On Some Systems of Three Nonlinear Difference Equations

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Abstract

We consider in this paper, the solution of the following systems of difference equation:

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{\pm 1 + y_{n-2}z_{n-1}x_n}, \; z_{n+1} = \frac{z_{n-2}}{\pm 1 + z_{n-2}x_{n-1}y_n}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0, z_{-2},\ z_{-1},\ z_0$ are arbitrary non zero real numbers.

Keywords: difference equations, recursive sequences, periodic solutions, system of difference equations, stability.
Mathematics Subject Classification: 39A10.

1 Introduction

Difference equations related to differential equations as discrete mathematics related to continuous mathematics. Most of these models are described by nonlinear delay difference equations; see, for example, [9], [10]. The subject of the qualitative study of the nonlinear delay population models is very extensive, and the current research work tends to center around the relevant global dynamics of the considered systems of difference equations such as oscillation, boundedness of solutions, persistence, global

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stability of positive steady sates, permanence, and global existence of periodic solutions. See [13], [17], [19]-[22], [26], [28], [29] and the references therein. In particular, Agarwal and Elsayed [1] deal with the global stability, periodicity character and gave the solution form of some special cases of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}.$$

Camouzis et al. [5] studied the global character of solutions of the difference equation

$$x_{n+1} = \frac{\delta x_{n-2} + x_{n-3}}{A + x_{n-3}}.$$

Clark and Kulenovic [7] investigated the global asymptotic stability of the system

$$x_{n+1} = \frac{x_n}{a + cy_n}, \qquad y_{n+1} = \frac{y_n}{b + dx_n}.$$

In [9], Din studied the boundedness character, steady-states, local asymptotic stability of equilibrium points, and global behavior of the unique positive equilibrium point of a discrete predator-prey model given by

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \qquad y_{n+1} = \frac{\delta x_n y_n}{x_n + \eta y_n}.$$

Elsayed et al. [23] discussed the global convergence and periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}}.$$

Elsayed and El-Metwally [24] discussed the periodic nature and the form of the solutions of the nonlinear difference equations systems

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1}\left(\pm 1 \pm x_n y_{n-2}\right)}, \qquad y_{n+1} = \frac{y_n x_{n-2}}{x_{n-1}\left(\pm 1 \pm y_n x_{n-2}\right)}.$$

Gelisken and Kara [25] studied some behavior of solutions of some systems of rational difference equations of higher order and they showed that every solution is periodic with a period depends on the order.

In [27] Kurbanli discussed a three-dimensional system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} y_n - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{y_{n-1} x_n - 1}, \qquad z_{n+1} = \frac{x_n}{z_{n-1} y_n}.$$

Touafek et al. [33] studied the sufficient conditions for the global asymptotic stability of the following systems of rational difference equations:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}}, \qquad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}.$$

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with a real number's initial conditions.

Our goal in this paper is to investigate the form of the solutions of the system of three difference equations

$$x_{n+1}=\frac{x_{n-2}}{\pm 1+x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{\pm 1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{\pm 1+z_{n-2}x_{n-1}y_n},\ (1)$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0,\ z_{-2},\ z_{-1},\ z_0$ are arbitrary real numbers. Moreover, we obtain some numerical simulation to the equation are given to illustrate our results.

2 The System

$$x_{n+1}=\frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n}$$

In this section, we study the solution of the following system of difference equations.

$$x_{n+1}=\frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n},\qquad (2)$$

where $n\in N_0$ and the initial conditions are arbitrary real numbers.

The following theorem is devoted to the form of the solutions of system (1).

Theorem 1. Suppose that $\{x_n,y_n,z_n\}$ are solutions of the system (1). Then for $n=0,1,2,...$, we have the following formulas

$$\begin{aligned}
x_{3n-2} &= x_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)},\quad x_{3n-1}=x_{-1}\prod_{i=0}^{n-1}\frac{(1+(3i+1)x_{-1}y_0z_{-2})}{(1+(3i+2)x_{-1}y_0z_{-2})},\\
x_{3n} &= x_0\prod_{i=0}^{n-1}\frac{(1+(3i+2)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})},
\end{aligned}$$

$$\begin{aligned}
y_{3n-2} &= y_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})},\quad y_{3n-1}=y_{-1}\prod_{i=0}^{n-1}\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)},\\
y_{3n} &= y_0\prod_{i=0}^{n-1}\frac{(1+(3i+2)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2})},
\end{aligned}$$

$$\begin{aligned}
z_{3n-2} &= z_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})},\quad z_{3n-1}=z_{-1}\prod_{i=0}^{n-1}\frac{(1+(3i+1)x_0y_{-2}z_{-1})}{(1+(3i+2)x_0y_{-2}z_{-1})},\\
z_{3n} &= z_0\prod_{i=0}^{n-1}\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)},
\end{aligned}$$

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Proof. For $n = 0$ the result holds. Suppose that the result holds for $n-1$.

$$x_{3n-5} = x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}, \quad x_{3n-4} = x_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_{-1}y_0z_{-2})}{(1+(3i+2)x_{-1}y_0z_{-2})},$$
$$x_{3n-3} = x_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})},$$

$$y_{3n-5} = y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}, \quad y_{3n-4} = y_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)},$$
$$y_{3n-3} = y_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2})},$$

$$z_{3n-5} = z_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}, \quad z_{3n-4} = z_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_0y_{-2}z_{-1})}{(1+(3i+2)x_0y_{-2}z_{-1})},$$
$$z_{3n-3} = z_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)}.$$

It follows from Eq.(1) that

$$\begin{aligned}
x_{3n-2} &= \frac{x_{3n-5}}{1+x_{3n-5}y_{3n-4}z_{3n-3}} \\
&= \frac{x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1+(x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)})(y_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)})(z_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)})} \\
&= \frac{x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1+x_{-2}y_{-1}z_0\prod_{i=0}^{n-2}((\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)})(\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)})(\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)}))} \\
&= \frac{x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1+x_{-2}y_{-1}z_0\prod_{i=0}^{n-2}(\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)})} \\
&= x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}\frac{1}{1+(\frac{x_{-2}y_{-1}z_0}{(1+(3n-3)x_{-2}y_{-1}z_0)})} \\
&= x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}(\frac{(1+(3n-3)x_{-2}y_{-1}z_0)}{(1+(3n-3)x_{-2}y_{-1}z_0)+x_{-2}y_{-1}z_0}) \\
&= x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}(\frac{(1+(3n-3)x_{-2}y_{-1}z_0)}{(1+(3n-2)x_{-2}y_{-1}z_0)}).
\end{aligned}$$

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Then, we see that

$$x_{3n-2} = x_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}.$$

Also, we see from Eq.(1) that

$$\begin{aligned}
y_{3n-2} &= \frac{y_{3n-5}}{1+y_{3n-5}z_{3n-4}x_{3n-3}} \\
&= \frac{y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}}{1+(y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})})(z_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_0y_{-2}z_{-1})}{(1+(3i+2)x_0y_{-2}z_{-1})})(x_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})})} \\
&= \frac{y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}}{1+x_0y_{-2}z_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})}} \\
&= y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}(\frac{1}{1+\frac{x_0y_{-2}z_{-1}}{1+(3n-3)x_0y_{-2}z_{-1}-}}) \\
&= y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}(\frac{1+(3n-3)x_0y_{-2}z_{-1}}{1+(3n-3)x_0y_{-2}z_{-1}+x_0y_{-2}z_{-1}}) \\
&= y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}(\frac{1+(3n-3)x_0y_{-2}z_{-1}}{1+(3n-2)x_0y_{-2}z_{-1}}).
\end{aligned}$$

Then, we see that

$$y_{3n-2} = y_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}$$

Finally, we see that

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$$
\begin{aligned}
z_{3n-2} &= \frac{z_{3n-5}}{1+z_{3n-5}x_{3n-4}y_{3n-3}} \\
&= \frac{z_{-2}\prod\limits_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}}{1+(z_{-2}\prod\limits_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})})(x_{-1}\prod\limits_{i=0}^{n-2}\frac{(1+(3i+1)x_{-1}y_0z_{-2})}{(1+(3i+2)x_{-1}y_0z_{-2})})(y_0\prod\limits_{i=0}^{n-2}\frac{(1+(3i+2)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2})})} \\
&= \frac{z_{-2}\prod\limits_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}}{1+x_{-1}y_0z_{-2}\prod\limits_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2}}} \\
&= z_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}\left(\frac{1}{1+\frac{x_{-1}y_0z_{-2}}{1+(3n-3)x_{-1}y_0z_{-2}}}\right) \\
&= z_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}\left(\frac{1+(3n-3)x_{-1}y_0z_{-2}}{1+(3n-2)x_{-1}y_0z_{-2}}\right).
\end{aligned}
$$

Then

$$z_{3n-2} = z_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}.$$

This completes the proof.

3 The System $x_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n}, y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$

In this section, we obtain the form of the solutions of the system of three difference equations

$$x_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, \; z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}, \quad (3)$$

where $n \in N_0$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 2. Suppose that $\{x_n, y_n, z_n\}$ are solutions of the system (2). Then for $n = 0, 1, 2, ...$, we have the following formulas

$$x_{3n-2} = \frac{x_{-2}}{1+nx_{-2}y_{-1}z_0}, \quad x_{3n-1} = \frac{x_{-1}(x_{-1}y_0z_{-2}-1)}{(n+1)x_{-1}y_0z_{-2}-1}, \quad x_{3n} = \frac{x_0}{1+nx_0y_{-2}z_{-1}},$$

$$
\begin{aligned}
y_{3n-2} &= \frac{(-1)^{n+1}y_{-2}(1+(n-1)x_0y_{-2}z_{-1})}{x_0y_{-2}z_{-1}-1}, \quad y_{3n-1} = (-1)^n y_{-1}(1+nx_{-2}y_{-1}z_0), \\
y_{3n} &= \frac{(-1)^n y_0((n+1)x_{-1}y_0z_{-2}-1)}{x_{-1}y_0z_{-2}-1},
\end{aligned}
$$

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$$z_{3n-2}=\frac{(-1)^{n+1}z_{-2}}{nx_{-1}y_0z_{-2}-1},\ z_{3n-1}=\frac{(-1)^{n+1}z_{-1}(x_0y_{-2}z_{-1}-1)}{(n-1)x_0y_{-2}z_{-1}+1},\ z_{3n}=\frac{(-1)^{n}z_0}{1+nx_{-2}y_{-1}z_0}.$$

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$.

$$x_{3n-5}=\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0},\ x_{3n-4}=\frac{x_{-1}(x_{-1}y_0z_{-2}-1)}{nx_{-1}y_0z_{-2}-1},\ x_{3n-3}=\frac{x_0}{1+(n-1)x_0y_{-2}z_{-1}},$$

$$\begin{aligned}y_{3n-5}&=\frac{(-1)^n y_{-2}(1+(n-2)x_0y_{-2}z_{-1})}{x_0y_{-2}z_{-1}-1},\ y_{3n-4}=(-1)^{n-1}y_{-1}(1+(n-1)x_{-2}y_{-1}z_0),\\ y_{3n-3}&=\frac{(-1)^{n-1}y_0(nx_{-1}y_0z_{-2}-1)}{x_{-1}y_0z_{-2}-1},\end{aligned}$$

$$z_{3n-5}=\frac{(-1)^n z_{-2}}{(n-1)x_{-1}y_0z_{-2}-1},\ z_{3n-4}=\frac{(-1)^n z_{-1}(x_0y_{-2}z_{-1}-1)}{(n-2)x_0y_{-2}z_{-1}+1},\ z_{3n-3}=\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0},$$

from system (2) we can prove as follow

$$\begin{aligned}x_{3n-2}&=\frac{x_{3n-5}}{1+x_{3n-5}y_{3n-4}z_{3n-3}}\\ &=\frac{\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0}}{1+(\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0})((-1)^{n-1}y_{-1}(1+(n-1)x_{-2}y_{-1}z_0))(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0})}\\ &=\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0+x_{-2}y_{-1}z_0}=\frac{x_{-2}}{1+nx_{-2}y_{-1}z_0}\end{aligned}$$

Also, we get

$$\begin{aligned}y_{3n-1}&=\frac{y_{3n-4}}{-1+y_{3n-4}z_{3n-3}x_{3n-2}}\\ &=\frac{(-1)^{n-1}y_{-1}(1+(n-1)x_{-2}y_{-1}z_0)}{-1+((-1)^{n-1}y_{-1}(1+(n-1)x_{-2}y_{-1}z_0))(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0})(\frac{x_{-2}}{1+nx_{-2}y_{-1}z_0})}\\ &=\frac{(-1)^n y_{-1}(1+(n-1)x_{-2}y_{-1}z_0)(1+nx_{-2}y_{-1}z_0)}{1+(n-1)x_{-2}y_{-1}z_0}=(-1)^n y_{-1}(1+nx_{-2}y_{-1}z_0)\end{aligned}$$

$$\begin{aligned}z_{3n}&=\frac{z_{3n-3}}{-1+z_{3n-3}x_{3n-2}y_{3n-1}}\\ &=\frac{\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0}}{-1+(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0})(\frac{x_{-2}}{1+nx_{-2}y_{-1}z_0})((-1)^n y_{-1}(1+nx_{-2}y_{-1}z_0))}\\ &=\frac{(-1)^n z_0}{1+(n-1)x_{-2}y_{-1}z_0+x_{-2}y_{-1}z_0}=\frac{(-1)^n z_0}{1+nx_{-2}y_{-1}z_0}.\end{aligned}$$

4 The System $x_{n+1}=\frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$

In this section, we study the solution of the following system of difference equations

$$x_{n+1}=\frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n},\quad (4)$$

where $n\in N_0$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 3. Suppose that $\{x_n,y_n,z_n\}$ are solutions of the system (3). Then for $n=0,1,2,...,$ we have the following formulas

$$x_{3n-2}=\frac{x_{-2}}{nx_{-2}y_{-1}z_0-1},\quad x_{3n-1}=\frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-1)x_{-1}y_0z_{-2}+1},\quad x_{3n}=\frac{(-1)^n x_0}{1+nx_0y_{-2}z_{-1}},$$

$$y_{3n-2}=\frac{y_{-2}}{nx_0y_{-2}z_{-1}+1},\quad y_{3n-1}=\frac{y_{-1}(x_{-2}y_{-1}z_0-1)}{(n+1)x_{-2}y_{-1}z_0-1},\quad y_{3n}=\frac{y_0}{nx_{-1}y_0z_{-2}+1},$$

$$z_{3n-2}=\frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1},\quad z_{3n-1}=(-1)^n z_{-1}(nx_0y_{-2}z_{-1}+1),$$

$$z_{3n}=\frac{(-1)^n z_0((n+1)x_{-2}y_{-1}z_0-1)}{x_{-2}y_{-1}z_0-1}.$$

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$

$$x_{3n-5}=\frac{x_{-2}}{(n-1)x_{-2}y_{-1}z_0-1},\quad x_{3n-4}=\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1},\quad x_{3n-3}=\frac{(-1)^{n-1}x_0}{1+(n-1)x_0y_{-2}z_{-1}},$$

$$y_{3n-5}=\frac{y_{-2}}{(n-1)x_0y_{-2}z_{-1}+1},\quad y_{3n-4}=\frac{y_{-1}(x_{-2}y_{-1}z_0-1)}{nx_{-2}y_{-1}z_0-1},\quad y_{3n-3}=\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1},$$

$$z_{3n-5}=\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1},\quad z_{3n-4}=(-1)^{n-1}z_{-1}((n-1)x_0y_{-2}z_{-1}+1),$$

$$z_{3n-3}=\frac{(-1)^{n-1}z_0(nx_{-2}y_{-1}z_0-1)}{x_{-2}y_{-1}z_0-1},$$

from system (3) we can prove as follow

$$\begin{aligned}
x_{3n-1} &= \frac{x_{3n-4}}{-1+x_{3n-4}y_{3n-3}z_{3n-2}}\\
&= \frac{\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1}}{-1+\left(\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1}\right)\left(\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1}\right)\left(\frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}\right)}\\
&= \frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{-((n-2)x_{-1}y_0z_{-2}+1)+((-1)^n x_{-1})((-1)^{n+1}y_0z_{-2})}\\
&= \frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-1)x_{-1}y_0z_{-2}+1}.
\end{aligned}$$

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Also, we get

$$\begin{aligned}
y_{3n} &= \frac{y_{3n-3}}{1+y_{3n-3}z_{3n-2}x_{3n-1}} \\
&= \frac{\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1}}{1+(\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1})(\frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1})(\frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-1)x_{-1}y_0z_{-2}+1})} \\
&= \frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1+y_0((-1)^{n+1}z_{-2})((-1)^{n+1}x_{-1})} \\
&= \frac{y_0}{nx_{-1}y_0z_{-2}+1}
\end{aligned}$$

$$\begin{aligned}
z_{3n-2} &= \frac{z_{3n-5}}{-1+z_{3n-5}x_{3n-4}y_{3n-3}} \\
&= \frac{\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}}{-1+(\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1})(\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1})(\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1})} \\
&= \frac{\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}}{\frac{-((n-2)x_{-1}y_0z_{-2}+1)}{(n-1)x_{-1}y_0z_{-2}+1}} \\
&= \frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}.
\end{aligned}$$

This completes the proof.

5 The System $x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n}$, $y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}$, $z_{n+1} = \frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n}$

In this section, we investigate the solution of the following system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, \; z_{n+1} = \frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n}, \quad (5)$$

where the initial conditions $n \in N_0$ are arbitrary non zero real numbers. The following theorem is devoted to the form of the solutions of system (4).

Theorem 4. Suppose that $\{x_n, y_n, z_n\}$ are solutions of the system (4). Then for $n = 0, 1, 2, ...$, we have the following formulas

$$\begin{aligned}
x_{3n-2} &= \frac{(-1)^{n+1}x_{-2}((n-1)x_{-2}y_{-1}z_0+1)}{x_{-2}y_{-1}z_0-1}, \; x_{3n-1} = (-1)^n x_{-1}(nx_{-1}y_0z_{-2}+1), \\
x_{3n} &= \frac{(-1)^n x_0((n+1)x_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1},
\end{aligned}$$

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$$y_{3n-2}=\frac{(-1)^{n+1}y_{-2}}{nx_0y_{-2}z_{-1}-1},\ y_{3n-1}=\frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-1)x_{-2}y_{-1}z_0+1},\ y_{3n}=\frac{(-1)^{n}y_0}{nx_{-1}y_0z_{-2}+1},$$
$$z_{3n-2}=\frac{z_{-2}}{nx_{-1}y_0z_{-2}+1},\quad z_{3n-1}=\frac{z_{-1}(x_0y_{-2}z_{-1}-1)}{(n+1)x_0y_{-2}z_{-1}-1},\quad z_{3n}=\frac{z_0}{nx_{-2}y_{-1}z_0+1}.$$

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$

$$\begin{aligned}x_{3n-5}&=\frac{(-1)^{n}x_{-2}((n-2)x_{-2}y_{-1}z_0+1)}{x_{-2}y_{-1}z_0-1},\ x_{3n-4}=(-1)^{n-1}x_{-1}((n-1)x_{-1}y_0z_{-2}+1),\\x_{3n-3}&=\frac{(-1)^{n-1}x_0(x_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1},\end{aligned}$$

$$y_{3n-5}=\frac{(-1)^{n}y_{-2}}{(n-1)x_0y_{-2}z_{-1}-1},\ y_{3n-4}=\frac{(-1)^{n}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1},\ y_{3n-3}=\frac{(-1)^{n-1}y_0}{(n-1)x_{-1}y_0z_{-2}+1},$$
$$z_{3n-5}=\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1},\ z_{3n-4}=\frac{z_{-1}(x_0y_{-2}z_{-1}-1)}{nx_0y_{-2}z_{-1}-1},\ z_{3n-3}=\frac{z_0}{(n-1)x_{-2}y_{-1}z_0+1},$$

from system (4) we can prove as follow

$$\begin{aligned}x_{3n}&=\frac{x_{3n-3}}{-1+x_{3n-3}y_{3n-2}z_{3n-1}}\\&=\frac{\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}}{-1+(\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1})(\frac{(-1)^{n+1}y_{-2}}{nx_0y_{-2}z_{-1}-1})(\frac{z_{-1}(x_0y_{-2}z_{-1}-1)}{(n+1)x_0y_{-2}z_{-1}-1})}\\&=\frac{\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}}{\frac{x_0y_{-2}z_{-1}-((n+1)x_0y_{-2}z_{-1}-1)}{((n+1)x_0y_{-2}z_{-1}-1)}}=\frac{(-1)^{n}x_0((n+1)x_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}\end{aligned}$$

Also, we get

$$\begin{aligned}y_{3n-1}&=\frac{y_{3n-4}}{-1+y_{3n-4}z_{3n-3}x_{3n-2}}\\&=\frac{\frac{(-1)^{n}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1}}{-1+(\frac{(-1)^{n}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1})(\frac{z_0}{(n-1)x_{-2}y_{-1}z_0+1})(\frac{(-1)^{n+1}x_{-2}((n-1)x_{-2}y_{-1}z_0+1)}{x_{-2}y_{-1}z_0-1})}\\&=\frac{\frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1}}{\frac{(n-2)x_{-2}y_{-1}z_0+1+x_{-2}y_{-1}z_0}{(n-2)x_{-2}y_{-1}z_0+1}}=\frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-1)x_{-2}y_{-1}z_0+1}\end{aligned}$$

$$\begin{aligned}z_{3n-2}&=\frac{z_{3n-5}}{1+z_{3n-5}x_{3n-4}y_{3n-3}}\\&=\frac{\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}}{1+(\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1})((-1)^{n-1}x_{-1}((n-1)x_{-1}y_0z_{-2}+1))(\frac{(-1)^{n-1}y_0}{(n-1)x_{-1}y_0z_{-2}+1})}\\&=\frac{\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}}{\frac{(n-1)x_{-1}y_0z_{-2}+1+x_{-1}y_0z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}}=\frac{z_{-2}}{nx_{-1}y_0z_{-2}+1}\end{aligned}$$

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This completes the proof.

The following cases can be proved similarly.

6 On The System

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\ y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n},\ z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$$

In this section we study the solution of the following system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, \ z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}, \quad (6)$$

where the initial conditions $n \in N_0$ are arbitrary non zero real numbers.

Theorem 5. Let $\{x_n, y_n, z_n\}_{n=-2}^{+\infty}$ be solutions of system (5). Then

1- $\{x_n\}_{n=-2}^{+\infty}$, $\{y_n\}_{n=-2}^{+\infty}$ and $\{z_n\}_{n=-2}^{+\infty}$ and are periodic with period six i.e.,

$$x_{n+6} = x_n, \quad y_{n+6} = y_n, \quad z_{n+6} = z_n.$$

2- We have the following form

$$x_{6n-2} = x_{-2}, \quad x_{6n-1} = x_{-1}, \quad x_{6n} = x_0,$$

$$x_{6n+1} = \frac{x_{-2}}{x_{-2}y_{-1}z_0 - 1}, \quad x_{6n+2} = x_{-1}(x_{-1}y_0z_{-2} - 1), \quad x_{6n+3} = \frac{x_0}{x_0y_{-2}z_{-1} - 1},$$

$$y_{6n-2} = y_{-2}, \quad y_{6n-1} = y_{-1}, \quad y_{6n} = y_0,$$

$$y_{6n+1} = \frac{y_{-2}}{x_0y_{-2}z_{-1} - 1}, \quad y_{6n+2} = y_{-1}(x_{-2}y_{-1}z_0 - 1), \quad y_{6n+3} = \frac{y_0}{x_{-1}y_0z_{-2} - 1},$$

$$z_{6n-2} = z_{-2}, \quad z_{6n-1} = z_{-1}, \quad z_{6n} = z_0,$$

$$z_{6n+1} = \frac{z_{-2}}{x_{-1}y_0z_{-2} - 1}, \quad z_{6n+2} = z_{-1}(x_0y_{-2}z_{-1} - 1), \quad z_{6n+3} = \frac{z_0}{x_{-2}y_{-1}z_0 - 1},$$

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{x_{-2}, x_{-1}, x_0, \frac{x_{-2}}{x_{-2}y_{-1}z_0 - 1}, x_{-1}(x_{-1}y_0z_{-2} - 1), \frac{x_0}{x_0y_{-2}z_{-1} - 1}\right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{y_{-2}, y_{-1}, y_0, \frac{y_{-2}}{x_0y_{-2}z_{-1} - 1}, y_{-1}(x_{-2}y_{-1}z_0 - 1), \frac{y_0}{x_{-1}y_0z_{-2} - 1}\right\}.$$

$$\{z_n\}_{n=-2}^{+\infty} = \left\{z_{-2}, z_{-1}, z_0, \frac{z_{-2}}{x_{-1}y_0z_{-2} - 1}, z_{-1}(x_0y_{-2}z_{-1} - 1), \frac{z_0}{x_{-2}y_{-1}z_0 - 1}\right\}.$$

7 On The System

$$x_{n+1}=\frac{x_{n-2}}{-1-x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{-1-y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{-1-z_{n-2}x_{n-1}y_n}$$

In this section we study the solution of the following system of difference equations

$$x_{n+1}=\frac{x_{n-2}}{-1-x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{-1-y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{-1-z_{n-2}x_{n-1}y_n}, \tag{7}$$

where the initial conditions $n\in N_0$ are arbitrary non zero real numbers.

Theorem 6. Let $\{x_n,y_n,z_n\}_{n=-2}^{+\infty}$ be solutions of system (6). Then

1- $\{x_n\}_{n=-2}^{+\infty}$, $\{y_n\}_{n=-2}^{+\infty}$ and $\{z_n\}_{n=-2}^{+\infty}$ and are periodic with period six i.e.,

$$x_{n+6}=x_n,\quad y_{n+6}=y_n,\quad z_{n+6}=z_n.$$

2- We have the following form

$$x_{6n-2}=x_{-2},\quad x_{6n-1}=x_{-1},\quad x_{6n}=x_0,$$

$$x_{6n+1}=-\frac{x_{-2}}{x_{-2}y_{-1}z_0+1},\quad x_{6n+2}=-x_{-1}(x_{-1}y_0z_{-2}+1),\quad x_{6n+3}=-\frac{x_0}{x_0y_{-2}z_{-1}+1},$$

$$y_{6n-2}=y_{-2},\quad y_{6n-1}=y_{-1},\quad y_{6n}=y_0,$$

$$y_{6n+1}=-\frac{y_{-2}}{x_0y_{-2}z_{-1}+1},\quad y_{6n+2}=-y_{-1}(x_{-2}y_{-1}z_0+1),\quad y_{6n+3}=-\frac{y_0}{x_{-1}y_0z_{-2}+1},$$

$$z_{6n-2}=z_{-2},\quad z_{6n-1}=z_{-1},\quad z_{6n}=z_0,$$

$$z_{6n+1}=-\frac{z_{-2}}{x_{-1}y_0z_{-2}+1},\quad z_{6n+2}=-z_{-1}(x_0y_{-2}z_{-1}+1),\quad z_{6n+3}=-\frac{z_0}{x_{-2}y_{-1}z_0+1},$$

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty}=\left\{x_{-2},x_{-1},x_0,-\frac{x_{-2}}{x_{-2}y_{-1}z_0+1},-x_{-1}(x_{-1}y_0z_{-2}+1),-\frac{x_0}{x_0y_{-2}z_{-1}+1}\right\},$$

$$\{y_n\}_{n=-2}^{+\infty}=\left\{y_{-2},y_{-1},y_0,-\frac{y_{-2}}{x_0y_{-2}z_{-1}+1},-y_{-1}(x_{-2}y_{-1}z_0+1),-\frac{y_0}{x_{-1}y_0z_{-2}+1}\right\}.$$

$$\{z_n\}_{n=-2}^{+\infty}=\left\{z_{-2},z_{-1},z_0,-\frac{z_{-2}}{x_{-1}y_0z_{-2}+1},-z_{-1}(x_0y_{-2}z_{-1}+1),-\frac{z_0}{x_{-2}y_{-1}z_0+1}\right\}.$$

8 The System

$$x_{n+1}=\frac{x_{n-2}}{1-x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{1-y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1-z_{n-2}x_{n-1}y_n}$$

In this section, we study the solution of the following system of difference equations.

$$x_{n+1}=\frac{x_{n-2}}{1-x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{1-y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1-z_{n-2}x_{n-1}y_n}\tag{8}$$

where $n\in N_0$ and the initial conditions are arbitrary nonzero real numbers.

The following theorem is devoted to the form of the solutions of system (7).

Theorem 7. Suppose that $\{x_n,y_n,z_n\}$ are solutions of the system (7). Then for $n=0,1,2,...$, we have the following formulas

$$\begin{aligned}
x_{3n-2} &= -x_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)},\quad x_{3n-1}=x_{-1}\prod_{i=0}^{n-1}\frac{(-1+(3i+1)x_{-1}y_0z_{-2})}{(-1+(3i+2)x_{-1}y_0z_{-2})},\\
x_{3n} &= x_0\prod_{i=0}^{n-1}\frac{(-1+(3i+2)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})},
\end{aligned}$$

$$\begin{aligned}
y_{3n-2} &= -y_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})},\quad y_{3n-1}=y_{-1}\prod_{i=0}^{n-1}\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)},\\
y_{3n} &= y_0\prod_{i=0}^{n-1}\frac{(-1+(3i+2)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2})},
\end{aligned}$$

$$\begin{aligned}
z_{3n-2} &= -z_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})},\quad z_{3n-1}=z_{-1}\prod_{i=0}^{n-1}\frac{(-1+(3i+1)x_0y_{-2}z_{-1})}{(-1+(3i+2)x_0y_{-2}z_{-1})},\\
z_{3n} &= z_0\prod_{i=0}^{n-1}\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)},
\end{aligned}$$

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$.

$$\begin{aligned}
x_{3n-5} &= -x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)},\quad x_{3n-4}=x_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_{-1}y_0z_{-2})}{(-1+(3i+2)x_{-1}y_0z_{-2})},\\
x_{3n-3} &= x_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})},
\end{aligned}$$

$$\begin{aligned}
y_{3n-5} &= -y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})},\quad y_{3n-4}=y_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)},\\
y_{3n-3} &= y_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2})},
\end{aligned}$$

$$z_{3n-5} = -z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}, \quad z_{3n-4} = z_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_0y_{-2}z_{-1})}{(-1+(3i+2)x_0y_{-2}z_{-1})},$$
$$z_{3n-3} = z_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)},$$

It follows from Eq.(7) that

$$\begin{aligned}
x_{3n-2} &= \frac{x_{3n-5}}{1+x_{3n-5}y_{3n-4}z_{3n-3}} \\
&= \frac{-x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1+(-x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)})(y_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)})(z_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)})} \\
&= \frac{-x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1-x_{-2}y_{-1}z_0\prod_{i=0}^{n-2}((\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)})(\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)})(\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)}))} \\
&= \frac{-x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1-x_{-2}y_{-1}z_0\prod_{i=0}^{n-2}(\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)})} \\
&= -x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}\frac{1}{1+(\frac{x_{-2}y_{-1}z_0}{(-1+(3n-3)x_{-2}y_{-1}z_0)})} \\
&= -x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}(\frac{(-1+(3n-3)x_{-2}y_{-1}z_0)}{(-1+(3n-3)x_{-2}y_{-1}z_0)+x_{-2}y_{-1}z_0}) \\
&= -x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}(\frac{(-1+(3n-3)x_{-2}y_{-1}z_0)}{(-1+(3n-2)x_{-2}y_{-1}z_0)})
\end{aligned}$$

Then, we see that

$$x_{3n-2} = -x_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}$$

Also, we see from Eq.(1) that

$$\begin{aligned}
y_{3n-2} &= \frac{y_{3n-5}}{1+y_{3n-5}z_{3n-4}x_{3n-3}} \\
&= \frac{-y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}}{1+(-y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})})(z_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_0y_{-2}z_{-1})}{(-1+(3i+2)x_0y_{-2}z_{-1})})(x_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})})} \\
&= \frac{-y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}}{1-x_0y_{-2}z_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})}} \\
&= -y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}\Big(\frac{1}{1+\frac{x_0y_{-2}z_{-1}}{-1+(3n-3)x_0y_{-2}z_{-1}-}}\Big) \\
&= -y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}\Big(\frac{-1+(3n-3)x_0y_{-2}z_{-1}}{-1+(3n-3)x_0y_{-2}z_{-1}+x_0y_{-2}z_{-1}}\Big) \\
&= -y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}\Big(\frac{-1+(3n-3)x_0y_{-2}z_{-1}}{-1+(3n-2)x_0y_{-2}z_{-1}}\Big)
\end{aligned}$$

Then, we see that

$$y_{3n-2} = -y_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}$$

Finally, we see that

$$\begin{aligned}
z_{3n-2} &= \frac{z_{3n-5}}{1+z_{3n-5}x_{3n-4}y_{3n-3}} \\
&= \frac{-z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}}{1+(-z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})})(x_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_{-1}y_0z_{-2})}{(-1+(3i+2)x_{-1}y_0z_{-2})})(y_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2})})} \\
&= \frac{-z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}}{1-x_{-1}y_0z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2}}} \\
&= -z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}\Big(\frac{1}{1+\frac{x_{-1}y_0z_{-2}}{-1+(3n-3)x_{-1}y_0z_{-2}-}}\Big) \\
&= -z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}\Big(\frac{-1+(3n-3)x_{-1}y_0z_{-2}}{-1+(3n-2)x_{-1}y_0z_{-2}}\Big)
\end{aligned}$$

Then,

$$z_{3n-2} = -z_{-2} \prod_{i=0}^{n-1} \frac{(-1 + (3i)x_{-1}y_0z_{-2})}{(-1 + (3i + 1)x_{-1}y_0z_{-2})}$$

This completes the proof

8.1 Numerical Examples

For confirming the results of this section, we consider the following numerical example which represent solutions to the previous systems.

Example 1. We consider interesting numerical example for the difference equations system (1) with the initial conditions $x_{-2} = 13$, $x_{-1} = 0.4$, $x_0 = 3$, $y_{-2} = 0.5$, $y_{-1} = 7$, $y_0 = 3.7$, $z_{-2} = 0.9$, $z_{-1} = 17$ *and* $z_0 = 0.72$. (See Fig. 1).

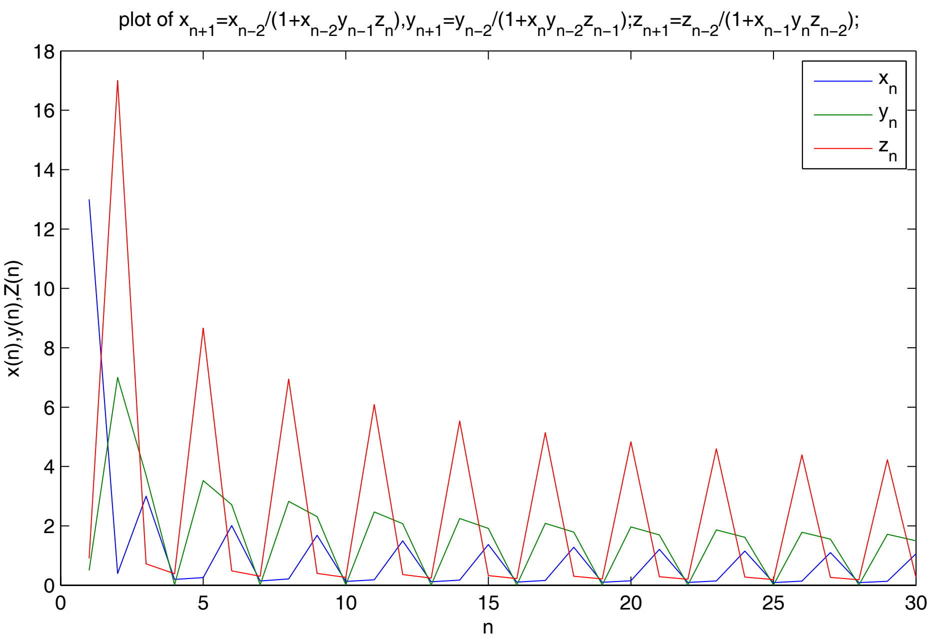

Figure 1.

Example 2. We put the initial conditions for system (2) as follows: $x_{-2} = 1.3$, $x_{-1} = -0.4$, $x_0 = 0.3$, $y_{-2} = 0.5$, $y_{-1} = 0.1$, $y_0 = -0.7$, $z_{-2} = -0.9$, $z_{-1} = 0.7$ *and* $z_0 = 0.2$. (See Fig. 2).

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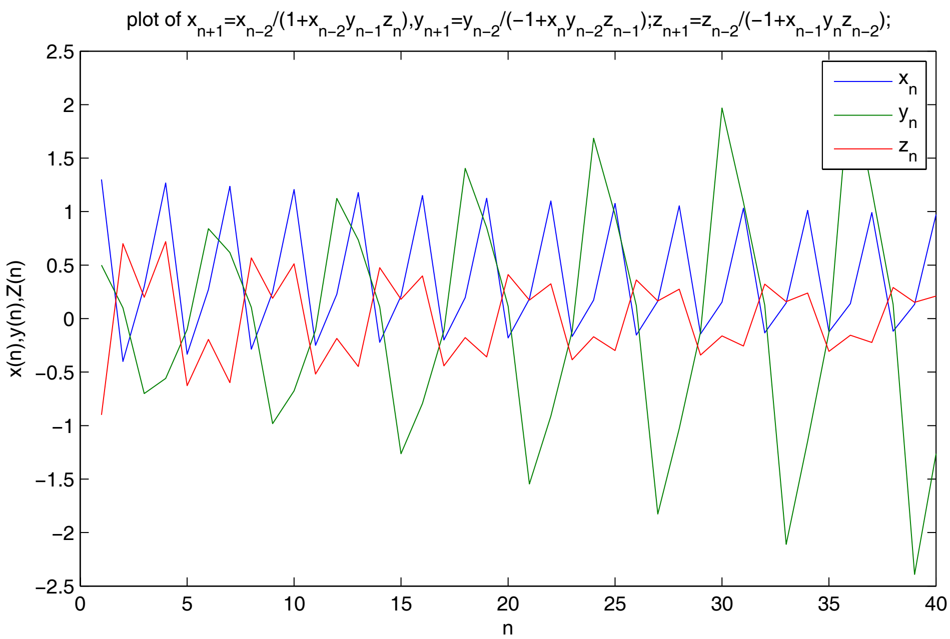

Figure 2.

Example 3. For the difference equations system (3) where the initial conditions $x_{-2} = 1.3,\ x_{-1} = 0.4,\ x_0 = 0.3,\ y_{-2} = 0.25,\ y_{-1} = 0.1,\ y_0 = 0.7,\ z_{-2} = 0.9,\ z_{-1} = 0.7$ *and* $z_0 = 0.2$. (See Fig. 3).

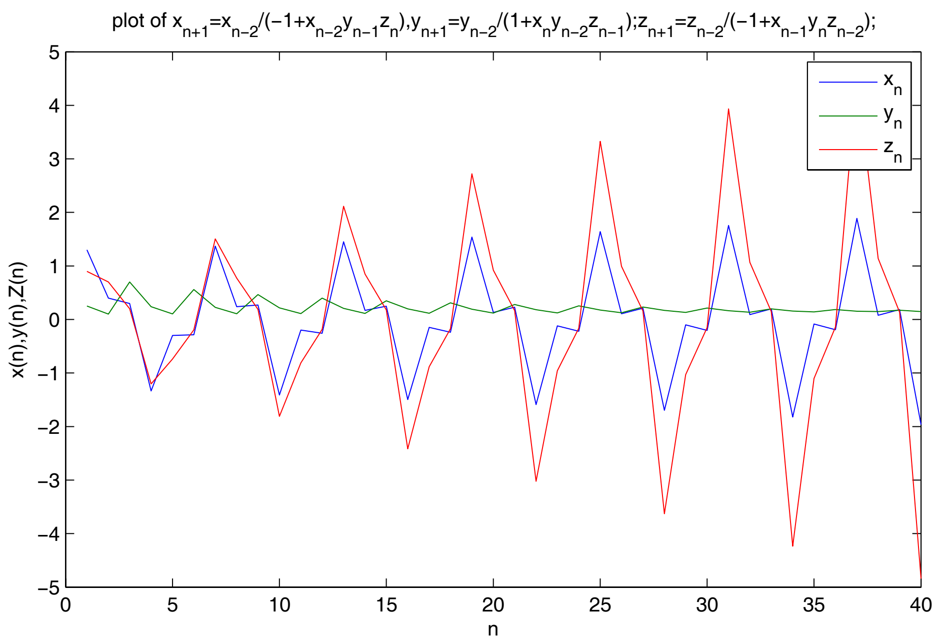

Figure 3.

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Example 4. We assume $x_{-2} = 1.3,\ x_{-1} = 0.4,\ x_0 = 0.3,\ y_{-2} = 0.25,\ y_{-1} = 0.1,$ $y_0 = 0.7,\ z_{-2} = 0.9,\ z_{-1} = 0.7$ *and* $z_0 = 0.2$ for system (4) see Fig. 4.

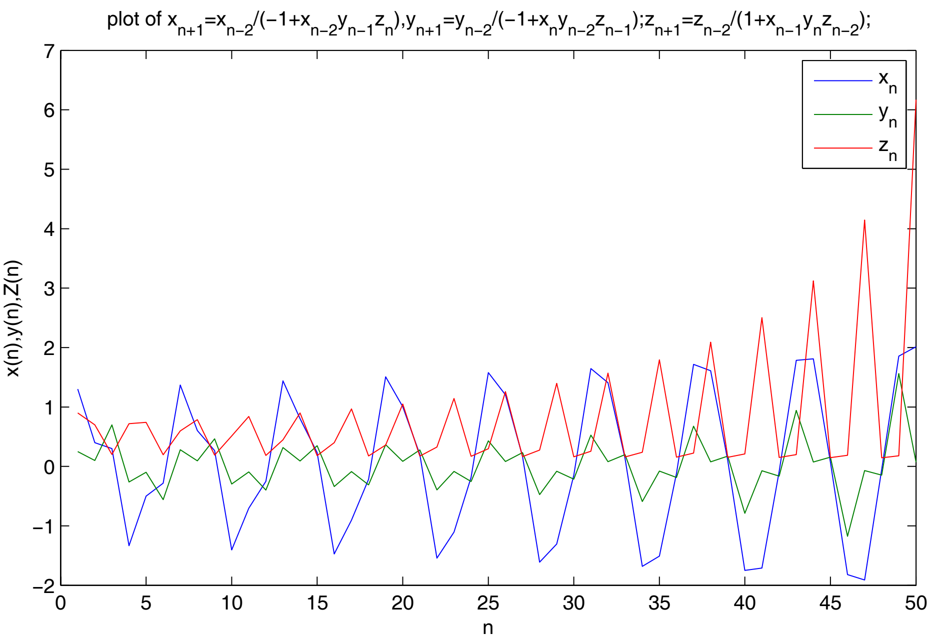

Figure 4.

Example 5. See Fig. 5, if we take system (5) with $x_{-2} = 3,\ x_{-1} = -0.4,\ x_0 = 2,\ y_{-2} = -0.5,\ y_{-1} = 0.9,\ y_0 = 0.7,\ z_{-2} = 0.19,\ z_{-1} = -0.4$ *and* $z_0 = 0.1$.

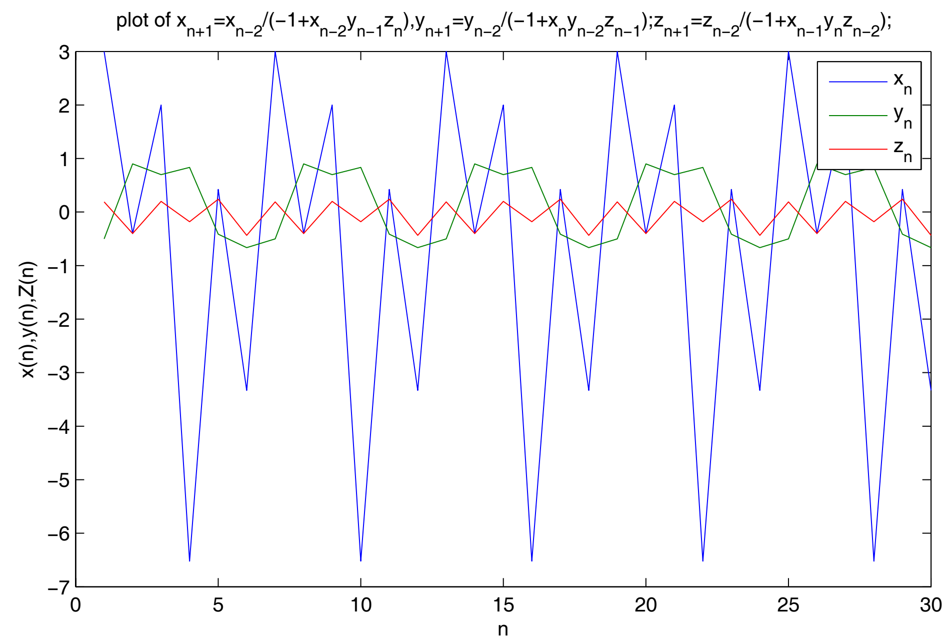

Figure 5.

Example 6. See Fig. 6, if we consider system (6) with $x_{-2} = -9,\ x_{-1} = 0.4,\ x_0 = -2,\ y_{-2} = 0.2,\ y_{-1} = 0.7,\ y_0 = 1.8,\ z_{-2} = 9,\ z_{-1} = -0.4$ *and* $z_0 = -2$.

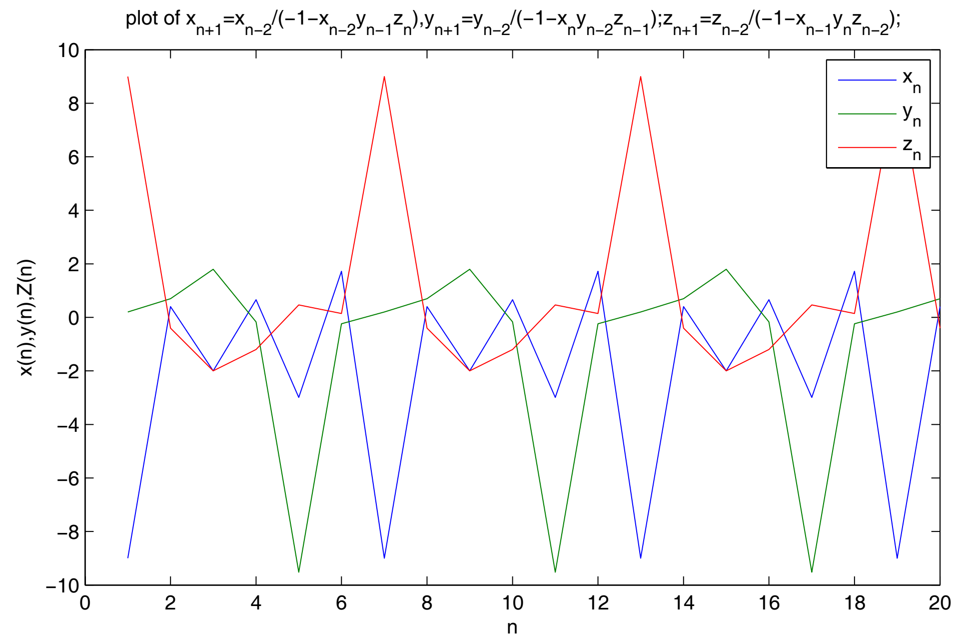

Figure 6.

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Example 7. We take the difference equations system (7) with the initial conditions $x_{-2} = 9,\ x_{-1} = 4,\ x_0 = 2,\ y_{-2} = 3,\ y_{-1} = 7,\ y_0 = 18,\ z_{-2} = 11,\ z_{-1} = -4$ *and* $z_0 = 5$. (See Fig. 7).

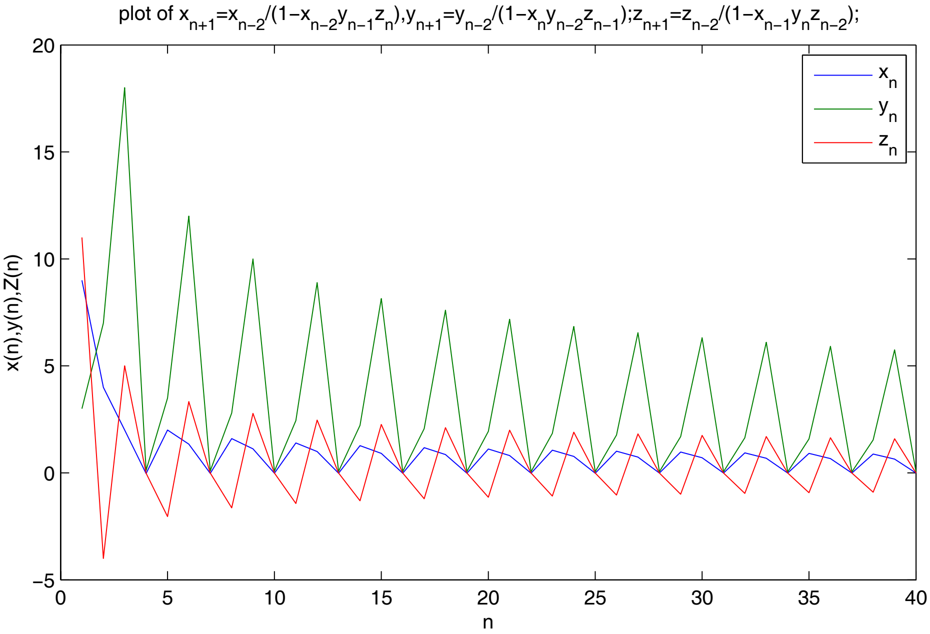

Figure 7.

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