Nwo, suppose$(v) = \frac{2}{3} + \frac{v\left(2-\frac{v^2}{2}\right)}{288},$

clearly, to obtain $\xi'(v) = \frac{5}{12} - \frac{21v^2}{12},$

take$\xi'(v) = 0$,where$v = \pm\sqrt{\frac{5}{21}}$,be critical point of the function $\xi(v)$are$v = \pm\sqrt{\frac{5}{21}}$,to obtain

$\xi''(v) = \xi''\left(\sqrt{\frac{5}{21}}\right) = -1.707825126 < 0,$

then$\xi(v)$ has the maximum value as

$|\alpha_4| \le \xi\left(\sqrt{\frac{5}{21}}\right) = \frac{2}{3} + \frac{v\left(2-\frac{v^2}{2}\right)}{288} = 0.802288343.$ (3.8)

$$|\alpha_5| = \left|\frac{\upsilon_4}{4} + \frac{17\upsilon_1^2\upsilon_2}{192} + \frac{13\upsilon_1\upsilon_3}{48} + \frac{3\upsilon_2^2}{32} - \frac{61\upsilon_1^4}{384}\right|$$

$= \left|\frac{1}{4}\left[\upsilon_4 - \frac{13\upsilon_1\upsilon_3}{12}\right] - \frac{61\upsilon_1^2}{192}\left[\upsilon_2 - \frac{\upsilon_1^2}{2}\right] - \frac{3\upsilon_2}{32}\left[\upsilon_2 - \frac{\upsilon_1^2}{2}\right] - \frac{25\upsilon_1^2\upsilon_2}{2304}\right|.$

Taking $\upsilon_1 = \upsilon, \upsilon \in [0,2]$ and taking Lemma (2.3) , to get

$|\alpha_5| \leq \frac{2}{4} + \frac{61\upsilon^2\left(2-\frac{\upsilon^2}{2}\right)}{192} + \frac{6}{32}\left(2 - \frac{\upsilon^2}{2}\right) + \frac{25\upsilon^2}{1252}.$

Suppose that$(\upsilon) = \frac{2}{4} + \frac{61\upsilon^2\left(2-\frac{\upsilon^2}{2}\right)}{192} + \frac{6}{32}\left(2 - \frac{\upsilon^2}{2}\right) + \frac{25\upsilon^2}{1252}$, to get

$\xi'(\upsilon) = \frac{14659}{30048}\upsilon - \frac{61\upsilon^3}{96}.$

Put $\xi'(\upsilon) = 0$, we have $\upsilon = 0.876223882 \; or \; \upsilon = 0$.

$\xi''(0.876223882) = -0.975705537 < 0$.

Now, at $\upsilon = 0.876223882$, thus $\xi(\upsilon)$ has maximum value: ,

$|\alpha_5| \leq \xi(0.876223882) = 0.968639472$.

$|\alpha_6| = \left|\frac{\upsilon_5}{5} + \frac{2\upsilon_2\upsilon_3}{60} + \frac{3\upsilon_1\upsilon_4}{10} + \frac{49\upsilon_1^3\upsilon_2}{24} + \frac{11\upsilon_1^2\upsilon_2}{80} + \frac{779\upsilon_1^5}{2400} - \frac{\upsilon_1^2\upsilon_3}{48} + \frac{\upsilon_2^2}{10} + \frac{49\upsilon_1^4}{640} + \frac{49\upsilon_1\upsilon_2^2}{480}\right|$.

$|\alpha_6| = \left|\frac{1}{5}\left[\upsilon_5 - \frac{3\upsilon_1\upsilon_4}{2}\right] + \frac{2}{60}\left[\upsilon_5 - \upsilon_2\upsilon_3\right] + \frac{779\upsilon_1^3}{12000}\left[\upsilon_2 - \frac{\upsilon_1^2}{2}\right] - \frac{49\upsilon_1\upsilon_2}{72}\left[\upsilon_2 - \frac{\upsilon_1^2}{2}\right] + \frac{11\upsilon_1\upsilon_2}{80}\left[\upsilon_2 - \frac{\upsilon_1^2}{2}\right] + \frac{\upsilon_2}{10}\left[\upsilon_2 - \frac{\upsilon_1^2}{2}\right] + \frac{49\upsilon_1^2}{640}\left[\upsilon_2 - \frac{\upsilon_1^2}{2}\right] + \frac{49\upsilon_1\upsilon_2}{480}\left[\upsilon_2 - \frac{\upsilon_1^2}{2}\right] + \frac{102\upsilon_1^3\upsilon_2}{51840} - \frac{\upsilon_1^2\upsilon_3}{48}\right|.$

Taking $\upsilon_1 = \upsilon, \upsilon \in [0,2]$ and taking Lemma (2.3) , to get

$|\alpha_6| \leq \frac{2}{5} + \frac{4}{600} + \frac{779\upsilon^3}{12000}\left[2 - \frac{\upsilon^2}{2}\right] - \frac{98\upsilon}{72}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{22\upsilon}{80}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{2}{10}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{49\upsilon^2}{640}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{98\upsilon}{480}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{204\upsilon^3}{160} - \frac{2\upsilon^2}{48},$

$|\alpha_6| \leq \frac{1}{72} + \frac{47}{2160} + \frac{\upsilon^3\left[2-\frac{\upsilon^2}{2}\right]}{800} - \frac{49\upsilon\left[2-\frac{\upsilon^2}{2}\right]}{25920} + \frac{37\left[2-\frac{\upsilon^2}{2}\right]}{8640} + \frac{\upsilon^3}{200}.$

Suppose that

$\xi(\upsilon) = \frac{2}{5} + \frac{4}{600} + \frac{779\upsilon^3}{12000}\left[2 - \frac{\upsilon^2}{2}\right] - \frac{98\upsilon}{72}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{22\upsilon}{80}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{2}{10}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{49\upsilon^2}{640}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{98\upsilon}{480}\left[2 - \frac{\upsilon^2}{2}\right] + \frac{204\upsilon^3}{160} - \frac{2\upsilon^2}{48},$

weget.

$\xi'(\upsilon) = \frac{14587\,\upsilon^2}{6000} - \frac{779\upsilon^4}{4800} - \frac{39\upsilon}{480} - \frac{49\upsilon^3}{320} - \frac{127}{72}$.

Thus,$\xi'(\upsilon) = 0$,we have $\upsilon = 0$on [0,2],also $\upsilon = 0$, $\xi(\upsilon)$ has maximum value as followes:

$|\alpha_6| \leq \xi(0) = \frac{38}{75} = 0.506666666$. (3.10)

$|\alpha_6| = \left|\frac{\upsilon_6}{6} + \frac{13\upsilon_1^2\upsilon_4}{320} - \frac{\upsilon_2^3\upsilon_3}{96} + \frac{29\upsilon_1\upsilon_5}{120} + \frac{7\upsilon_1^3\upsilon_3}{768} + \frac{1313\upsilon_1^2\upsilon_2^2}{3840} + \frac{7843\upsilon_1^4\upsilon_2}{3072} - \frac{743\upsilon_1^6\upsilon_3}{276480} + \frac{23\upsilon_2^2}{576} + \frac{\upsilon_2\upsilon_4}{12} + \frac{353\upsilon_1\upsilon_2\upsilon_3}{1440} + \frac{11\upsilon_1^3\upsilon_2}{80} - \frac{10039\upsilon_1^6}{921600} + \frac{3\upsilon_1\upsilon_2^2}{30} + \frac{147\upsilon_1^5}{1920} - \frac{\upsilon_2^2\upsilon_3}{3} - \frac{5\upsilon_1\upsilon_2^3}{12} + \frac{7\upsilon_1^3\upsilon_2^2}{32} - \frac{7\upsilon_1^2\upsilon_2\upsilon_3}{24} + \frac{49\upsilon_1^5\upsilon_2}{96}\right|$.

$$= \left|\frac{10039\upsilon_1^6}{921600} - \frac{1313\upsilon_1^2\upsilon_2^2}{3840} - \frac{29\upsilon_1\upsilon_5}{120} + \frac{13\upsilon_1^2[\upsilon_4 - \upsilon_1^2]}{3360} + \frac{1777\upsilon_1\upsilon_2[\upsilon_3 - \upsilon_1\upsilon_2]}{241920} + \right.$$

$\frac{7\upsilon_1^3[\upsilon_3-\upsilon_1\upsilon_2]}{768} - \frac{7843\upsilon_1^4\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{3072} - \frac{743\upsilon_1^4[\upsilon_3-\upsilon_1^2]}{276480} + \frac{23\upsilon_2\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{276480} + \frac{353\upsilon_1\upsilon_2[\upsilon_3-\upsilon_1\upsilon_2]}{1440} + \frac{\left[\upsilon_6-\frac{3\upsilon_2\upsilon_4}{2}\right]}{6} + \frac{11\upsilon_1^3\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{80} + \frac{3\upsilon_2\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{30} + \frac{147\upsilon_1^3\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{1920} - \frac{\upsilon_2\upsilon_3\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{3} - \frac{5\upsilon_{1\upsilon_2^2}\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{12} + \frac{7\upsilon_1^3[\upsilon_3-\upsilon_1\upsilon_2]}{32} - \frac{7\upsilon_1^2\upsilon_2[\upsilon_3-\upsilon_1\upsilon_2]}{24} + \frac{49\upsilon_1^3\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{12} - \frac{\upsilon_2^2[\upsilon_3-\upsilon_1\upsilon_2]}{96} - \frac{7\upsilon_2[\upsilon_4-\upsilon_2^2]}{672} + \left.\frac{23\upsilon_2^2}{576}\right|.$

Assume$\upsilon_1 = \upsilon$, $\upsilon \in [0,2]$, by using Lemma (2.3) we obtain

$|\alpha_7| \leq \frac{10039\upsilon^6}{921600} - \frac{1313\upsilon_1^2\upsilon_2^2}{3840} - \frac{29\upsilon_1\upsilon_5}{120} + \frac{13\upsilon_1^2[\upsilon_4-\upsilon_1^2]}{3360} + \frac{1777\upsilon_1\upsilon_2[\upsilon_3-\upsilon_1\upsilon_2]}{241920} + \frac{7\upsilon_1^3[\upsilon_3-\upsilon_1\upsilon_2]}{768} - \frac{7843\upsilon_1^4\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{3072} - \frac{743\upsilon_1^4[\upsilon_3-\upsilon_1^2]}{276480} + \frac{23\upsilon_2\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{276480} + \frac{353\upsilon_1\upsilon_2[\upsilon_3-\upsilon_1\upsilon_2]}{1440} + \frac{\left[\upsilon_6-\frac{3\upsilon_2\upsilon_4}{2}\right]}{6} + \frac{11\upsilon_1^3\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{80} + \frac{3\upsilon_2\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{30} + \frac{147\upsilon_1^3\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{1920} - \frac{\upsilon_2\upsilon_3\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{3} - \frac{5\upsilon_{1\upsilon_2^2}\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{12} + \frac{7\upsilon_1^3[\upsilon_3-\upsilon_1\upsilon_2]}{32} - \frac{7\upsilon_1^2\upsilon_2[\upsilon_3-\upsilon_1\upsilon_2]}{24} + \frac{49\upsilon_1^3\left[\upsilon_2-\frac{\upsilon_1^2}{2}\right]}{12} - \frac{\upsilon_2^2[\upsilon_3-\upsilon_1\upsilon_2]}{96} - \frac{7\upsilon_2[\upsilon_4-\upsilon_2^2]}{672} + \left.\frac{23\upsilon_2^2}{576}\right|$.

Suppose that

$$\xi(\upsilon) = -\frac{341}{90} - \frac{4099\upsilon}{720} - \frac{1387\upsilon^2}{504} - \frac{7697\upsilon^3}{1920} + \frac{49665281\upsilon^4}{9815040} - \frac{2371\upsilon^5}{3840} - \frac{3491803\upsilon^6}{2764800}$$

We obtain
$\xi^{'}(\upsilon) = -\frac{4099}{720} - \frac{1387\,\upsilon}{252} - \frac{7697\,\upsilon^2}{640} + \frac{49665281\,\upsilon^3}{2453760} - \frac{2371\,\upsilon^4}{768} - \frac{3491803\,\upsilon^5}{460800}$.
We set $\xi^{'}(\upsilon) = 0$, we get $\upsilon = 0.315510$,
$\xi^{''}(\upsilon) = -\frac{1387}{252} - \frac{7697\,\upsilon}{320} + \frac{49665281\,\upsilon^2}{817920} - \frac{2371\,\upsilon^3}{192} - \frac{3491803\,\upsilon^4}{92160}$.
$\xi^{''}(\upsilon = 0.315510) = -7.826,$
$|\alpha_7| \le \xi(\upsilon = 0.315510) = 0.311130508.$ (3.11)
Therefore, we get the result .

Theorem(3.2): Suppose that the function$f \in \Lambda_\sigma^{\circ}$ and be taking (1.1), then
$|\alpha_3 - \alpha_2^2| \le 1.$ (3.12)
Proof: According to equation (3.6), we have
$|\alpha_3 - \alpha_2^2| = \left|\frac{\upsilon_2}{2} - \frac{9\upsilon_1^2}{16}\right|$.
Thus, by using Lemma (2.1), to obtain
$|\alpha_3 - \alpha_2^2| = \left|\frac{\upsilon_1^2+\chi(4-\upsilon_1^2)}{4} - \frac{\upsilon_1^2}{16}\right|$.
Suppose that $\upsilon_1 = \upsilon, \upsilon \in [0,1], |\chi| = \varrho\ and\ \varrho \in [0,1]$.Thus by taking the triangle inequality, to obtain
$|\alpha_3 - \alpha_2^2| = \frac{\varrho(4-\upsilon_1^2)}{4} - \frac{\upsilon^2}{8}$.
Assume$(\upsilon, \varrho) = \frac{\tau(4-\upsilon^2)}{4} - \frac{\upsilon^2}{8}$,
Therefor , we have$\frac{\xi(\upsilon)}{\partial\varrho} = \frac{4-\upsilon^2}{0} > 0$,
On the interval [0,1]the function $\xi(\upsilon, \varrho)$be increasing and $\xi(\upsilon, \varrho)$ has the maximum value at $\varrho = 1$,hence
$\max \xi(\upsilon, \varrho) = \xi(\upsilon, 1) = \frac{4-\upsilon^2}{4} + \frac{9\upsilon^2}{32} = \frac{8(4-\upsilon^2)-9\upsilon^2}{32} = \frac{32-17\upsilon^2}{32}$.
Define$\rho(\upsilon) = 1 - \frac{17\upsilon^2}{32}$,
$\rho^{'}(\upsilon) = -\frac{34\upsilon^2}{32} \le 0$.
Therefore,$\rho(\upsilon)$ has a maximum value at $\upsilon = 0$, thus
$|\alpha_3 - \alpha_2^2| \le, \rho(\upsilon) = 1$.
The proof is complete.

Theorem (3.3): Suppose that the function$f \in \Lambda_\sigma^{\circ}$ and be taking (1.1), then
$|\alpha_2\alpha_3 - \alpha_4| \le 0.185767863.$ (3.13)
Proof: Taking (3.6) ,to get
$|\alpha_2\alpha_3 - \alpha_4| = \left|\frac{\upsilon_1\upsilon_2}{2} + \frac{5\upsilon_1\upsilon_2}{12} - \frac{\upsilon_3}{3} + \frac{49\upsilon_1^3}{48}\right| = \left|\frac{\upsilon_1\upsilon_2}{12} - \frac{\upsilon_3}{3} + \frac{49\upsilon_1^3}{48}\right|$.
Therefore,by Lemma (2.1), to get
$|\alpha_2\alpha_3 - \alpha_4| = \left|\frac{\upsilon_1^3}{16} + \frac{7\upsilon_1\varrho(4-\upsilon_1^2)}{96} + \frac{(4-\upsilon_1^2)\varrho^2\upsilon_1}{36} - \frac{(4-\upsilon_1^2)(1-|\chi|^2)z}{6}\right|$.
Suppose that $\upsilon_1 = \upsilon, \upsilon \in [0,2], |\chi| = \varrho\ and\ \varrho \in [0,1]$. Addition to that we applying the triangle inequality to obtain
$|\alpha_2\alpha_3 - \alpha_4| \le \frac{\upsilon^3}{16} + \frac{7\upsilon\varrho(4-\upsilon^2)}{96} + \frac{(4-\upsilon^2)\varrho^2\upsilon}{36} - \frac{(4-\upsilon^2)}{6}$.
Let$\xi(\upsilon, \varrho) = \frac{\upsilon^3}{16} + \frac{7\upsilon\varrho(4-\upsilon^2)}{96} + \frac{(4-\upsilon^2)\varrho^2\upsilon}{36} - \frac{(4-\upsilon^2)}{6}$.
Thus, for all $\upsilon \in (0,2)\ and\ \varrho \in (0,1)$, to obtain
$\frac{\partial\xi(\upsilon,\varrho)}{\partial\varrho} = +\frac{7\upsilon\varrho(4-\upsilon^2)}{96} + \frac{(4-\upsilon^2)\varrho^2\upsilon}{36} \ge 0$,
On the interval [0,1]the function $\xi(\upsilon, \varrho)$be increasing and $\xi(\upsilon, \varrho)$ has the maximum value at $\varrho = 1$,hence
$\max \xi(\upsilon, \varrho) = \xi(\upsilon, 1) = \frac{\upsilon^3}{16} + \frac{7\upsilon(4-\upsilon^2)}{96} + \frac{(4-\upsilon^2)\upsilon}{36} - \frac{(4-\upsilon^2)}{6}$.
Taking$(\upsilon) = \frac{\upsilon^3}{16} + \frac{7\upsilon(4-\upsilon^2)}{96} + \frac{(4-\upsilon^2)\upsilon}{36} - \frac{(4-\upsilon^2)}{6}$,
Now, we have$\rho^{'}(\upsilon) = \frac{116}{288} - \frac{\upsilon}{3} + \frac{33\upsilon^2}{288}$.
Put, $\rho^{'}(\upsilon) = 0$ to find $\upsilon = \frac{96-\sqrt{24528}}{66}$,
$\rho^{'}\left(\frac{96-\sqrt{24528}}{66}\right) = -\frac{1}{3} - \frac{66\upsilon}{288} = -0.9183966$.
Then, we have$|\alpha_2\alpha_3 - \alpha_4| \le \rho\left(\frac{96-\sqrt{24528}}{66}\right) \le 0.185767863$.

Hence,the proof is complete .

Theorem(3.4): Suppose that the function$f \in \Lambda_{\sigma}^{\circ}$ and be taking (1.1), then
$|\alpha_2\alpha_4 - \alpha_3^2| \le \frac{7}{9} = 0.999999999$. (3.14)
Proof: Taking (3.6) , to get$|\alpha_2\alpha_4 - \alpha_3^2| = \left|\frac{v_1v_3}{4} - \frac{595v_1^4}{768} - \frac{v_1^2v_2}{48} - \frac{v_2^2}{4}\right|$.
Applying Lemma (2.1), to get
$|\alpha_2\alpha_4 - \alpha_3^2| = \left|\frac{147v_1^4}{384} - \frac{\varrho v_1^2(4-v_1^2)}{16} + \frac{\varrho^2 v_1^2(4-v_1^2)}{92} - \frac{\varrho^2(4-v_1^2)^2}{36} + \frac{v_1(4-v_1^2)(1-|\chi|^2)z}{96}\right|$.
Suppose that $v_1 = v, v \in [0,2], |\chi| = \varrho \; and \; \varrho \in [0,1]$. Addition to that we applying the triangle inequality to obtain
$|\alpha_2\alpha_4 - \alpha_3^2| \le \frac{147v_1^4}{384} + \frac{\varrho v^2(4-v^2)}{16} + \frac{\varrho^2 v^2(4-v^2)}{92} + \frac{\varrho^2(4-v^2)^2}{36} + \frac{(4-v^2)}{12}$.
Suppose$\xi(v,\varrho) = \frac{147v_1^4}{384} + \frac{\varrho v^2(4-v^2)}{16} + \frac{\varrho^2 v^2(4-v^2)}{92} + \frac{\varrho^2(4-v^2)^2}{36} + \frac{(4-v^2)}{12}$.
Visibly, to obtain $\frac{\partial\xi(v,\varrho)}{\partial\varrho} = \frac{v^2(4-v^2)}{16} + \frac{\varrho v^2(4-v^2)}{46} + \frac{\varrho(4-v^2)^2}{18} > 0$,
On the interval [0,1]the function $\xi(v,\varrho)$be increasing and $\xi(v,\varrho)$ has the maximum value at $\varrho = 1$,hence
$\max\xi(v,\varrho) = \xi(v,1) = \frac{v^2(4-v^2)}{16} + \frac{\varrho v^2(4-v^2)}{46} + \frac{\varrho(4-v^2)^2}{18}$.
Let$\rho(v) = \frac{147v_1^4}{384} + \frac{\varrho v^2(4-v^2)}{16} + \frac{\varrho^2 v^2(4-v^2)}{92} + \frac{\varrho^2(4-v^2)^2}{36} + \frac{(4-v^2)}{12}$,
thus, we obtain $\rho'(v) = -\frac{128v}{414} - \frac{1139v^3}{6624} \le 0$.
Let $v = 0$ thus $\rho(v)$ can extent its maximum value:
$|\alpha_2\alpha_4 - \alpha_3^2| \le \rho(0) \le \frac{7}{9} = 0.999999999$.
Hence ,the proof is complete .

Theorem (3.5): Suppose that the function$f \in \Lambda_{\sigma}^{\circ}$ and be taking (1.1), then
$|\alpha_2\alpha_5 - \alpha_3\alpha_4| \le 0.875$.
Proof: Taking(3.6), to get

$$|\alpha_2\alpha_5 - \alpha_3\alpha_4| = \left|\frac{37v_1^5}{384} + \frac{v_1v_4}{4} - \frac{11v_1v_2^2}{96} + \frac{6v_1^2v_3}{48} + \frac{38v_1^3v_2}{192} - \frac{v_2v_3}{6}\right|$$

$$= \left|\frac{38v_1^3\left[v_2 - \frac{v_1^2}{2}\right]}{192} - \frac{v_3\left[v_2 - \frac{v_1^2}{2}\right]}{6} + \frac{9v_1[v_4 - v_1v_3]}{16} + \frac{407v_1^5}{1920} + \frac{7v_1\left[v_4 - \frac{1}{4}v_2^2\right]}{16}\right|.$$

By applying Lemma (2.3), to get
$|\alpha_5| \le \frac{38v^3\left[2 - \frac{v^2}{2}\right]}{192} + \frac{\left[2 - \frac{v^2}{2}\right]}{6} + \frac{7v}{8} + \frac{407v^5}{1920}$.

$$|\alpha_5| = \frac{2}{3} + \frac{7v}{8} - \frac{v^2}{6} + \frac{152v^3}{384} + \frac{217v^5}{1920}$$

Let$\xi(v) = \frac{2}{3} + \frac{7v}{8} - \frac{v^2}{6} + \frac{152v^3}{384} + \frac{217v^5}{1920}$.
Visibly, to obtain $\xi'(v) = \frac{7}{8} - \frac{v}{3} + \frac{152v^2}{128} + \frac{217v^4}{384} > 0$.
Hence ,put$\xi'(v,\varrho) = 0$,to get$v = 0, v = 1.5899$to find
$\xi''(v) = -\frac{1}{3} + \frac{152v}{64} + \frac{217v^3}{96}, \xi''(v) = -\frac{1}{3} < 0$,
Now, the function $\xi(v)$ has maximum value as followes
$|\alpha_2\alpha_5 - \alpha_3\alpha_4| \le \xi(0) = \frac{7}{8} = 0.875$.
Hence, the proof is complete.

Theorem(3.6): Suppose that the function$f \in \Lambda_{\sigma}^{\circ}$ and be taking (1.1), then
$|\alpha_5 - \alpha_2\alpha_4| \le 0.24726105$. (3.16)
Proof: Taking (2.6), to get

$$|\alpha_5 - \alpha_2\alpha_4| = \left|\frac{v_4}{4} - \frac{v_1v_3}{16} + \frac{163v_1^4}{384} - \frac{63v_1^2v_2}{192} - \frac{3v_2^2}{32}\right|$$

$$= \left| \frac{v_1^4+3v_2^2+4v_1v_3-59v_1^2v_2-v_4}{160} - \frac{35v_1^2\left[v_2-\frac{v_1^2}{2}\right]}{640} - \frac{163v_1^4}{7680} + \frac{\left[v_4-\frac{v_1v_3}{2}\right]}{16} \right|.$$

By applying Lemma (2.3), to get

$|\alpha_5 - \alpha_2\alpha_4| \leq \frac{2}{16} + \frac{35v^2\left[2-\frac{v^2}{2}\right]}{60} + \frac{2}{8} + \frac{63v^4}{7680}.$

$|\alpha_5 - \alpha_2\alpha_4| \leq \frac{6}{16} + \frac{140v^2}{120} + \frac{2177v^4}{7680}.$

Let$\xi(v) = \frac{6}{16} + \frac{140v^2}{120} + \frac{2177v^4}{7680}$,

therefore ,we have$\xi'(v) = \frac{140v}{60} - \frac{2177v^4}{7680}$and $\xi''(v) = \frac{140}{60} - \frac{2177v^3}{640}$.

Let$\xi'(v) = 0$, we have$v = \pm 1.06242543 \ \ or \ \ v = 0$

To find $\xi''(1.06242543) = -1.506172836$.

Now, the function $\xi(v)$ has maximum value as followes

$|\alpha_5 - \alpha_2\alpha_4| \leq \xi(1.06242543) = 0.24761051.$

Hence, the proof is complete.

Theorem(3.7): Suppose that the function$f \in \Lambda_\sigma^\circ$ and be taking (1.1), then

$|\alpha_5\alpha_3 - \alpha_4^2| \leq \frac{17}{18} = 0.944444444$. (3.17)

Proof: Taking (2.6) , we get

$$|\alpha_5\alpha_3 - \alpha_4^2| = \left| \frac{v_2v_4}{8} + \frac{41v_1v_2v_3}{288} + \frac{4345v_1^4v_2}{9216} + \frac{407v_1^2v_2^2}{4608} + \frac{3v_2^3}{64} - \frac{v_3^2}{9} + \frac{1169v_1^3v_3}{2304} - \frac{7553v_1^6}{18432} \right|$$

$$= \left| \frac{v_2\left[v_4-\frac{41v_1v_3}{2}\right]}{32} - \frac{v_3\left[v_3-\frac{41v_1v_2}{2}\right]}{9} + \frac{3v_2^2\left[v_2-\frac{v_1^2}{2}\right]}{64} - \frac{395v_1^2v_2\left[v_2-\frac{v_1^2}{2}\right]}{1536} + \frac{1169v_1^3\left[v_3-\frac{5845v_1v_2}{331776}\right]}{2304} + \frac{7v_1^2v_4}{13824} - \frac{7553v_1^6}{18432} \right|.$$

By applying Lemma (2.3),(2.4), to get

$|\alpha_5\alpha_3 - \alpha_4^2| \leq \frac{1}{8} + \frac{4}{9} + \frac{12\left[2-\frac{v^2}{2}\right]}{64} + \frac{395v^2\left[2-\frac{v^2}{2}\right]}{768} + \frac{1169v^3}{1152} + \frac{7v^4}{64} + \frac{7553v^6}{18432}.$

Suppose that

$\xi(v) = \frac{1}{8} + \frac{4}{9} + \frac{12\left[2-\frac{v^2}{2}\right]}{64} - \frac{395v^2\left[2-\frac{v^2}{2}\right]}{768} + \frac{1169v^3}{1152} + \frac{7v^4}{64} + \frac{7553v^6}{18432}.$

$= \frac{17}{18} - \frac{1724v^2}{1536} + \frac{1295v^3}{1152} - \frac{395v^4}{1536} - \frac{7553v^6}{18432}.$

$\xi'(v) = -\frac{1724v}{768} + \frac{1295v^2}{384} - \frac{395v^3}{384} - \frac{7553v^5}{3072}$.

Thus, we have$v = 0$, where$\xi'(v) = 0$,such that only root belong to[0,2], to obtain

$\xi''(v) = -\frac{1724}{768} + \frac{1295v}{192} - \frac{395v^2}{128} - \frac{37765v^4}{3072}and\xi''(0) = -\frac{1724}{768}$.

Thus$|\alpha_5\alpha_3 - \alpha_4^2| \leq \xi(0) = \frac{17}{18}$.

Hence, the proof is complete.

Theorem (3.8): Suppose that the function$f \in \Lambda_\sigma^\circ$ and be taking (1.1), then

$|H_4(1)| \leq 4.131346765$. (3.18)

Proof: By applying equation (1.4) .

Therefore by taking the definition (1,2) and by using the triangle inequality ,we have

$H_4(1) = \Xi_1\alpha_7 + \Xi_2\alpha_6 + \Xi_3\alpha_5 + \Xi_4\alpha_4$. (3.19)

Now, replace the equation (3.1) and (3.12)- (3.17) into (3.19), we suddenly to obtain the result.

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