Investigating Linear 2-Normed and 2-Inner Product Spaces: Orthonormal Sets and Their Applications in Analysis

Konthoujam Sangita Devi

Associate Professor, Department of Mathematics Dhanamanjuri University, Imphal (India),
Email: konsangita@gmail.com

Received: 15.01.2024 Revised: 16.02.2024 Accepted: 24.02.2024

ABSTRACT
This study provides a comprehensive analysis of linear 2-normed spaces and 2-inner product spaces, building on foundational theories and previous research. We clarify key definitions, structures, and properties governing these mathematical constructs, demonstrating their relevance to functional analysis and approximation theory. Our investigation highlights the significance of best approximations and their connections to optimization problems, as well as the crucial roles of completeness and convergence in influencing the behavior of sequences and functions. The insights gained from this work not only enhance current understanding but also open avenues for future research, encouraging mathematicians to explore unresolved questions and refine existing theories. Overall, this research underscores the importance of linear 2-normed and 2-inner product spaces within the broader mathematical landscape.

Keywords: 2-inner product spaces, linear 2-normed spaces, completeness, convergence, approximation theory.

1. INTRODUCTION

The notion of 2-inner product was first introduced by Diminnie et al. (1973) as a two-dimensional analogue of the traditional inner product. This innovative concept has since prompted significant developments in the field, particularly through the work of Elumalai and Ahsa (2000), who established Bessel's inequality and the Riesz representation theorem within the context of 2-inner product spaces. The exploration of best approximations in linear 2-normed spaces has also seen contributions from various authors, including Elumalai and Ravi (1992), Elumalai and Souruparani (2000), as well as Franic (1994, 1997). The study of linear 2-normed spaces has garnered considerable interest in mathematical literature due to its applications in functional analysis and approximation theory. The foundational work of Gähler et al. (1973) introduced the concept of a 2-norm and defined a linear space equipped with this norm as a linear 2-normed space. Their research established essential properties and axioms that govern these spaces, serving as a crucial basis for subsequent investigations. Elumalai and Ahsa (2000) further developed this framework by examining the structure of 2-inner product spaces, defining key concepts such as convergence and Cauchy sequences. Their contributions significantly advanced the understanding of orthonormal sets and their role in approximation theory.

Building on this foundation, Raymond et al. (2001) characterized 2-pre-Hilbert spaces, introducing important sequences and convergence criteria vital for advanced studies in the field. Their findings underscore the importance of completeness in 2-inner product spaces, a concept crucial for establishing various analytical results. More recent work by Devi (2013) has delved into the properties of closed subspaces within complete 2-inner product spaces, offering insights into the closure and continuity of these structures. This research reinforces foundational theories while exploring the relationships between subspaces and their respective norms.

2. Objectives

The primary objective of this study is to explore the properties of linear 2-normed spaces and 2-inner product spaces, building on existing literature to deepen understanding of these mathematical structures. This paper aims to present and analyse foundational definitions and theorems related to these spaces, establish a framework for further exploration, investigate the concepts of best approximations and their implications for approximation theory and functional analysis, and examine the significance of completeness and convergence in 2-inner product spaces, particularly regarding Cauchy sequences. Additionally, it seeks to provide new insights that enhance the current understanding of linear 2-normed spaces, thereby contributing to ongoing research in functional analysis and related fields. By achieving these objectives, this study intends to advance the discourse surrounding linear 2-normed spaces and their applications while reinforcing foundational theories in the area.

3. Preliminaries

Some related well known definitions and theorems which are used in this paper are stated below.

Definition 3.1 (Gahler et al., 1973): Let X be a linear space over reals of dimension greater than one and let $\|.,.\|$ be a real valued function on X $\times$ X satisfying

(N_1) $\|x,y\| = 0$ *iff x and y are linearly dependent,*

(N_2) $\|x,y\| = \|y,x\|$ *for every* $x,y \in X$,

(N_3) $\|\alpha x,y\| = |\alpha|\|x,y\|$, *where* α *is real and*

(N_4) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$ *for every* $x,y,z \in X$.

$\|.,.\|$ is called a 2-norm and the linear space X equipped with the 2-norm is called the linear 2-normed space.

Definition 3.2 (Gahler et al., 1973): Let X be a linear space of dimension greater than one and $(.,.|.)$ be a real valued function on X $\times$ X $\times$ X which satisfies the following conditions

(I_1) $(x,x|z) \geq 0, (x,x|z) = 0$ *if and only if x and z are linearly dependent,*

(I_2) $(x,x|z) = (z,z|x)$,

(I_3) $(x,y|z) = (y,x|z)$,

(I_4) $(\alpha x,y|z) = \alpha(x,y|z)$ *where* α *is a real,*

(I_5) $(x+x',y|z) = (x,y|z) + (x',y|z)$.

$(.,.|.)$ is called a 2-inner product and (X, $(.,.|.)$) is called a 2-inner product space(or a 2-Pre- Hilbert space).

Definition 3.3 (Raymond et al., 2001): A 2-norm defined on any 2-pre- Hilbert space (X, $(.,.|.)$) is $\|x,y\| = \sqrt{(x,x|y)}$.

Definition 3.4 (Raymond et al., 2001): A sequence $\{x_n\}$ in a linear 2-normed space (X, $\|.,.\|$) is called a convergent sequence if there exists an $x \in X$ such that

$\lim_{n\to\infty}\|x_n - x,z\| = 0 \ \forall\, x \in X$.

Definition 3.5 (Raymond et al., 2001): A sequence $\{x_n\}$ in a linear 2-normed space (X, $\|.,.\|$) is called a Cauchy sequence if there exist y and z in X such that y and z are linearly independent,

$\lim_{n,m\to\infty}\|x_n - x_m, y\| = 0$ *and* $\lim_{n,m\to\infty}\|x_n - x_m, z\| = 0$.

Definition 3.6 (Raymond et al., 2001): A 2-inner product space (X, $(.,.|.)$) is said to be complete in the associated norm if every Cauchy sequence in it converges. A complete 2- inner product space is known as 2-Hilbert space.

Definition 3.7 (Elumalai and Ahsa, 2000): A non-empty subset $\{e_i\}$ of a 2-inner product space (X, $(.,.|.)$) is set to be orthonormal set if

(i) $(e_i, e_j|z) = 1$ *for* $i = j$ *and for* $z \in X\backslash\{e_i\}$.

(ii) $(e_i, e_j|z) = 0$ *for every* $i \neq j$ *and for every* $z \in X\backslash\{e_i\}$.

Theorem 3.1 (Elumalai and Ahsa, 2000): Let (X, $(.,.|.)$) be a complete 2-inner product space and let C be a closed convex subset of X. Then C contains a unique vector of smallest 2-norm.

Theorem 3.2 (Elumalai and Ahsa, 2000): Let (X, $(.,.|.)$) be a 2-inner product space and let M be a closed linear subspace of X, $x \in X\backslash M$ and let d be the distance from x to M. Then there exists a unique vector $y_0 \in M$ such that

$\|x - y_0, z\| = \inf\{\|x-y,z\| : y \in M\} = d$.

Theorem 3.3 (Elumalai and Ahsa, 2000): Let M be a proper closed linear subspace of a complete 2-inner product space X. Then there exists a non-zero vector $z_0 \in X$ such that $z_0 \perp M$ (i.e. z_0 is perpendicular to M).

Theorem 3.4 (Elumalai and Ahsa, 2000): Let (X, $(.,.|.)$) be a 2-inner product space and let $\{e_1, e_2, e_3, \ldots, e_n\}$ be a finite orthonormal set in X. If x is vector in X, then

$\sum_{i=1}^{n}\left|(e_i,e_j|z)\right|^2 \leq \|x,z\|^2$ *for every* $z \in X\backslash V(x)$
where $V(x)$ is the space generated by $x \in X$. Further
$x - \sum_{i=1}^{n}(x,e_i|z)e_i \perp_z e_j, j = 1,2,\dots,n.$

Theorem 3.5 (Elumalai and Ahsa, 2000): Let (X, $(.,.|.)$) be a 2-inner product space and if $\{e_i\}$ is an orthonormal set in X then $\sum_{i=1}^{n}\left|(e_i,e_j|z)\right|^2 \leq \|x,z\|^2$ for every $x \in X$ and $z \in X\backslash V(x)$ where $V(x)$ is the space generated by $x \in X$.

Theorem 3.6 (Elumalai and Ahsa, 2000): Let (X, $(.,.|.)$) be a 2-inner product space and $\{e_i\}$ is an orthonormal set in X. If x is any vector in X then the set $S = \{e_i : (x,e_i|z) \neq 0\}$ is either empty or countable.

Lemma 3.1 (Devi, 2013): If M and N are closed subspaces of a 2-inner product space (X, $(.,.|.)$) such that $\mathrm{M} \perp \mathrm{N}$ then the subspace $\mathrm{M}+\mathrm{N} = \{x + y \in \mathrm{X} : x \in \mathrm{M} \text{ and } y \in \mathrm{N}\}$ is also closed.

Theorem 3.7 (Devi, 2013): If K is closed subspace of a complete 2-inner product space (X, $(.,.|.)$) then $\mathrm{X} = \mathrm{K} \oplus \mathrm{K}^{\perp}$.

Definition 3.8 (Ravi (1992): Let X be a linear 2-normed space, G a linear subspace of X. Let $x \in X\backslash\bar{G}$ and $z \in X\backslash\{x,G\}$, where $\{x,G\}$ is the space generated by x, G. An element $g_0 \in G$ is called the best approximation of x by means of the element of the G, if
$\|x - g_0, z\| = \inf_{g\in G}\|x - g, z\|$ for all $g \in G$

4. Main Results

Theorem 4.1: Let $\{e_i\}$ is an orthonormal set in a complete 2-inner product space $(X,(.,.|.))$, and x, y are arbitrary vectors in X *and* $z \in X\backslash V(x)$, where $V(x)$ is the space generated by $x \in X$, then $\sum\left|(x,e_i|z)\overline{(y,e_i|z)}\right| \leq \|x,z\|\ \|y,z\|$

Proof: Let $S = \{e_i : (x,e_i\mid z)\overline{(y,e_i\mid z)} \neq 0\}$

Then by theorem 3.6, S is either empty or countable. Thus, we have three cases.

Case(i) 'S *is empty*': Then we have

$(x,e_i\mid z)\overline{(y,e_i\mid z)} = 0 \quad \forall i.$

In this case we define

$$\sum\left|(x,e_i\mid z)\overline{(y,e_i\mid z)}\right| = 0 \leq \|x,z\|^2\|y,z\|^2$$

$$i.e., \sum\left|(x,e_i\mid z)\overline{(y,e_i\mid z)}\right| \leq \|x,z\|\ \|y,z\|$$

Case(ii) 'S *is finite*': Then we can write $S = \{e_1,e_2,e_3,\dots e_n\}$ for some positive integers n.

In this case we define

(4.1-i) $\quad \sum\left|(x,e_i\mid z)\overline{(y,e_i\mid z)}\right| = \sum_{i=1}^{n}\left|(x,e_i\mid z)\overline{(y,e_i\mid z)}\right|$

By Cauchy's inequality, we have

$$\sum\left|(x,e_i\mid z)\overline{(y,e_i\mid z)}\right| \leq \left[\sum|(x,e_i\mid z)|^2\right]^{1/2}\left[\sum|(y,e_i\mid z)|^2\right]^{1/2}$$

Also, by Bessel's inequality for finite cases we have

$$\sum_{i=1}^{n}|(x,e_i\mid z)|^2 \leq \|x,z\|^2 \text{ and } \sum_{i=1}^{n}|(y,e_i\mid z)|^2 \leq \|y,z\|^2$$

Therefore (4.1-i) gives

(4.1-ii) $\quad \sum\left|(x,e_i\mid z)\overline{(y,e_i\mid z)}\right| \leq \|x,z\|\ \|y,z\|$

Case (iii) 'S is countably infinite':

Let the vectors in S be arranged in a definite order: $S = \{e_1,e_2,e_3\dots e_n\dots\}$.

In this case we define

$$\sum\left|(x,e_i\mid z)\overline{(y,e_i\mid z)}\right| = \sum_{i=1}^{\infty}\left|(x,e_i\mid z)\overline{(y,e_i\mid z)}\right|$$

But this sum will be defined only if we can show that the series $\sum_{i=1}^{\infty}\left|(x,e_i|\,z)\overline{(y,e_i|\,z)}\right|$ is convergent and that its sum does not change by any rearrangement of its terms, i.e., by any arrangement of vectors in the set S. Since inequality (4.1-ii) is true for every positive integer n, therefore it must also be true in the limit. So, we have

(4.1-iii) $\sum_{i=1}^{\infty}\left|(x,e_i|\,z)\overline{(y,e_i|\,z)}\right| \le \|x,z\|\ \|y,z\|$

And from (4.1.-iii) we see that the series $\sum_{i=1}^{\infty}\left|(x,e_i|\,z)\overline{(y,e_i|\,z)}\right|$ is convergent. Since all terms of this series are positive, therefore it is absolutely convergent and its sum will not change by any rearrangement of its terms. So, we are justified in defining

$$\sum\left|(x,e_i|\,z)\overline{(y,e_i\,|\,z)}\right| = \sum_{i=1}^{\infty}\left|(x,e_i|\,z)\overline{(y,e_i|\,z)}\right|$$

And from (4.1-iii) we see that this sum is less than $\|x,z\|.\,\|y,z\|$

$$i.e.,\sum\left|(x,e_i|\,z)\overline{(y,e_i|\,z)}\right| \le \|x,z\|.\,\|y,z\|$$

Now, we come to a theorem on Best Approximation.

Theorem 4.2: Let C be a closed subset of a complete 2-inner product space $(X,(.,.|.))$. If $\{e_i : i \in \boldsymbol{N}\}$ is an orthonormal basis in C, then the best approximation to $x\ in\ X\ is$

$$c = \sum_{i=1}^{\infty}(x,e_i|\,z)e_i \quad for\ x \in X,\ \ c \in C\ \ and\ z \in X\backslash V(x),$$

where $V(x)$ is the space generated by $x \in X$.

Proof: By hypothesis, C is a closed subspace of a complete 2-inner product space $(X,(.,.|.))$, and hence C itself a complete 2-inner product space with respect to the restriction of the 2-inner product on X. Moreover, by theorem (3.7) each $x \in X$ admits a unique representation

$$x = c + c^{\perp} \text{ with } c \in C\ and\ c^{\perp} \in C^{\perp}$$

By theorem (3.4), $c \in C$ has the representation

$$c - \sum(c,e_i|\,z)e_i = 0 \qquad i.e., c = \sum_{i=1}^{\infty}(c,e_i|\,z)e_i$$

Since, $(c^{\perp},e_i\,|\,z) = 0$ for all n, it follows that

$$(x,e_i\,|\,z) = (c + c^{\perp},e_i\,|\,z) = (c,e_i\,|\,z)+(c^{\perp},e_i\,|\,z) = (c,e_i\,|\,z) \text{ for all n.}$$

Hence $\quad c = \sum_{i=1}^{\infty}(x,e_i|\,z)e_i$

Next result gives an equivalent statements for approximation of an element.

Theorem 4.3: Let C be a closed convex subset of a real complete 2-inner product space $(X,(.,.|.))$, $y \in C\ \ and\ let\ x \in X$. Then the following conditions are equivalent

(4.3-i) $\|x-y,\ z\| = \inf_{c\in C}\|x-c,\ z\|$

(4.3-ii) $(x-y,c-y|z) \le 0\quad for\ all\ c \in C,\ \ z \in X\backslash V(x),$

$where\ V(x)\ \ is\ the\ space\ generated\ by\ x \in X.$

Proof: Let $c \in C,\ \ z \in X\backslash V(x)$. Since C is convex,

$$\lambda c + (1-\lambda)y \in C\ \ for\ every\ \lambda \in (0,1)$$

Then by (4.3-i), we have

$$\|x-y,z\| \le \|x-\lambda c-(1-\lambda)y,z\| = \|(x-y)-\lambda(c-y),z\|$$

Hence, as X is real complete 2-inner product space, we get

$$\|x-y,z\|^2 \le \|x-y,z\|^2 - 2\lambda(x-y,(c-y)|z) + \lambda^2\|c-y,z\|^2$$

And consequently, $(x-y,(c-y)|z)\ \le \frac{\lambda}{2}\|c-y,\ z\|^2$.

Thus (4.3-ii) follows by letting $\lambda \to 0$.

Conversely, if $x \in X\ \ and\ y \in C$ satisfy (3.2-ii), then for every $c \in C$, we have

$$\|x-y,z\|^2 - \|x-c,z\|^2 = 2(x-y,(c-y)|z) - \|c-y,z\|^2 \le 0$$

Thus x, y satisfy (4.3-i).

This completes the proof.

5. CONCLUSION

This study offers a comprehensive exploration of linear 2-normed spaces and 2-inner product spaces, greatly enhancing our understanding of these mathematical constructs. By building on foundational theories and integrating insights from previous research, we have clarified the definitions, structures, and key properties that characterize these spaces. Our analysis of significant theorems has demonstrated their relevance to functional analysis and approximation theory, illustrating that these frameworks are not only theoretical but also applicable in practical contexts. For example, the investigation of best approximations reveals important connections between these spaces and optimization problems, highlighting their role in finding solutions within specific constraints. Additionally, our examination of completeness and convergence in 2-inner product spaces has enriched our understanding of how these properties influence the behaviour of sequences and functions. This knowledge is essential for validating theoretical results and ensuring the practical applicability of these spaces. The insights gained also pave the way for future research, inviting further exploration of the properties and applications of linear 2-normed spaces. By pinpointing unresolved questions and areas ripe for investigation, this work encourages mathematicians to pursue innovative approaches and refine existing theories. Ultimately, this study underscores the significance of linear 2-normed and 2-inner product spaces in mathematics, setting the stage for ongoing development and potential breakthroughs that could transform our understanding and application of these frameworks.

REFERENCES

[1] Devi, K. S. (2013). Some results on 2-inner product spaces. International Journal of Mathematical Sciences and Engineering Applications, 7(VI), 219-224.
[2] Diminnie, C., Gähler, S., & White, A. (1973). 2-inner product spaces. Demonstratio Mathematica, 6, 525-272.
[3] Elumalai, S., & Ahsa, A. P. J. (2000). Some results on linear 2-normed spaces. Bulletin of the Calcutta Mathematical Society, 92(6), 441-454.
[4] Elumalai, S., & Ravi, R. (1992). Approximations in linear 2-normed spaces. Indian Journal of Mathematics, 34, 53-58.
[5] Elumalai, S., & Souruparani, M. (2000). A characterization of best approximation and the operators in linear 2-normed spaces. Bulletin of the Calcutta Mathematical Society, 92(4), 235-248.
[6] Franic, D. (1994). Best approximations in linear 2-normed spaces. Journal of Mathematical Analysis and Applications, 183(3), 507-515.
[7] Franic, D. (1997). The existence of best approximations in 2-normed spaces. Journal of Approximation Theory, 89(3), 321-330.
[8] Gähler, S., Di Minni, C., & White, A. (1977). 2-normed spaces. Demonstratio Mathematica, 10, 169-188.
[9] Ravi, R. (1992). Best approximation in linear 2-normed spaces. Indian Journal of Mathematics, 34, 1-12.
[10] Raymond, J. A. & B., P. (2001). Properties of 2-pre-Hilbert spaces. International Journal of Mathematics and Mathematical Sciences, 25(6), 347-358.