APPROXIMATION OF SOLUTIONS OF THE INHOMOGENEOUS GAUSS DIFFERENTIAL EQUATIONS BY HYPERGEOMETRIC FUNCTION

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Abstract. In this paper, we solve the inhomogeneous Gauss differential equation and apply this result to estimate the error bound occurring when an analytic function is approximated by an appropriate hypergeometric function.

1. Introduction

More than a half century ago, Ulam [22] posed the famous Ulam stability problem which was partially solved by Hyers [7] in the framework of Banach spaces. The Hyers' theorem was generalized by Aoki [4] for additive mappings. In 1978, Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy difference inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p), \qquad (\varepsilon \geq 0, p \in [0,1)).$$

Since then, the stability problems of various functional equations have been studied by many authors (see [1, 6, 8, 9, 13, 15, 17, 18, 19, 20]).

Alsina and Ger [3] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved that if a differentiable function $f: I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t)-y(t)| \leq \epsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t)-f_0(t)| \leq 3\epsilon$ for any $t \in I$. From then on, many research papers about the Hyers-Ulam stability of differential equations have appeared in the literature, see [2, 5, 10, 11, 12, 21, 23] for instance.

The form of the homogeneous Gauss differential equation has the form

$$x(1-x)y'' + [r-(1+s+t)x]y' - sty = 0. \tag{1.1}$$

It is easy to see that

$$y_1 = 1 + \frac{st}{1!r}x + \frac{(st)(s+1)(t+1)}{2!r(r+1)}x^2 + \frac{(st)(s+1)(s+2)(t+1)(t+2)}{3!r(r+1)(r+2)}x^5 + \cdots$$

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and

$$\begin{aligned} y_2 =& x^{1-r}[1 + \frac{(s-r+1)(t-r+1)}{1!(2-r)}x \\ &+ \frac{(s-r+1)(s-r+2)(t-r+1)(t-r+2)}{2!(2-r)(3-r)}x^2 \\ &+ \frac{(s-r+1)(s-r+2)(s-r+3)(t-r+1)(t-r+2)(t-r+3)}{3!(2-r)(3-r)(4-r)}x^5 + ...] \end{aligned}$$

are a fundamental set of solutions of equation (1.1) (if $r \neq 1$). The series y_1 known the hypergeometric function is convergent for $|x| < 1$ and is represented by

$$y_1 = F(s, t, r, x).$$

Note that

$$y_2 = x^{1-r}F(s-r+1, t-r+1, 2-r, x)$$

is of the same type. Thus the general solution is

$$y_c = c_1y_1 + c_2y_2 = c_1F(s, t, r, x) + c_2x^{1-r}F(s-r+1, t-r+1, 2-r, x).$$

2. Inhomogeneous Gauss differential equation

In this section, we consider the solution of inhomogeneous Gauss differential equation of the form

$$x(1-x)y'' + [r - (1+s+t)x]y' - sty = \sum_{m=0}^{+\infty} a_m x^m, \tag{2.1}$$

where the coefficients a_n's of the power series are given such that the radius of convergence is positive.

Theorem 2.1. *Assume that the radius of convergence of the power series $\sum_{m=0}^{+\infty} a_m x^m$ is $R_0 > 0$ and*

$$R_1 = \lim_{k\to\infty} |\frac{c_k}{c_{k+1}}| > 0. \tag{2.2}$$

Let ρ be a positive number defined by $\rho = min\{1, R_0, R_1\}$. Then every solution $y : (-\rho, \rho) \to \mathbb{C}$ of differential equation (2.1) can be expressed by

$$y(x) = y_c(x) + \sum_{m=1}^{+\infty} c_m x^m, \tag{2.3}$$

where $c_1 = \frac{1}{r}a_0$ and

$$\begin{aligned} c_m =& \frac{a_{m-1}}{m(m-1+r)} \\ &+ \frac{1}{m!}\sum_{i=1}^{m-1} a_{m-i-1} \prod_{j=1}^{i+1} \frac{1}{m-j+r} \prod_{j=1}^{i} (m+s-j)(m+t-j) \end{aligned} \tag{2.4}$$

for any $m \in \{2, 3, \cdots\}$.

Proof. We will show that each function $y : (-\rho, \rho) \to \mathbb{C}$ defined by (2.3) is a solution of the inhomogeneous Gauss differential equation (2.1), where y_c is a solution of homogeneous Gauss differential equation (1.1). For this purpose, it is only necessary to show that $y_p(x) = \sum_{m=1}^{\infty} c_m x^m$ satisfies differential equation (2.1). Therefore, letting $y_p(x) = \sum_{m=1}^{\infty} c_m x^m$ in differential equation (2.1), we obtain

$$\sum_{m=1}^{+\infty} m(m+1)c_{m+1}x^m + r\sum_{m=0}^{+\infty}(m+1)c_{m+1}x^m - \sum_{m=2}^{+\infty} m(m-1)c_m x^m - (1+s+t)\sum_{m=1}^{+\infty} mc_m x^m - st\sum_{m=1}^{+\infty} c_m x^m = \sum_{m=0}^{+\infty} a_m x^m.$$

Hence

$$rc_1 + \sum_{m=1}^{+\infty}[(m+1)(m+r)c_{m+1} - (m+s)(m+t)c_m]x^m = \sum_{m=0}^{+\infty} a_m x^m.$$

Therefore, we get $c_1 = \frac{1}{r}a_0$ and

$$c_{m+1} = \frac{1}{(m+1)(m+r)}a_m + \frac{(m+s)(m+t)}{(m+1)(m+r)}c_m, \quad (m = 1, 2, ...).$$

By some manipulations, we obtain

$$c_m = \frac{a_{m-1}}{m(m-1+r)} + \frac{1}{m!}\sum_{i=1}^{m-1} a_{m-i-1}\prod_{j=1}^{i+1}\frac{1}{m-j+r}\prod_{j=1}^{i}(m+s-j)(m+t-j) \tag{2.5}$$

for any $m \in \{2, 3, ...\}$. The condition (2.2) implies that the radius of convergence of $y_p(x) = \sum_{m=1}^{+\infty} c_m x^m$ is R_1. By using the ratio test, we can easily show that the radius of convergence of y_c is 1. Thus y is certainly defined on $(-\rho, \rho)$. □

Corollary 2.2. *Assume that the assumptions of Theorem* 2.1 *hold. Then there exists* $C > 0$ *such that*

$$\sum_{m=1}^{+\infty} c_m x^m \le \sum_{m=1}^{+\infty}\frac{a_{m-1}}{m(m-1+r)}x^m + \sum_{i=1}^{+\infty}\sum_{m=2}^{+\infty}\frac{Ca_{m-2}}{(m+i-1)^2}\prod_{j=0}^{i}(1 - \frac{-st}{(m+i-j-1)(m+i-j+s+t-1)})x^{m+i-1}.$$

Proof. Since there exists a constant $C > 0$ with

$$\frac{1}{m!}\prod_{j=1}^{i+1}\frac{1}{m-j+r} \le \frac{C}{m^2}\prod_{j=0}^{i}\frac{1}{(m-j)(m-j+s+t)}$$

for any $m = 2, 3, ...$ and for any $i = 1, 2, ...$, it follows from (2.5) that

$$\begin{aligned}
\sum_{m=1}^{+\infty} c_m x^m &= c_1 x + \sum_{m=2}^{+\infty} c_m x^m = \frac{1}{r} a_0 x + \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m \\
&\quad + \sum_{m=2}^{+\infty} \frac{1}{m!} \sum_{i=1}^{m-1} a_{m-i-1} \prod_{j=1}^{i+1} \frac{1}{m-j+r} \prod_{j=0}^{i} (m+s-j)(m+t-j) x^m \\
&\le \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{m=2}^{+\infty} \sum_{i=1}^{m-1} \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^{i} \frac{(m+s-j)(m+t-j)}{(m-j)(m-j+s+t)} x^m \\
&= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{m=2}^{+\infty} \sum_{i=1}^{m-1} \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^{i} (1 - \frac{-st}{(m-j)(m-j+s+t)}) x^m \\
&= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{i=1}^{+\infty} \sum_{m=i+1}^{+\infty} A_{mi} x^m \\
&= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{i=1}^{+\infty} \sum_{m=2}^{+\infty} A_{m+i-1 i} x^{m+i-1},
\end{aligned}$$

where we define

$$A_{mi} := \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^{i} (1 - \frac{-st}{(m-j)(m-j+s+t)})$$

for all $i = 1, 2, \cdots$ and $m = 2, 3, \cdots$. □

3. Approximation property of hypergeometric function

In this section, we investigate an approximation property of hypergeometric functions. More precisely, we will prove that if an analytic function satisfies the condition (2.2), then it can be approximated by a hypergeometric function. Suppose that y is a given function expressed as a power series of the form

$$y(x) = \sum_{m=0}^{\infty} b_m x^m, \tag{3.1}$$

whose radius of convergence is $R_0 > 0$. Then we obtain

$$\begin{aligned}
x(1-x)y'' &+ [r - (1+s+t)x]y' - sty \\
&= \sum_{m=0}^{\infty} [(m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m] x^m \\
&= \sum_{m=0}^{\infty} a_m x^m,
\end{aligned} \tag{3.2}$$

where we define

$$a_m := (m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m \tag{3.3}$$

for all $m \in \{0, 1, 2, 3, \cdots\}$.

Lemma 3.1. *If the a_m's, the b_m's and the c_m's are as defined in* (3.3), (3.1) *and* (2.4), *then*

$$c_m = b_m - \frac{b_0}{m!}\prod_{j=1}^{m}\frac{1}{m-j+r}\prod_{j=1}^{m-1}(m+s-j)(m+t-j) \tag{3.4}$$

for all $m \in \{0,1,2,3,...\}$.

Proof. The proof is clear by induction on m. For $m = 1$ and by (3.3) we have

$$c_1 = \frac{a_0}{r} = \frac{1}{r}(rb_1 - stb_0) = b_1 - \frac{st}{r}b_0. \tag{3.5}$$

Assume now that formula (3.3) is true for some m. It follows from (2.4), (3.3) and (3.4) that

$$\begin{aligned} c_{m+1} &= \frac{a_m}{(m+1)(m+r)} + \frac{(m+s)(m+t)}{(m+1)(m+r)}c_m \\ &= \frac{1}{(m+1)(m+r)}\Big((m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m\Big) \\ &+ \frac{(m+s)(m+t)}{(m+1)(m+r)}\Big(b_m - \frac{b_0}{m!}\prod_{j=1}^{m}\frac{1}{m-j+r}\prod_{j=1}^{m-1}(m+s-j)(m+t-j)\Big) \\ &= b_{m+1} - \frac{b_0}{(m+1)!}\prod_{j=1}^{m+1}\frac{1}{m+1-j+r}\prod_{j=1}^{m}(m+1+s-j)(m+1+t-j), \end{aligned}$$

as desired. □

Theorem 3.2. *Let R and R_0 be positive constants with $R < R_0$. Assume that $y : (-R,R) \to \mathbb{C}$ is a function of the form* (3.1) *whose radius of convergence is R_1. Also, b_m's and c_m's are given by* (3.3) *and* (3.4), *respectively. If $R < min\{1, R_0, R_1\}$, then there exist a hypergeometric function $y_h : (-R,R) \to \mathbb{C}$ and a constant $d > 0$ such that $|y(x) - y_h(x)| \le d\frac{x}{1-x}$ for all $x \in (-R,R)$.*

Proof. We assume that y can be represented by a power series (3.1) whose radius of convergence is $R < R_0$. So

$$x(1-x)\sum_{m=2}^{+\infty} m(m-1)b_m x^{m-2} + [r-(1+s+t)x]\sum_{m=1}^{+\infty} mb_m x^{m-1} - st\sum_{m=0}^{+\infty} mb_m x^m$$

is also a power series whose radius of convergence is R_0, more precisely, in view of (3.2) and (3.3), we have

$$\begin{aligned} x(1-x)\sum_{m=0}^{+\infty} m(m-1)b_m x^{m-2} + [r-(1+s+t)x]\sum_{m=0}^{+\infty} mb_m x^{m-1} \\ - st\sum_{m=0}^{+\infty} mb_m x^m = \sum_{m=0}^{+\infty} a_m x^m \end{aligned}$$

for all $x \in (-R, R)$. Since the power series $\sum_{m=0}^{+\infty} a_m x^m$ is absolutely convergent on its interval of convergence, which includes the interval $[-R, R]$ and the power series $\sum_{m=0}^{+\infty} |a_m x^m|$ is continuous on $[-R, R]$. So there exists a constant $d_1 > 0$ with

$$\sum_{m=0}^{n} |a_m x^m| \leq d_1$$

for all integers $n \geq 0$ and for any $x \in (-R, R)$.

On the other hand, since

$$\sum_{k=1}^{+\infty} |\frac{-st}{(m-k-1)(m-k-1+t+s)}| \leq \frac{st\pi^2}{6} =: d_2, \quad (m = 2, 3, ...),$$

we have

$$|\prod_{k=1}^{+\infty}(1 - \frac{-st}{(m-k-1)(m-k-1+t+s)}| \leq d_2, \quad (m = 2, 3, \cdots)$$

(see [16, Theorem 6.6.2]). Hence, substituting $i - j$ for k in the above infinite product, there exists a constant d_3 with

$$|\prod_{j=0}^{i}(1 - \frac{-st}{(m-i-j-1)(m-i-j-1+t+s)}| \leq d_3$$

for all $i = 1, 2, \cdots$ and $m = 2, 3, \cdots$. Therefore, it follows Lemma 2.2 that

$$|\sum_{m=0}^{\infty} c_m x^m| \leq d_1 d_3 \frac{x}{1-x} \tag{3.6}$$

for all $x \in (-R_0, R_0)$. This completes the proof of our theorem. □

Corollary 3.3. *Assume that R and R_0 are positive constants with $R < R_0$. Let $y : (R, R_0) \to C$ be a function which can be represented by a power series of the form (3.1) whose radius of convergence is R_0. Moreover, assume that there exists a positive number R_1 satisfying the condition (2.2) with b_m's and c_m's given in (3.1) and Lemma 3.1. If $R < min\{1, R_0, R_1\}$ then there exists a hypergeometric function $y_h : (-R, R) \to \mathbb{C}$ such that $|y(x) - y_h(x)| = O(x)$ as $x \to 0$.*

Example 3.4. *Now, we will introduce an example concerning the hypergeometric function for differential equation (2.1) with $st = \frac{1}{16}$. Given a constant R with $0 < R < 1$ and assume that a function $y : (-R, R) \longrightarrow \mathbb{C}$ can be expressed as a power series of the form (3.1), where*

$$b_m = \{^{0, \quad m=0}_{\frac{1}{4^m}, \quad m \geq 1.}$$

It is easy to see that the radius of convergence of the above power series is $R_1 = 4$. Since $b_0 = 0$ it follows from Lemma 3.1 that $c_m = b_m$ for each $m \in \{0, 1, 2, 3, ...\}$. Moreover, there exists a positive constant R_1 such that the condition (2.2) is satisfied

$$R_1 = \lim_{k \to +\infty} |\frac{c_k}{c_{k+1}}| = \lim_{k \to +\infty} |\frac{b_k}{b_{k+1}}| = 4.$$

Now we assume $r = s = t = \frac{1}{4}$. Then we get

$$\begin{aligned}\sum_{m=0}^{+\infty} |a_m x^m| &\le \frac{1}{16} + \frac{15}{64}|x| + \sum_{m=2}^{+\infty} \frac{4(m+\frac{1}{4})^2 - (m+1)(m+\frac{1}{4})}{4^{m+2}}|x|^m \\ &\le \frac{1}{16} + \frac{15}{64} + \sum_{m=2}^{+\infty} \frac{3m(m+\frac{1}{4})}{4^{m+2}} \\ &\le \frac{1}{16} + \frac{15}{64} + \sum_{m=2}^{+\infty} \frac{1}{2^{m+2}} \le \frac{1}{16} + \frac{15}{64} + \frac{1}{8} = \frac{27}{64}\end{aligned}$$

for all $x \in (-R, R)$. Since $R < min\{1, R_0, R_1\} = 1$, we can conclude from (3.6) that there exists a solution function $y_h : (-R, R) \to \mathbb{C}$ of the Gauss differential equation (2.1) with $r = s = t = \frac{1}{4}$ satisfying $|y(x) - y_h(x)| \le \frac{27}{64}\frac{x}{1-x}$ for all $x \in (-R, R)$.

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