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Mittag-Leffler-Hyers-Ulam Stability of Linear Differential Equations using Fourier Transforms

J.M. Rassias, R. Murali, and A. Ponmana Selvan

Abstract. In this paper, we are going to study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the general Linear Differential Equations of Higher order with constant coefficients using Fourier Transforms method. Moreover, the Mittag-Leffler-Hyers-Ulam stability constants of these differential equations are obtained. Some examples are given to illustrate the main results.

AMS Subject Classification: 34K20, 26D10, 34A40, 39B82.

Key Words and Phrases: Mittag-Leffler-Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam-Rassias stability, Linear Differential equation, Fourier Transforms.

1. Introduction

A classical question in the theory of functional equation is the following : "when is it true that a function which approximately satisfies a functional equation (g) must be close to an exact solution of (g)?" If the problem accepts a solution, we say that the equation (g) is stable.

A simulating and famous talk presented by Ulam [**40**] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range of talks before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. One of his question was that when is it true that a mapping that approximately satisfies a functional equation must be close to an exact solution of the equation? If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [**9**] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings on Banach spaces. In the course of time, the Theorem formulated by Hyers was generalized by Rassias Th. [**36**], Aoki [**4**], Bourgin [**6**] and J.M.Rassias [**29**] for additive mappings. Then a number of authors has studied the Ulam problem for various functional equations by different methods in [**2, 21, 30, 31, 32, 33, 34, 35**].

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

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has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', ...x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon\to 0} K(\epsilon) = 0$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability. Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [**25, 26**]. Thereafter, In 1998, C. Alsina and R. Ger [**3**] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved in [**3**] the following Theorem.

Theorem 1.1. *Assume that a differentiable function $f : I \to R$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open sub interval of $\mathbb{R}$. Then there exists a solution $g : I \to R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of C. Alsina and R. Ger [**3**] has been generalized by Takahasi [**39**]. They proved in [**39**] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [**11, 12, 13, 17, 18, 19, 20**]. Using the approach as in [**40**], Miura, Takahasi and Choda [**19**], Miura [**20**], Takahasi, Miura and Miyajima [**39**] and Miura, Jung and Takahasi are [**17**] proved that the Hyers-Ulam stability holds true for the differential equation $x' = \lambda x$, while Jung [**11**] proved a similar result for the differential equation $\phi(t)x'(t) = x$.

In 2006, S.M. Jung [**14**] investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients by using matrix method. In 2007, G. Wang, M. Zhou, L. Sun [**42**] studied the Hyers-Ulam stability of a class of first-order linear differential equations. I. A. Rus [**37**] discussed four types of Ulam stability: Ulam-Hyers stability, Generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and Generalized Ulam-Hyers-Rassias stability of the Ordinary Differential Equation $u'(t) = A(u(t)) + f(t, u(t)), t \in [a, b]$. In 2014, Q. H. Alqifiary and S. M. Jung [**5**] proved the Generalized Hyers-Ulam stability of linear differential equation by using the Laplace Transforms. These days the Hyers-Ulam stability of differential equations are investigated [**1, 7, 8, 15, 16, 22, 24, 27, 28, 43**] and the investigation is ongoing.

Recently, Vida Kalvandi, N. Eghbali and J.M. Rassias [**41**] studied the Mittag-Leffler-Hyers-Ulam stability of a fractional differential equation of second order. In this paper, with the help of Fourier Transforms, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation

$$x'(t) + l\ x(t) = 0 \tag{1.1}$$

and the non-homogeneous linear differential equation

$$x'(t) + l\ x(t) = r(t) \tag{1.2}$$

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where l is a scalar, $x(t)$ and $r(t)$ are the continuously differentiable functions. Also, by using Fourier Transforms, we establish the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order homogeneous linear differential equation

$$x''(t) + l\ x'(t) + m\ x(t) = 0 \tag{1.3}$$

and the non-homogeneous second order differential equation

$$x''(t) + l\ x'(t) + m\ x(t) = r(t) \tag{1.4}$$

where l and m are scalars, $x(t)$ is a twice continuously differentiable function and $r(t)$ is a continuously differentiable function.

2. Preliminaries

In this section, we introduce some standard notations, Definitions and Theorems, it will be very useful to prove our main results.

Throughout this paper, $\mathbb{F}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{F}$ of exponential order if there exists a constants $A, B \in \mathbb{R}$ such that $|f(t)| \leq Ae^{tB}$ for all $t > 0$.

For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order. Let g denote the Fourier Transform of f so that

$$g(u) = \int_{-\infty}^{\infty} f(t)\ e^{-itu}\ dt.$$

Then, at points of continuity of f, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u)\ e^{-ixu}\ du,$$

this is called the inverse Fourier transforms. The Fourier transform of f is denoted by $\mathcal{F}(\xi)$. We also introduce a notion, the convolution of two functions.

Definition 2.1. (Convolution). Given two functions f and g, both Lebesgue integrable on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t)\ g(x-t)\ dt$$

exists. This integral defines a function h on S called the convolution of f and g. We also write $h = f * g$ to denote this function.

Theorem 2.2. *The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of the Fourier transform of $f(x)$ and $g(x)$. That is,*

$$\mathcal{F}\{f(x) * g(x)\} = \mathcal{F}\{f(x)\}\ \mathcal{F}\{g(x)\} = F(s)\ G(s)$$

or

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f(t)\ g(x-t)\ dt\right\} = F(s)\ G(s),$$

where $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively.

Definition 2.3. [**41**] The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} z^k$$

where $z, \alpha \in \mathbb{C}$ and $Re(\alpha) > 0$. If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k} = e^z.$$

Definition 2.4. [**41**] The generalization of $E_\alpha(z)$ is defined as a function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} z^k$$

where $z, \alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.

Now, we give the definition of Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equations (1.1), (1.2), (1.3) and (1.4).

Definition 2.5. The linear differential equation (1.1) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l\ x(t)| \le \epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.1) such that $|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.1).

Definition 2.6. The linear differential equation (1.2) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l\ x(t) - r(t)| \le \epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.2) such that $|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.2).

Definition 2.7. The linear differential equation (1.3) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function satisfying

$$|x''(t) + l\ x'(t) + m\ x(t)| \le \epsilon E_\alpha(t^\alpha),$$

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where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \to \mathbb{F}$ satisfies the differential equation (1.3) such that $|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.3).

Definition 2.8. The linear differential equation (1.4) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function satisfying

$$|x''(t) + l\ x'(t) + m\ x(t) - r(t)| \leq \epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \to \mathbb{F}$ satisfies the differential equation (1.4) such that $|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.4).

Definition 2.9. We say that the homogeneous linear differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function, if there exists $\phi : (0, \infty) \to (0, \infty)$ satisfies the inequality

$$|x'(t) + l\ x(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \to \mathbb{F}$ satisfies the differential equation (1.1) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.1).

Definition 2.10. We say that the non-homogeneous linear differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function, if there exists $\phi : (0, \infty) \to (0, \infty)$ satisfies the inequality

$$|x'(t) + l\ x(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \to \mathbb{F}$ satisfies the differential equation (1.2) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.2).

Definition 2.11. We say that the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function, if there exists $\phi : (0, \infty) \to (0, \infty)$ satisfies the inequality

$$|x''(t) + l\ x'(t) + m\ x(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \to \mathbb{F}$ satisfies the differential equation (1.3) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.3).

Definition 2.12. We say that the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function, if there exists $\phi : (0, \infty) \to (0, \infty)$ satisfies the inequality

$$|x''(t) + l\ x'(t) + m\ x(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.4) such that $|x(t) - y(t)| \le K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.4).

3. Mittag-Leffler-Hyers-Ulam Stability

In the following theorems, we prove the Mittag-Leffler-Hyers-Ulam stability of the homogeneous and non-homogeneous linear differential equations (1.1), (1.2), (1.3) and (1.4). Firstly, we prove the Mittag-Leffler-Hyers-Ulam stability of first order homogeneous differential equation (1.1).

Theorem 3.1. *The differential equation* (1.1) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let l be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0,\infty) \to \mathbb{F}$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l\ x(t)| \le \epsilon E_\alpha(t^\alpha) \tag{3.1}$$

for all $t > 0$. We will prove that, there exists a solution $y : (0,\infty) \to \mathbb{F}$ satisfying the differential equation $y'(t) + l\ y(t) = 0$ such that

$$|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha)$$

for any $t > 0$. Let us define a function $p : (0,\infty) \to \mathbb{F}$ such that $p(t) =: x'(t) + l\ x(t)$ for each $t > 0$. In view of (3.1), we have $|p(t)| \le \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\{x'(t) + l\ x(t)\} \\ P(\xi) &= \mathcal{F}\{x'(t)\} + l\ \mathcal{F}\{x(t)\} = -i\xi X(\xi) + l\ X(\xi) = (l - i\xi)X(\xi) \\ X(\xi) &= \frac{P(\xi)}{(l - i\xi)}. \end{aligned}$$

Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)\ (l + i\xi)}{l^2 - \xi^2}. \tag{3.2}$$

Taking $Q(\xi) = \dfrac{1}{(l - i\xi)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l - i\xi)} \quad \Rightarrow \quad q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(l - i\xi)}\right\}.$$

Now, we set $y(t) = e^{-lt}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt}\ e^{ist}\ dt = \int_{-\infty}^{0} e^{-lt}\ e^{ist}\ dt + \int_{0}^{\infty} e^{-lt}\ e^{ist}\ dt = 0. \tag{3.3}$$

Now,

$$\mathcal{F}\{y'(t) + l\ y(t)\} = \mathcal{F}\{y'(t)\} + l\ \mathcal{F}\{y(t)\} = -i\xi Y(\xi) + l\ Y(\xi) = (l - i\xi)Y(\xi).$$

Then by using (3.3), we have $\mathcal{F}\{y'(t) + l\ y(t)\} = 0$, since $\mathcal{F}$ is one-to-one operator, thus $y'(t) + l\ y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.1). Then by using (3.2) and (3.3) we can obtain

$$\begin{aligned}\mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} = X(\xi) - Y(\xi) = \frac{P(\xi)\ (l+i\xi)}{l^2-\xi^2} = P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\}\\ \Rightarrow \qquad \mathcal{F}\{x(t)-y(t)\} = \mathcal{F}\{p(t) * q(t)\}.\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t)-y(t)| = |p(t)*q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \le |p(t)| \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right| \le K\epsilon E_\alpha(t^\alpha).$$

Where $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$ exists for each value of t. Then by virtue of Definition 2.5 the homogeneous linear differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam stability. □

Now, we are going prove the Mittag-Leffler-Hyers-Ulam stability of the non-homogeneous linear differential equation (1.2) using Fourier transform method.

Theorem 3.2. *The differential equation* (1.2) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let l be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0,\infty) \to \mathbb{F}$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l\ x(t) - r(t)| \le \epsilon E_\alpha(t^\alpha) \tag{3.4}$$

for all $t > 0$. We have to show that there exists a solution $y : (0,\infty) \to \mathbb{F}$ satisfying the non-homogeneous differential equation $y'(t) + l\ y(t) = r(t)$ such that $|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$.

Let us define a function $p : (0,\infty) \to \mathbb{F}$ such that $p(t) =: x'(t) + l\ x(t) - r(t)$ for each $t > 0$. In view of (3.4), we have $|p(t)| \le \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned}\mathcal{F}\{p(t)\} &= \mathcal{F}\{x'(t) + l\ x(t) - r(t)\}\\ P(\xi) &= \mathcal{F}\{x'(t)\} + l\ \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\}\\ &= -i\xi X(\xi) + l\ X(\xi) - R(\xi) = (l - i\xi)X(\xi) - R(\xi)\\ X(\xi) &= \frac{P(\xi) + R(\xi)}{(l - i\xi)}.\end{aligned}$$

Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{\{P(\xi) + R(\xi)\}\ (l+i\xi)}{l^2 - \xi^2}. \tag{3.5}$$

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Let us choose $Q(\xi)$ as $\dfrac{1}{(l-i\xi)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l-i\xi)} \quad \Rightarrow \quad q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(l-i\xi)}\right\}.$$

Now, we set $y(t) = e^{-lt} + (r(t) * q(t))$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int\limits_{-\infty}^{\infty} e^{-lt}\ e^{ist}\ dt + \frac{R(\xi)}{(l-i\xi)} = \frac{R(\xi)}{(l-i\xi)} \tag{3.6}$$

Now, $\mathcal{F}\{y'(t) + l\ y(t)\} = -i\xi Y(\xi) + l\ Y(\xi) = R(\xi)$. Then by using (3.6), we have

$$\mathcal{F}\{y'(t) + l\ y(t)\} = F\{r(t)\},$$

since $\mathcal{F}$ is one-to-one operator, thus $y'(t) + l\ y(t) = r(t)$, Hence $y(t)$ is a solution of the differential equation (1.2). Then by using (3.5) and (3.6) we have

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} = X(\xi) - Y(\xi) &= \frac{\{P(\xi) + R(\xi)\}\ (l+i\xi)}{l^2 - \xi^2} - \frac{R(\xi)}{(l-i\xi)} \\ &= P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\} \\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int\limits_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \leq |p(t)|\ \left|\int\limits_{-\infty}^{\infty} q(t-s)\ ds\right| \leq K\epsilon E_\alpha(t^\alpha).$$

Where $K = \left|\int\limits_{-\infty}^{\infty} q(t-s)\ ds\right|$, the integral exists for each value of t. Hence, by the virtue of Definition 2.6 the non-homogeneous differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam stability. □

Now, we prove the Mittag-Leffler-Hyers-Ulam stability of the homogeneous and non-homogeneous second order linear differential equations (1.3) and (1.4).

Theorem 3.3. *The differential equation* (1.3) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu + \nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ be a twice continuously differentiable function satisfying the inequality

$$|x''(t) + l\ x'(t) + m\ x(t)| \leq \epsilon E_\alpha(t^\alpha) \tag{3.7}$$

for all $t > 0$. We will show that there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the homogeneous differential equation $y''(t) + l\ y'(t) + m\ y(t) = 0$ such that

$$|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha),$$

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for any $t > 0$. Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x''(t) + l\, x'(t) + m\, x(t)$ for each $t > 0$. In view of (3.7), we have $|p(t)| \leq \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\{x''(t) + l\ x'(t) + m\ x(t)\} \\ P(\xi) &= \mathcal{F}\{x''(t)\} + l\ \mathcal{F}\{x'(t)\} + m\ \mathcal{F}\{x(t)\} = (\xi^2 - i\xi l + m)\ X(\xi) \\ X(\xi) &= \frac{P(\xi)}{\xi^2 - i\xi l + m}. \end{aligned}$$

Since l, m are constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)\ (i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{3.8}$$

Let $Q(\xi) = \dfrac{1}{(i\xi - \mu)\ (i\xi - \nu)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(i\xi - \mu)\ (i\xi - \nu)} \quad \Rightarrow \quad q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(i\xi - \mu)\ (i\xi - \nu)}\right\}.$$

Now, setting $y(t)$ as $\dfrac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}$ and taking Fourier transform, we obtain

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}\ e^{ist}\ dt = 0. \tag{3.9}$$

Now,

$$\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = (\xi^2 - i\xi l + m)\ Y(\xi).$$

Then by using (3.9), we have $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = 0$. Since $\mathcal{F}$ is one-to-one operator, then $y''(t) + l\ y'(t) + m\ y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.3). Then by using (3.8) and (3.9) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} = X(\xi) - Y(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m} = P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\} \\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \leq |p(t)|\ \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right| \leq K\epsilon E_\alpha(t^\alpha).$$

Where $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$, the integral exists for each value of t. Then by virtue of Definition 2.7 the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam stability. $\square$

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Theorem 3.4. *The differential equation* (1.4) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu+\nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0,\infty) \to \mathbb{F}$ is a twice continuously differentiable function satisfying the inequality

$$|x''(t) + l\ x'(t) + m\ x(t) - r(t)| \leq \epsilon E_\alpha(t^\alpha) \tag{3.10}$$

for all $t > 0$. We have to prove that there exists a solution $y : (0,\infty) \to \mathbb{F}$ satisfying the non-homogeneous differential equation $y''(t) + l\ y'(t) + m\ y(t) = r(t)$ such that

$$|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha),$$

for any $t > 0$. Assume that $x(t)$ is a continuously differentiable function satisfying the inequality (3.10). Let us define a function $p : (0,\infty) \to \mathbb{F}$ such that $p(t) =: x''(t) + l\ x'(t) + m\ x(t) - r(t)$ for each $t > 0$. In view of (3.10), we have $|p(t)| \leq \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned}
\mathcal{F}\{p(t)\} &= \mathcal{F}\{x''(t) + l\ x'(t) + m\ x(t) - r(t)\}\\
P(\xi) &= \mathcal{F}\{x''(t)\} + l\ \mathcal{F}\{x'(t)\} + m\ \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\} = (\xi^2 - i\xi l + m)\ X(\xi) - R(\xi)\\
X(\xi) &= \frac{P(\xi) + R(\xi)}{\xi^2 - i\xi l + m}.
\end{aligned}$$

Since l, m are constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)\ (i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{3.11}$$

Taking

$$Q(\xi) = \mathcal{F}\{q(t)\} = \frac{1}{(i\xi - \mu)\ (i\xi - \nu)},$$

setting

$$y(t) = \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} + (r(t) * q(t))$$

and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}\ e^{ist}\ dt + \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} = \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{3.12}$$

Now,

$$\begin{aligned}
\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} &= \mathcal{F}\{y''(t)\} + l\ \mathcal{F}\{y'(t)\} + m\ \mathcal{F}\{y(t)\}\\
&= (\xi^2 - i\xi l + m)\ Y(\xi) = R(\xi).
\end{aligned}$$

Then by using (3.12), we have $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = \mathcal{F}\{r(t)\}$, since $\mathcal{F}$ is one-to-one operator, thus $y''(t) + l\ y'(t) + m\ y(t) = r(t)$, Hence $y(t)$ is a solution of the differential

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equation (1.4). Then by using (3.11) and (3.12) we can obtain

$$\begin{aligned}\mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} = X(\xi) - Y(\xi) &= \frac{P(\xi) + R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} - \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} \\ &= P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\} \\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\}\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds \right| \leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right| \leq K\epsilon E_\alpha(t^\alpha).$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right|$, the integral exists for each value of t. Then by virtue of Definition 2.8 the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam stability. ☐

4. Mittag-Leffler-Hyers-Ulam-Rassias Stability

In the following theorems, we are going to investigate the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equations (1.1), (1.2), (1.3) and (1.4).

Theorem 4.1. *The differential equation* (1.1) *has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Let l be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ be a continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ be an integrable function satisfies

$$|x'(t) + l\ x(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.1}$$

for all $t > 0$. We will prove that, there exists a solution $y : (0, \infty) \to \mathbb{F}$ which satisfies the differential equation $y'(t) + l\ y(t) = 0$ such that

$$|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$$

for any $t > 0$. Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x'(t) + l\ x(t)$ for each $t > 0$. In view of (4.1), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)\ (l + i\xi)}{l^2 - \xi^2}. \tag{4.2}$$

Choosing $Q(\xi) = \dfrac{1}{(l - i\xi)}$, then we have $q(t) = \mathcal{F}^{-1}\left\{\dfrac{1}{(l - i\xi)}\right\}$. Now, we set $y(t) = e^{-lt}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt}\ e^{ist}\ dt = 0. \tag{4.3}$$

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Hence

$$\mathcal{F}\{y'(t) + l\ y(t)\} = -i\xi Y(\xi) + l\ Y(\xi) = (l - i\xi)Y(\xi)$$

Then by using (4.3), we have $\mathcal{F}\{y'(t) + l\ y(t)\} = 0$, since $\mathcal{F}$ is one-to-one operator, thus $y'(t) + l\ y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.1). Then by using (4.2) and (4.3) we can obtain

$$\mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds \right| \leq |p(t)| \ \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right| \leq K\phi(t)\epsilon E_\alpha(t^\alpha).$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right|$, the integral exists for each value of t and $\phi(t)$ is an integrable function. Then by virtue of Definition 2.9 the differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. □

Now, we prove the Mittag-Leffler-Hyers-Ulam-Rassias stability of the non-homogeneous linear differential equation (1.2) with the help of Fourier Transforms.

Theorem 4.2. *The differential equation* (1.2) *has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Let l be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ an integrable function satisfying

$$|x'(t) + l\ x(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.4}$$

for all $t > 0$. We will now prove that, there exist a solution $y : (0, \infty) \to \mathbb{F}$, which satisfies the differential equation $y'(t) + l\ y(t) = r(t)$ such that

$$|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha),$$

for any $t > 0$. Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x'(t) + l\ x(t) - r(t)$ for each $t > 0$. In view of (4.4), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{\{P(\xi) + R(\xi)\}\ (l + i\xi)}{l^2 - \xi^2}. \tag{4.5}$$

Now, let us take $Q(\xi)$ as $\dfrac{1}{(l - i\xi)}$; then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l - i\xi)} \quad \Rightarrow \quad q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(l - i\xi)}\right\}.$$

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We set $y(t) = e^{-lt} + (r(t) * q(t))$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt}\, e^{ist}\, dt + \frac{R(\xi)}{(l - i\xi)} = \frac{R(\xi)}{(l - i\xi)} \tag{4.6}$$

Now,

$$\mathcal{F}\{y'(t) + l\; y(t)\} = \mathcal{F}\{y'(t)\} + l\; \mathcal{F}\{y(t)\} = -i\xi Y(\xi) + l\; Y(\xi) = R(\xi)$$

Then by using (4.6), we have $\mathcal{F}\{y'(t) + l\; y(t)\} = F\{r(t)\}$, since $\mathcal{F}$ is one-to-one operator, thus $y'(t) + l\; y(t) = r(t)$. Hence $y(t)$ is a solution of the differential equation (1.2). Then by using (4.5) and (4.6) we can obtain

$$\mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}.$$

Since the operator $\mathcal{F}$ is one-to-one and linear, it gives $x(t)-y(t) = p(t)*q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t)\; q(t-s)\; ds \right| \le |p(t)| \; \left| \int_{-\infty}^{\infty} q(t-s)\; ds \right| \le K\; \phi(t)\epsilon E_\alpha(t^\alpha).$$

If $K = \left| \int\limits_{-\infty}^{\infty} q(t-s)\; ds \right|$ the integral exists for each value of t and $\phi(t)$ is an integrable function. Hence by the virtue of Definition 2.10 the differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. □

Now, we are going to establish the Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order homogeneous differential equation (1.3).

Theorem 4.3. *The second order linear differential equation* (1.3) *has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Let l, m are constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu+\nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ is a twice continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ an integrable function satisfying the inequality

$$|x''(t) + l\; x'(t) + m\; x(t)| \le \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.7}$$

for all $t > 0$. We will now prove that there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the homogeneous differential equation (1.3) such that

$$|x(t) - y(t)| \le K\phi(t)\epsilon E_\alpha(t^\alpha),$$

for any $t > 0$. Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x''(t)+l\; x'(t)+m\; x(t)$ for each $t > 0$. In view of (4.7), we have $|p(t)| \le \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$P(\xi) = \mathcal{F}\{x''(t)\} + l\; \mathcal{F}\{x'(t)\} + m\; \mathcal{F}\{x(t)\} = (\xi^2 - i\xi l + m)\; X(\xi)$$

$$X(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m}.$$

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Since l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)\ (i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{4.8}$$

Choosing $Q(\xi)$ as $\dfrac{1}{(i\xi - \mu)\ (i\xi - \nu)}$, then we have $\mathcal{F}\{q(t)\} = \dfrac{1}{(i\xi - \mu)\ (i\xi - \nu)}$ and we define a function $y(t) = \dfrac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}\ e^{ist}\ dt = 0. \tag{4.9}$$

Now, $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = (\xi^2 - i\xi l + m)\ Y(\xi)$. Then by using (4.9), we have $\mathcal{F}\{y''(t) + l\, y'(t) + m\, y(t)\} = 0$, since $\mathcal{F}$ is one-to-one operator, thus $y''(t) + l\, y'(t) + m\, y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.3). Then by using (4.8) and (4.9) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= X(\xi) - Y(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m} \\ &= P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\} \\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$\begin{aligned} |x(t) - y(t)| = |p(t) * q(t)| &= \left| \int_{-\infty}^{\infty} p(t)\ q(t - s)\ ds \right| \\ &\leq |p(t)| \left| \int_{-\infty}^{\infty} q(t - s)\ ds \right| \leq K\phi(t)\epsilon E_\alpha(t^\alpha). \end{aligned}$$

Where $K = \left| \int_{-\infty}^{\infty} q(t - s)\ ds \right|$ exists for each value of t and $\phi(t)$ is an integrable function. Then by the virtue of Definition 2.11 the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. □

Finally, we are going to investigate the Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order non-homogeneous differential equation (1.4).

Theorem 4.4. *The second order linear differential equation* (1.4) *has the Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Let l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu + \nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ is

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a twice continuously differentiable function and $\phi : (0,\infty) \to (0,\infty)$ an integrable function satisfying the inequality

$$|x''(t) + l\ x'(t) + m\ x(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.10}$$

for all $t > 0$. We have to prove that there exists a solution $y : (0,\infty) \to \mathbb{F}$ satisfying the non-homogeneous differential equation (1.4) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$.

Let us define a function $p : (0,\infty) \to \mathbb{F}$ such that $p(t) =: x''(t) + l\ x'(t) + m\ x(t) - r(t)$ for each $t > 0$. In view of (4.10), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking the Fourier transform to $p(t)$, we have

$$\begin{aligned} P(\xi) &= \mathcal{F}\{x''(t)\} + l\ \mathcal{F}\{x'(t)\} + m\ \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\} \\ &= (\xi^2 - i\xi l + m)\ X(\xi) - R(\xi) \\ X(\xi) &= \frac{P(\xi) + R(\xi)}{\xi^2 - i\xi l + m}. \end{aligned}$$

Since l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)\ (i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{4.11}$$

Assuming $Q(\xi) = \mathcal{F}\{q(t)\} = \dfrac{1}{(i\xi - \mu)\ (i\xi - \nu)}$ and defining a function

$$y(t) = \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} + (r(t) * q(t))$$

and also taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}\ e^{ist}\ dt + \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} = \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{4.12}$$

Now, $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = (\xi^2 - i\xi l + m)\ Y(\xi) = R(\xi)$. Then by using (4.12), we have $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = \mathcal{F}\{r(t)\}$, since $\mathcal{F}$ is one-to-one operator; thus

$$y''(t) + l\ y'(t) + m\ y(t) = r(t).$$

Hence $y(t)$ is a solution of the differential equation (1.4). Then by using (4.11) and (4.12) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= \frac{P(\xi) + R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} - \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} \\ &= P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\} \\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

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Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$\begin{aligned} |x(t) - y(t)| &= |p(t) * q(t)| \\ &= \left| \int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds \right| \\ &\leq |p(t)| \ \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right| \leq K\phi(t)\epsilon E_{\alpha}(t^{\alpha}). \end{aligned}$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right|$, the integral exists for each value of t. Then by the virtue of Definition 2.12 the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. □

Conclusion: We have proved the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equations of first order and second order with constant co-efficients using the Fourier Transforms method. That is, we established the sufficient criteria for Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation of first order and second order with constant co-efficients using Fourier Transforms method. Additionally, this paper also provides another method to study the Mittag-Leffler-Hyers-Ulam stability of differential equations. Also, this paper shows that the Fourier Transform method is more convenient to study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation with constant co-efficients.

References

[1] M.R. Abdollahpoura, R. Aghayaria, M.Th. Rassias, Hyers-Ulam stability of associated Laguerre Differential equations in a subclass of analytic functions, Journal of Mathematical Analysis and Applications, 437 (2016), 605-612.

[2] Abbas Najati, Jung Rye Lee, Choonkil Park, and Themistocles M. Rassias, On the stability of a Cauchy type functional equation, Demonstr. Math. 51 (2018) 323-331.

[3] Alsina, C., Ger, R.: On Some inequalities and stability results related to the exponential function. Journal of Inequalities Appl. **2**, 373–380 (1998)

[4] Aoki, T.: On the stability of the linear transformation in Banach Spaces. J. Math. Soc. Japan, **2**, 64–66 (1950)

[5] Alqifiary, Q.H., Jung, S.M.: Laplace Transform And Generalized Hyers-Ulam stability of Differential equations. Elec., J. Diff., Equations, **2014 (80)**, 1–11 (2014)

[6] Bourgin, D.G.: Classes of transformations and bordering transformations. Bull. Amer. Math. Soc. **57**, 223–237 (1951)

[7] I. Fakunle, P.O. Arawomo, Hyers-Ulam stability of certain class of Nonlinear second order differential equations, Global Journal of Pure and Applied Mathematics, 14 (8) (2018) 1029-1039.

[8] P. Gavruta, S. M. Jung and Y. Li, Hyers - Ulam Stability for Second order linear differential equations with boundary conditions, Elec. J. of Diff. Equations, Vol. 2011 (2011), No. 80, pp. 1-5.

[9] Hyers, D.H.: On the Stability of a Linear functional equation. Proc. Natl. Acad. Sci. USA, **27**, 222–224 (1941)

[10] Iz-iddine EL-Fassi, Choonkil Park, and Gwang Hui Kim, Stability and hyperstability of a quadratic functional equation and a characterization of inner product spaces, Demonstr. Math. 51 (2018) 295-303.

[11] Jung, S.M.: Hyers-Ulam stability of linear differential equation of first order. Appl. Math. Lett. **17**, 1135–1140 (2004)

[12] Jung, S.M.: Hyers-Ulam stability of linear differential equations of first order (III). J. Math. Anal. Appl. **311**, 139–146 (2005)

[13] Jung, S.M.: Hyers-Ulam stability of linear differential equations of first order (II). Appl. Math. Lett. **19**, 854–858 (2006)

[14] Jung, S.M.: Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients. J. Math. Anal. Appl. **320**, 549–561 (2006)

[15] S. M. Jung, Approximate solution of a Linear Differential Equation of Third Order, Bull. of the Malaysian Math. Sciences Soc. (2) 35 (4), 2012, 1063-1073.

[16] Y. Li, Y. Shen, Hyers-Ulam stability of linear differential equations of second order, Appl. Math. Lett. 23 (2010), 306-309.

[17] Miura, T.: On the Hyers-Ulam stability of a differentiable map. Sci. Math. Japan, **55**, 17–24 (2002)

[18] Miura, T., Jung, S.M., Takahasi, S.E.: Hyers-Ulam-Rassias stability of the Banach space valued linear differential equation $y' = \lambda y$. J. Korean Math. Soc. **41**, 995–1005 (2004)

[19] Miura, T., Miyajima, S., Takahasi, S.E.: A Characterization of Hyers-Ulam Stability of first order linear differential operators. J. Math. Anal. Appl. **286**, 136–146 (2003)

[20] Miura, T., Takahasi, S.E., Choda, H.: On the Hyers-Ulam stability of real continuous function valued differentiable map. Tokyo J. Math. **24**, 467–476 (2001)

[21] Nazarianpoor, M., Rassias, J.M., Sadeghi, GH.: Solution and stability of Quattuorvigintic Functional equation in Intuitionistic Fuzzy Normed spaces. Iranian Journal of Fuzzy system, **15 (4)**, 13–30 (2018)

[22] R. Murali and A. Ponmana Selvan, On the Generalized Hyers-Ulam Stability of Linear Ordinary Differential Equations of Higher Order, International Journal of Pure and Applied Mathematics, 117 (12) (2017) 317-326.

[23] R. Murali and A. Ponmana Selvan, Hyers-Ulam-Rassias Stability for the Linear Ordinary Differential Equation of Third order, Kragujevac Journal of Mathematics, **42 (4)** (2018) 579-590.

[24] R. Murali and A. Ponmana Selvan, *Hyers-Ulam Stability of Linear Differential Equation*, Computational Intelligence, Cyber Security and Computational Models, Models and Techniques for Intelligent Systems and Automation, Communications in Computer and Information Science, **844**. Springer, Singapore (2018), 183-192.

[25] Obloza, M.: Hyers stability of the linear differential equation. Rockznik Nauk-Dydakt. Prace Math. **13**, 259–270 (1993)

[26] Obloza, M.: Connection between Hyers and Lyapunov stability of the ordinary differential equations. Rockznik Nauk-Dydakt. Prace Math. **14**, 141–146 (1997)

[27] M. N. Qarawani, Hyers-Ulam stability of Linear and Nonlinear differential equation of second order, Int. Journal of Applied Mathematical Research, 1 (4), 2012, 422-432.

[28] M. N. Qarawani, Hyers-Ulam stability of a Generalized second order Nonlinear Differential equation, Applied Mathematics, 2012, 3, pp. 1857-1861.

[29] Rassias, J.M.: On approximation of approximately linear mappings by linear mappings. J. Funct., Anal. **46**, 126–130 (1982)

[30] Eshaghi Gordji, M., Javadian, A., Rassias, J.M.: Stability of systems of bi-quadratic and additive-cubic functional equations in Frechet's spaces. Functional Analysis, Approximation and Computation, **4 (1)**, 85–93 (2012)

[31] Rassias, J.M., Murali, R., Rassias, M.J., Antony Raj, A.: General Solution, stability and Non-stability of Quattuorvigintic functional equation in Multi-Banach spaces. Int. J. Math. And Appl. **5**, 181–194 (2017)

[32] Ravi, K., Rassias, J.M., Senthil Kumar, B.V.: Ulam-Hyers stability of undecic functional equation in quasi-*beta*-normed spaces fixed point method. Tbilisi Mathematical Science, **9 (2)**, 83–103 (2016)

[33] Ravi, K., Rassias, J.M., Pinelas, S., Suresh, S.: General solution and stability of Quattuordecic functional equation in quasi-*beta*-normed spaces. Advances in pure mathematics, **6**, 921–941 (2016) doi: 10.4236/apm.2016.612070

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

[34] Xu, T.Z., Rassias, J.M., Xu, W.X.: A fixed point approach to the stability of a general mixed additive-cubic functional equation in quasi fuzzy normed spaces. International Journal of the Physical Sciences, **6(2)**, 313–324 (2011)

[35] Ravi, K., Rassias, J.M., Senthil Kumar, B.V.: Generalized Ulam-Hyers stability of the Harmonic Mean functional equation in two variables. International Journal of Analysis and Applications, **1 (1)**, 1–17 (2013)

[36] Rassias, Th.M.: On the stability of the linear mappings in Banach Spaces. Proc. Amer. Math. Soc. **72**, 297–300 (1978)

[37] Rus, I.A.: Ulam Stabilities of Ordinary Differential Equations in a Banach Space. Carpathian J. Math. **26 (1)**, 103–107 (2010)

[38] K. Ravi, R. Murali and A. Ponmana Selvan, Hyers - Ulam stability of nth order linear differential equation with initial and boundary condition, Asian Journal of Mathematics and Computer Research, 11 (3), 2016, 201-207.

[39] Takahasi, S.E., Miura, T., Miyajima, S.: On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \alpha y$. Bulletin Korean Math. Soc. **39**, 309-315 (2002)

[40] Ulam, S.M.: Problem in Modern Mathematics, Chapter IV, Science Editors, Willey, New York (1960)

[41] Vida Kalvandi, Nasrin Eghbali and John Micheal Rassias, Mittag-Leffler-Hyers-Ulam stability of fractional differential equations of second order, J. Math. Extension, **13 (1)** (2019) 1 - 15.

[42] Wang, G., Zhou, M., Sun, L.: Hyers-Ulam stability of linear differential equations of first order. Appl. Math. Lett. **21**, 1024–1028 (2008)

[43] J. Xue, Hyers - Ulam stability of linear differential equations of second order with constant coefficient, Italian Journal of Pure and Applied Mathematics, No. 32, (2014) 419-424.

Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrain University of Athens, Athens 15342, Greece.

E-mail address: jrassias@primedu.uoa.gr, Ioannis.Rassias@Primedu.uoa.gr

PG and Research Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur - 635 601, Vellore Dt., Tamil Nadu, India

E-mail address: shcrmurali@yahoo.co.in

PG and Research Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur - 635 601, Vellore Dt., Tamil Nadu, India

E-mail address: selvaharry@yahoo.com